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No. 1

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3次元オゼーン流れについて

内容

1. 運動方程式, 境界条件, 境界力

オゼーン流れの運動方程式は

$$U\frac{\partial}{\partial x_1}u_j = -\frac{1}{\rho}\frac{\partial}{\partial x_j}p + \nu\nabla^2 u_j, \quad j=1,2,3. \quad (\nu=\mu/\rho) \qquad (1.1)$$

連続の方程式は

$$\sum_j \frac{\partial u_j}{\partial x_j} = 0, \qquad (1.2)$$

(1.1)と(1.2)より

$$\nabla^2 p = 0, \qquad (1.3)$$

(1.1)のUの符号を変えると随伴方程式が得られ, その解は一様流れを逆にしたものと考えられるから以下逆流れと呼び(~)印でそれを示すものとする。

$$-U\frac{\partial}{\partial x_1}\tilde{u}_j = -\frac{1}{\rho}\frac{\partial}{\partial x_j}p + \nu\nabla^2 \tilde{u}_j, \qquad (1.4)$$

$$\sum_j^3 \frac{\partial}{\partial x_j}\tilde{u}_j = 0, \qquad (1.5)$$

境界条件は物体表面S上で

$$u_1 = -U, \quad u_2 = u_3 = 0, \quad \text{on } S, \qquad (1.6)$$

$$\tilde{u}_1 = U, \quad \tilde{u}_2 = \tilde{u}_3 = 0, \quad \text{on } S, \qquad (1.7)$$

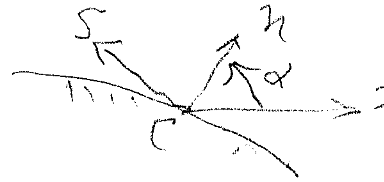
軸対称流れでは

$$x_1 \equiv x, \quad x_2 = r\cos\theta, \quad x_3 = r\sin\theta, \qquad (1.8)$$

なる円筒座標を導入し

$$\left.\begin{aligned} u_r &= u_2\cos\theta + u_3\sin\theta \\ u_\theta &= -u_2\sin\theta + u_3\cos\theta \end{aligned}\right\} \qquad (1.9)$$

回転体の切線をCとし αを図のようにとり



定義すると u の法線成分 u_n,

切線成分 u_s は.

$$\left.\begin{aligned} u_n &= u_1\cos\alpha + u_2\sin\alpha \\ u_s &= -u_1\sin\alpha + u_2\cos\alpha \end{aligned}\right\} \quad \cdots \quad (1.10)$$

(1.6), (1.7) を法線成分と切線成分で書き直すと上式より

$$\left.\begin{aligned} u_n &= -U\cos\alpha = -U\frac{\partial x}{\partial n} = -U\frac{\partial y}{\partial s} \\ u_s &= U\sin\alpha = +U\frac{\partial y}{\partial n} = -U\frac{\partial x}{\partial s} \end{aligned}\right\} \quad \cdots (1.11)$$

もう一度一般的記号に戻って法線の方向余弦を(l_j)とするとその面に働く流体力の各成分 τ_j は

$$\tau_j = \sum_{i=1}^{3} \tau_{ji} l_i \qquad (1.12)$$

$$\tau_{ji} = \mu\gamma_{ji} - p\,\delta_{ij}\ , \qquad (1.13)$$

$$\gamma_{ji} = \gamma_{ij} = \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\ , \quad \cdots (1.14)$$

今渦度 ζ_j を導入しよう。

$$\zeta_j = \frac{\partial u_{j+2}}{\partial x_{j+1}} - \frac{\partial u_{j+1}}{\partial x_{j+2}}\ , \quad \cdots (1.15)$$

$j+2$ は mod(3) で u_j に等しいと解する。

$$\zeta_1 = \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\ , \quad \zeta_2 = \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\ , \quad \zeta_3 = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\ ,$$

(1.1) から

$$\left.\begin{aligned} U\frac{\partial}{\partial x}\zeta_j &= \nu\nabla^2\zeta_j \\ -U\frac{\partial}{\partial x}\tilde{\zeta}_j &= \nu\nabla^2\tilde{\zeta}_j \end{aligned}\right\} \quad \cdots \quad (1.16)$$

である。

さて物体表面上で (1.6) が成立つとすれば，夫々の速度を s に沿って微分すれば 0 であるから

$$\frac{\partial u_j}{\partial x_i} = \frac{\partial u_j}{\partial n} l_i \quad , \quad \cdots \quad (1.17)$$

(1.15) に代入すると

$$\zeta_j = \frac{\partial u_{j+2}}{\partial n} l_{j+1} - \frac{\partial u_{j+1}}{\partial n} l_{j+2} \quad , \quad (1.18)$$

又 (1.2) から

$$\sum_{j=1}^{3} \frac{\partial u_j}{\partial n} l_j = 0 \quad , \quad \cdots \quad (1.19)$$

$$\therefore \quad \frac{\partial u_j}{\partial n} = \zeta_{j+1} l_{j+2} - \zeta_{j+2} l_{j+1} \quad , \quad (1.20)$$

これを (1.14) に代入すると

$$\sum_i \sigma_{ij} l_i = \frac{\partial u_j}{\partial n} + l_j \sum_i \frac{\partial u_i}{\partial n} l_i$$

$$= \zeta_{j+1} l_{j+2} - \zeta_{j+2} l_{j+1} \quad , \quad \cdots \quad (1.21)$$

故に

$$\tau_j = -p l_j + \mu\left[\zeta_{j+1} l_{j+2} - \zeta_{j+2} l_{j+1}\right] \quad , \quad (1.22)$$

その法線成分は

$$\tau_n = \sum_j \tau_j l_j = -p \quad , \cdots\cdots \quad (1.23)$$

ついで再び軸対称流れを考え

$$\left.\begin{aligned} \tau_r &= \tau_2 \cos\theta + \tau_3 \sin\theta \\ 0 &= -\tau_2 \sin\theta + \tau_3 \cos\theta \end{aligned}\right\} \cdots\cdots \quad (1.24)$$

$$\left.\begin{aligned} \zeta_2 \cos\theta + \zeta_3 \sin\theta &= 0 \\ -\zeta_2 \sin\theta + \zeta_3 \cos\theta &= \zeta \end{aligned}\right\} \cdots \quad (1.25)$$

とおくと $\left(\begin{aligned} l_2\cos\theta + l_3 \sin\theta &= \sin\alpha \\ l_1 &= \cos\alpha \end{aligned} \right)$

$$\left.\begin{aligned} \tau_1 &= -p\cos\alpha - \mu\zeta\sin\alpha \\ \tau_r &= -p\sin\alpha + \mu\zeta\cos\alpha \end{aligned}\right\} \cdots \quad (1.26)$$

法線成分と切線成分に変換すると

$$\left.\begin{aligned} \tau_n &= \tau_1 \cos\alpha + \tau_r \sin\alpha = -p \\ \tau_s &= -\tau_1 \sin\alpha + \tau_r \cos\alpha = \mu\zeta \end{aligned}\right\} \cdots \quad (1.27)$$

特に

$$\begin{aligned} \frac{\partial u_1}{\partial x_1} &= \frac{\partial u_1}{\partial n} l_1 = (\zeta_2 l_3 - \zeta_3 l_2) l_1 \\ &= -\zeta (l_3 \sin\theta + l_2 \cos\theta) l_1 = -\zeta \sin\alpha\cos\alpha , \quad (1.28) \end{aligned}$$

$$\gamma_{ij}\Big|_S = \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} = \frac{\partial u_j}{\partial n} l_i + \frac{\partial u_i}{\partial n} l_j$$

$$= (\zeta_{j+1} l_{j+2} - \zeta_{j+2} l_{j+1}) l_i + (\zeta_{i+1} l_{i+2} - \zeta_{i+2} l_{i+1}) l_j$$

$$\sum_j \sum_i \gamma_{ij} \tilde{\gamma}_{ij}\Big|_S = \sum_i \sum_j \Big[(\zeta_{j+1} l_{j+2} - \zeta_{j+2} l_{j+1})(\tilde{\zeta}_{j+1} l_{j+2} - \tilde{\zeta}_{j+2} l_{j+1}) l_i^2$$
$$+ (\zeta_{i+1} l_{i+2} - \zeta_{i+2} l_{i+1})(\tilde{\zeta}_{i+1} l_{i+2} - \tilde{\zeta}_{i+2} l_{i+1}) l_j^2$$
$$+ 2 (\zeta_{j+1} l_{j+2} - \zeta_{j+2} l_{j+1})(\tilde{\zeta}_{i+1} l_{i+2} - \tilde{\zeta}_{i+2} l_{i+1}) l_i l_j \Big]$$

$$= 2 \sum_j \Big[\zeta_{j+1} \tilde{\zeta}_{j+1} l_{j+2}^2 + \zeta_{j+2} \tilde{\zeta}_{j+2} l_{j+1}^2$$
$$- \zeta_{j+2} \tilde{\zeta}_{j+1} l_{j+1} l_{j+2} - \zeta_{j+1} \tilde{\zeta}_{j+2} l_{j+1} l_{j+2} \Big]$$

$$+ 2 \sum_i \sum_j \Big[\zeta_{j+1} \tilde{\zeta}_{i+1} l_i l_j l_{i+2} l_{j+2} + \zeta_{j+2} \tilde{\zeta}_{i+2} l_i l_j l_{i+1} l_{j+1}$$
$$- \zeta_{j+2} \tilde{\zeta}_{i+1} l_i l_j l_{i+2} l_{j+1} - \zeta_{j+1} \tilde{\zeta}_{i+2} l_i l_j l_{i+1} l_{j+2} \Big]$$

$$= 2 \sum \Big[\zeta_j \tilde{\zeta}_j (l_{j+1}^2 + l_{j-1}^2) - \zeta_j \tilde{\zeta}_{j-1} l_j l_{j-1} - \zeta_j \tilde{\zeta}_{j+1} l_j l_{j+1} \Big]$$

$$= 2 \sum_j \zeta_j \tilde{\zeta}_j \quad , \qquad \cdots \quad (1.29)$$

$$\because \sum_j \zeta_j l_j = 0$$

$$\Big[E = \frac{\mu}{2} \iiint \gamma_{ij} \tilde{\gamma}_{ij} \, d\tau \Big]$$

2. 速度場の表現 （以下 $\nu=1$ とする）

附録Aの核関数について 同Bの可逆定理を使うと

$$u_j(Q) = \frac{1}{\mu}\iint_S \left[\sum_i \left(u_i \tilde{T}_i^{(j)} - \tau_i \tilde{U}_i^{(j)}(P,Q)\right) + \rho U\left(\sum_i u_i \tilde{U}_i^{(j)}\right)\frac{\partial x}{\partial n}\right] dS(P), \qquad (2.1)$$

なる表現をうる。

ついで、Sの内側で $u_1=-U$, $u_2=u_3=0$ なる速度場を考えると それについても上と同様な式が成立ち、かつ外側では0 であるからそれを差引くと

$$u_j(Q) = \frac{1}{\mu}\iint_S \sum_i \tau_i \tilde{U}_i^{(j)}(P,Q)\, dS(P), \quad \cdots \qquad (2.2)$$

のように簡単な表現が成立つ。

核の対称性から

$$\tilde{U}_i^{(j)}(P,Q) = U_i^{(j)}(Q,P), \quad \cdots \qquad (2.3)$$

であるから

$$u_j(Q) = -\frac{1}{\mu}\iint_S \sum_i \tau_i(P)\, U_i^{(j)}(Q,P)\, dS(P), \qquad (2.4)$$

と書く事も出来る。

（附録Aのは $U_i^{(j)}(P,Q)$ であるから注意 ）

軸対称流れでは、

$$
\left.\begin{array}{l}
\left.\begin{array}{l}
u_2\cos\theta + u_3\sin\theta = u_r \\
-u_2\sin\theta + u_3\cos\theta = 0
\end{array}\right\} \\
\tau_1 = \frac{\mu}{r}\tau_x, \quad \tau_2 = \frac{\mu}{r}\tau_r\cos\theta, \quad \tau_3 = \frac{\mu}{r}\tau_r\sin\theta,
\end{array}\right\} \quad (2.5)
$$

とおくと

$$
\left.\begin{array}{l}
u_1(Q) = -\int_C \left[\tau_x V_1^{(1)}(Q,P) + \tau_r V_r^{(1)}(Q,P)\right] ds(P), \\
u_r(Q) = -\int_C \left[\tau_x V_1^{(r)}(Q,P) + \tau_r V_r^{(r)}(Q,P)\right] ds,
\end{array}\right\} \quad (2.6)
$$

$$
\left.\begin{array}{l}
\left.\begin{array}{l}
u_n = u_1\cos\alpha + u_r\sin\alpha \\
u_s = -u_1\sin\alpha + u_r\cos\alpha
\end{array}\right\} \\
\left.\begin{array}{l}
\tau_n = \tau_x\cos\alpha + \tau_r\sin\alpha = -\frac{r}{\mu}p \\
\tau_s = \tau_x\sin\alpha + \tau_r\cos\alpha = r\zeta
\end{array}\right\}
\end{array}\right\} \cdots (2.7)
$$

とおくと

$$
\left.\begin{array}{l}
u_n(Q) = -\int_C \left[\tau_n U_n^{(n)}(Q,P) + \tau_s U_s^{(n)}(Q,P)\right] ds, \\
u_s(Q) = -\int_C \left[\tau_n U_n^{(s)}(Q,P) + \tau_s U_s^{(s)}(Q,P)\right] ds,
\end{array}\right\} \quad (2.8)
$$

$$
\left.
\begin{aligned}
U_n^{(n)} &= V_1^{(1)} \cos\alpha \cos\alpha' + V_r^{(1)} \sin\alpha \cos\alpha' + V_1^{(r)} \sin\alpha' \cos\alpha + V_r^{(r)} \sin\alpha \sin\alpha' \\
U_s^{(n)} &= -V_1^{(1)} \sin\alpha \cos\alpha' + V_r^{(1)} \cos\alpha \cos\alpha' - V_1^{(r)} \sin\alpha \sin\alpha' + V_r^{(r)} \cos\alpha \sin\alpha' \\
U_n^{(s)} &= -V_1^{(1)} \cos\alpha \sin\alpha' - V_r^{(1)} \sin\alpha \sin\alpha' + V_1^{(r)} \cos\alpha \cos\alpha' + V_r^{(r)} \sin\alpha \cos\alpha' \\
U_s^{(s)} &= V_1^{(1)} \sin\alpha \sin\alpha' - V_r^{(1)} \cos\alpha \sin\alpha' - V_1^{(r)} \sin\alpha \cos\alpha' + V_r^{(r)} \cos\alpha \cos\alpha',
\end{aligned}
\right\} \quad (2.9)
$$

3. 速度場の漸近性

レイノルズ数が充分小さいと (A·11) より，$U_j^{(k)}$ は

ストークス核 $U_{sj}^{(k)}$ に一致するので

$$u_j(Q) \underset{R\to 0}{\doteqdot} -\frac{1}{\mu}\iint_S \sum \tau_i U_{si}^{(j)}(Q,P)\, ds_P \quad , \cdots (3\cdot 1)$$

なおストークス核では

$$U_{si}^{(j)}(P,Q) = U_{si}^{(j)}(Q,P) \quad , \cdots (3\cdot 2)$$

である。

(2·4) において充分遠方では

$$u_j(Q) \doteqdot -\frac{1}{\mu}\sum_i X_i U_i^{(j)}(Q,0) \quad , \cdots (3\cdot 3)$$

但し

$$X_i = \iint_S \tau_i\, dS \quad , \cdots (3\cdot 4)$$

軸対称流れでは $X_2 = X_3 = 0$ から

$$u_j(Q) \doteqdot -\frac{X_1}{\mu} U_1^{(j)}(Q,0) , \cdots (3\cdot 5)$$

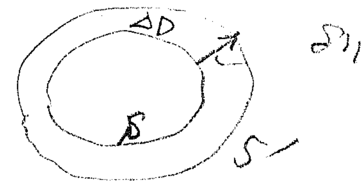
附録 A の式より x 軸の近傍で上流側では

吹出しによる

ポテンシャル流れとなり，後流では wake を持っている。

4. 物体の微少変形 (U=1とする)

物体Sが法線方向に僅かにδnだけ滑らかに変形しS'となったとする。 $D'(\equiv D-\Delta D)$
(領域はDからD'に変る)
速度場は(u_j)から(u_j')に変化し



抗力はX_1からX_1'に変わったとしよう。

境界条件は

$$\left.\begin{aligned} u_1' &= -1,\quad u_2' = u_3' = 0 \\ \hat{u}_1' &= 1,\quad \hat{u}_2' = \hat{u}_3' = 0 \end{aligned}\right\}\ \text{on } S', \qquad (4.1)$$

$$X_1' = \iint_{S'} \tau_1' \, dS' = -\rho \iiint_{D'} \hat{u}_j' \frac{\partial u_j'}{\partial x} d\tau - E_{D'}(u_j', \hat{u}_j'), \qquad (4.2)$$

$(d\tau \equiv dx\,dy\,dz)$

Eの下添字は積分領域を意味する。

$$X_1 = \iint_S \tau_1 \, dS = -\rho \iiint_D \hat{u}_j \frac{\partial u_j}{\partial x} d\tau - E_D(u_j, \hat{u}_j), \qquad (4.3)$$

$$\delta X_1 = X_1' - X_1 \fallingdotseq \mathrm{I} - \mathrm{II}, \qquad (4.4)$$

$$\mathrm{I} = E_{\Delta D}(u_j, \hat{u}_j) + \rho \iiint_{\Delta D} \hat{u}_j \frac{\partial u_j}{\partial x} d\tau, \qquad (4.5)$$

$$\mathrm{II} = E_D(\delta u_j, \hat{u}_j) + E_D(u_j, \delta \hat{u}_j) + \rho \iiint_D \left[\delta \hat{u}_j \frac{\partial u_j}{\partial x} + \hat{u}_j \frac{\partial \delta u_j}{\partial x} \right] d\tau, \qquad (4.6)$$

但し $u_j' - u_j = \delta u_j$,

ΔDは充分小さいから近似的に(1.29)より

$$\mathrm{I} = \sum_j \iint_S \left[\mu \zeta_j \tilde{\zeta}_j + \rho \tilde{u}_j \frac{\partial u_j}{\partial x} \right] \delta n \, dS \quad , \cdots\cdots \quad (4.7)$$

次に(B.3)より

$$\mathrm{II} = - \sum_j \iint_S \left(\tau_j \delta \tilde{u}_j + \tilde{u}_j \delta \tau_j \right) dS \quad , \cdots \quad (4.8)$$

となり，また(B.5)により，

$$\iint_S \tilde{u}_j \delta \tau_j \, dS = \iint_S \left[\tilde{\tau}_j \delta u_j - \rho \tilde{u}_j \delta u_j \frac{\partial x}{\partial n} \right] dS \, , \quad (4.9)$$

（和記号は時々省略する）

それでは

$$\mathrm{II} = - \iint_S \left(\tau_j \delta \tilde{u}_j + \tilde{\tau}_j \delta u_j - \rho \tilde{u}_j \delta u_j \frac{\partial x}{\partial n} \right) dS \, , \quad (4.10)$$

所で

$$\delta u_j = u'_j \big|_{\cancel{S}} - u_i \big|_S = - \frac{\partial u_j}{\partial n} \Big|_S \delta n \quad , \cdots (4.11)$$

であるから(1.20)より

$$\sum_j \left(\tau_j \delta \tilde{u}_j + \tilde{\tau}_j \delta u_j \right) = - 2\mu \left(\sum_j \zeta_j \tilde{\zeta}_j \right) \delta n \, , \quad (4.12)$$

つまり

$$\mathrm{II} = + \iint_S \left[2\mu \zeta_j \tilde{\zeta}_j - \rho \tilde{u}_j \frac{\partial u_j}{\partial x} \right] \delta n \, dS \, , \quad (4.13)$$

$$\because \quad \delta u_j \frac{\partial x}{\partial n} = - \frac{\partial u_j}{\partial n} \frac{\partial x}{\partial n} \Big|_S \delta n = - \frac{\partial u_j}{\partial x} \delta n \, , \quad (4.14)$$

$$\therefore \delta X_1 = \sum_j \iint_S \left[2\rho \tilde{u}_j \frac{\partial u_j}{\partial x} - \mu \zeta_j \tilde{\zeta}_j \right] \delta n \, dS, \quad (4.15)$$

$$= \iint_S \left[2\rho \frac{\partial u_1}{\partial x} - \mu \sum_j \zeta_j \tilde{\zeta}_j \right] \delta n \, dS,$$

同様にして

$$\delta \tilde{X}_1 = \tilde{X}_1' - \tilde{X}_1 = -\tilde{\mathrm{I}} + \tilde{\mathrm{II}}, \quad (4.16)$$

$$\tilde{\mathrm{I}} = \iint_S \left[\mu \sum_j \zeta_j \tilde{\zeta}_j + \rho \frac{\partial \tilde{u}_1}{\partial x} \right] \delta n \, dS, \quad (4.17)$$

$$\tilde{\mathrm{II}} = \iint_S \left[2\mu \sum_j \zeta_j \tilde{\zeta}_j - \rho \frac{\partial \tilde{u}_1}{\partial x} \right] \delta n \, dS, \quad (4.18)$$

$$\delta \tilde{X}_1 = \iint_S \left[\mu \sum_j \zeta_j \tilde{\zeta}_j - 2\rho \frac{\partial \tilde{u}_1}{\partial x} \right] \delta n \, dS, \quad (4.19)$$

それ故

$$\delta X_1 + \delta \tilde{X}_1 = 2\rho \iint_S \frac{\partial}{\partial x}(u_1 - \tilde{u}_1) \delta n \, dS = 0, \quad (4.20)$$

$$(\because \quad X_1 = \tilde{X}_1).$$

$$\frac{\delta X_1 - \delta \tilde{X}_1}{2} = \delta X_1 = -\delta \tilde{X}_1$$

$$= \iint_S \left[\rho \frac{\partial}{\partial x}(u_1 + \tilde{u}_1) - \mu \sum_j \zeta_j \tilde{\zeta}_j \right] \delta n \, dS, \quad (4.21)$$

軸対称な流れでは (1.28) 等により

$$
\left.
\begin{aligned}
\delta X_1 &= -\iint_S \left[\mu\zeta\tilde{\zeta} + 2\rho\zeta\sin\alpha\cos\alpha\right]\delta n\, dS \\
\delta \tilde{X}_1 &= -\iint_S \left[\mu\zeta\tilde{\zeta} + 2\rho\tilde{\zeta}\sin\alpha\cos\alpha\right]\delta n\, dS
\end{aligned}
\right\} \quad \cdots (4.22)
$$

これは形式的に2次元流れと全く同じであるから

最少抵抗問題についても全く同じ取扱いでよい。

附録 A　核関数

$$P_k(P,Q) = \frac{\mu}{4\pi}\frac{\partial}{\partial x_j}\left(\frac{1}{R}\right), \quad R = \overline{PQ} = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2}, \tag{A·1}$$

$$P \equiv (x,y,z),\ Q \equiv (x',y',z'),$$

$$U_j^{(k)} = \frac{1}{8\pi}\frac{\partial^2}{\partial x_j \partial x_k}\Phi - \frac{\delta_{jk}}{8\pi}\Delta\Phi, \tag{A·2}$$

$$\Phi = \frac{1}{k}\int_0^{k\{R-(x-x')\}} \frac{1-e^{-\alpha}}{\alpha}\,d\alpha, \tag{A·3}$$

$$\Delta\Phi = \frac{2}{R}e^{-k\{R-(x-x')\}}, \tag{A·4}$$

$$\Delta\Phi - 2k\frac{\partial}{\partial x}\Phi = \frac{2}{R}, \tag{A·5}$$

Φ の微分を $Q\equiv 0$ において書き下すと

$$\left.\begin{aligned}
\Phi &= \frac{1}{k}\int_0^{k(r-x)} \frac{1-e^{-\alpha}}{\alpha}\,d\alpha \doteqdot \frac{1}{k}\int_0 \left[1 - \frac{\alpha}{2} + \frac{\alpha^2}{6}\ \right]d\alpha \\
&\underset{k\to 0}{=} (r-x)\left[1 - \frac{\alpha}{4} + \frac{\alpha^2}{18} + \right], \quad \underline{\alpha = r-x} \\
\Phi_x &= \left(\frac{x}{r}-1\right)\frac{1-e^{-k(r-x)}}{k(r-x)} = -\frac{1-e^{-k(r-x)}}{kr} \underset{k\to 0}{\doteqdot} \left(\frac{x}{r}-1\right), \\
\Phi_y &= \frac{y}{r}\frac{1-e^{-k(r-x)}}{k(r-x)} \doteqdot \frac{y}{r} \\
\Phi_z &= \frac{z}{r}\frac{1-e^{-\alpha}}{k(r-x)} \doteqdot \frac{z}{r}.
\end{aligned}\right\} \tag{A-6}$$

$$\frac{1}{\alpha}\left[\ -\alpha\left(1-\alpha+\frac{\alpha^2}{2}\right) - \left(\frac{\alpha}{2} - \frac{\alpha^2}{3} + \right)\right.$$

$$\Phi_{xx} = \frac{1}{r}\left(1-\frac{x^2}{r^2}\right)\frac{1-e^{-\alpha}}{\alpha} + k\left(\frac{x}{r}-1\right)^2\left[\frac{e^{-\alpha}}{\alpha} - \frac{1-e^{-\alpha}}{\alpha^2}\right]$$

$$\underset{k\to 0}{\doteqdot} \frac{1}{r}\left(1-\frac{x^2}{r^2}\right)\left[1-\frac{\alpha}{2}+\cdots\right] + \frac{1}{kr^2}\left[-\frac{\alpha^2}{2}+\cdots\right]$$

$$-\frac{k}{2}\left(1-\frac{x}{r}\right)^2$$

$$\frac{\partial\alpha}{\partial y} = \frac{ky}{r}$$

$$\left.\begin{aligned}
\Phi_{xy} &= -\frac{xy}{r^3}\left(\frac{1-e^{-\alpha}}{\alpha}\right) + \frac{ky}{r}\left(\frac{x}{r}-1\right)\left(\frac{e^{-\alpha}}{\alpha} - \frac{1-e^{-\alpha}}{\alpha^2}\right),\\
\Phi_{xz} &= -\frac{xz}{r^3}\left(\frac{1-e^{-\alpha}}{\alpha}\right) + \frac{kz}{r}\left(\frac{x}{r}-1\right)\left(\frac{e^{-\alpha}}{\alpha} - \frac{1-e^{-\alpha}}{\alpha^2}\right),\\
\Phi_{yy} &= \left(\frac{1}{r} - \frac{y^2}{r^3}\right)\left(\frac{1-e^{-\alpha}}{\alpha}\right) + \frac{ky^2}{r^2}\left(\frac{e^{-\alpha}}{\alpha} - \frac{1-e^{-\alpha}}{\alpha^2}\right),\\
\Phi_{yz} &= -\frac{yz}{r^3}\left(\frac{1-e^{-\alpha}}{\alpha}\right) + \frac{kyz}{r^2}\left(\frac{e^{-\alpha}}{\alpha} - \frac{1-e^{-\alpha}}{\alpha^2}\right),\\
\Phi_{zz} &= \left(\frac{1}{r} - \frac{z^2}{r^3}\right)\left(\frac{1-e^{-\alpha}}{\alpha}\right) + \frac{kz^2}{r^2}\left(\frac{e^{-\alpha}}{\alpha} - \frac{1-e^{-\alpha}}{\alpha^2}\right),
\end{aligned}\right\} \quad (A.7)$$

$$\left.\begin{aligned}
\frac{1-e^{-\alpha}}{\alpha} &\fallingdotseq 1 - \left(1-\alpha+\frac{\alpha^2}{2}-\frac{\alpha^3}{6}\right) \fallingdotseq 1 - \frac{\alpha}{2} + \frac{\alpha^2}{6}\\
\frac{e^{-\alpha}}{\alpha} - \frac{1-e^{-\alpha}}{\alpha^2} &= \frac{\partial}{\partial\alpha}\left(\frac{1-e^{-\alpha}}{\alpha}\right) \fallingdotseq -\frac{1}{2} + \frac{\alpha}{3} + \cdots
\end{aligned}\right\}$$

これから

$$\Delta\Phi - \frac{2}{R} = 2k\Phi_x = -\frac{2}{r}\left(1-e^{-\alpha}\right), \quad (A.8)$$

又

$$\left.\begin{aligned}
r_{xx} &= \frac{1}{r} - \frac{x^2}{r^3}, & r_{xy} &= -\frac{xy}{r^3}, & r_{xz} &= -\frac{xz}{r^3}\\
r_{yy} &= \frac{1}{r} - \frac{y^2}{r^3}, & r_{yz} &= -\frac{yz}{r^3}, & r_{zz} &= \frac{1}{r} - \frac{z^2}{r^3}
\end{aligned}\right\} \quad (A.9)$$

であるから、ストークス流れの核を

$$U_{s_j}^{(k)} = \frac{1}{8\pi}\frac{\partial^2}{\partial x_j \partial x_k}(R) - \frac{\delta_{jk}}{4\pi R}. \quad (A.10)$$

とすると

$$U_j^{(k)}(P,Q) - U_{s_j}^{(k)}(P,Q) = \frac{1}{8\pi}\frac{\partial^2}{\partial x_j \partial x_k}\left[\Phi - R\right] - \frac{\delta_{jk}}{8\pi}\left[\Delta\Phi - \frac{2}{R}\right], \quad (A.11)$$

となって右辺は $P \to Q$ で常に0となる。つまりストークス核に一致する。

$$V_1^{(1)} = \int_0^{2\pi} U_1^{(1)} d\theta = \frac{1}{8\pi} \frac{\partial^2}{\partial x^2} \int_0^{2\pi} \bar{\Phi}\, d\theta - \frac{1}{8\pi} \int_0^{2\pi} \Delta \bar{\Phi}\, d\theta, \quad (A\cdot 12)$$

$$V_r^{(1)} = \int_0^{2\pi} \left(U_2^{(1)} \cos\theta + U_3^{(1)} \sin\theta \right) d\theta$$

$$= \frac{1}{8\pi} \int_0^{2\pi} d\theta \left[\frac{\partial^2}{\partial x \partial r} \bar{\Phi} \right]$$

$$V_1^{(r)} = \int_0^{2\pi} \left[U_1^{(2)} \cos\theta' + U_1^{(3)} \sin\theta' \right] d\theta' = \frac{1}{8\pi} \int_0^{2\pi} \frac{\partial^2}{\partial x \partial r'} \bar{\Phi}\, d\theta,$$

$$V_r^{(r)} = \int_0^{2\pi} d\theta \left[\left\{ U_2^{(2)} \cos\theta + U_3^{(2)} \sin\theta \right\} \cos\theta' + \left\{ U_2^{(3)} \cos\theta + U_3^{(3)} \sin\theta \right\} \sin\theta' \right]$$

$$= -\frac{1}{8\pi} \int_0^{2\pi} d\theta \left[\Delta \bar{\Phi} \cos(\theta - \theta') \right] \quad (A\cdot 13)$$

$$+ \frac{1}{8\pi} \int_0^{2\pi} d\theta \left[\frac{\partial^2 \bar{\Phi}}{\partial y \partial r} \cos\theta' + \frac{\partial^2 \bar{\Phi}}{\partial z \partial r} \sin\theta' \right]$$

$$= -\frac{1}{8\pi} \int_0^{2\pi} \Delta \bar{\Phi} \cos(\theta - \theta')\, d\theta' - \frac{1}{8\pi} \int_0^{2\pi} \frac{\partial^2 \bar{\Phi}}{\partial r' \partial r}\, d\theta,$$

$$\bar{\Phi}_{xr} = -\frac{(x-x')}{R^3} (r - r' \cos\overline{\theta - \theta'}) \left(\frac{1 - e^{-\alpha}}{\alpha} \right) + \frac{k(r - r'\cos(\theta-\theta'))}{R} \left(\frac{x-x'}{R} - 1 \right) \left(\frac{e^{-\alpha}}{\alpha} - \frac{1 - e^{-\alpha}}{\alpha^2} \right)$$

$$R^2 = (x-x')^2 + r^2 + r'^2 - 2 r r' \cos(\theta - \theta'),$$

$$\bar{\Phi}_r = \frac{\{r - r'\cos(\theta-\theta')\}}{R} \left(\frac{1 - e^{-\alpha}}{\alpha} \right), \quad (A\cdot 14)$$

$$\bar{\Phi}_{rr'} = \left\{ \frac{-\cos(\theta-\theta')}{R} - \frac{\{r - r'\cos(\theta-\theta')\}\{r' - r\cos(\theta-\theta')\}}{R^3} \right\} \left(\frac{1 - e^{-\alpha}}{\alpha} \right)$$

$$+ \frac{k\{r - r'\cos(\theta-\theta')\}\{r' - r\cos(\theta-\theta')\}}{R^2} \left\{ \frac{e^{-\alpha}}{\alpha} - \frac{1 - e^{-\alpha}}{\alpha^2} \right\},$$

$R \to \infty$ の時は

$$e^{-k[R-(x-x')]} \doteqdot \left\{ \begin{array}{ll} 0 & \text{for } x \ll x' \\ e^{-\frac{k\varpi^2}{2R}} & \text{for } x \gg x', \ \varpi^2 = (y-y')^2 + (z-z')^2, \end{array} \right\} \quad (A\cdot 15)$$

まず $x' \gg x$ （Q点が後流側）では

$$\left.\begin{aligned}
\overline{\Phi}_{xx} &\doteqdot \frac{\{R+(x-x')\}}{kR^3} - \frac{1}{kR^2} = \frac{x-x'}{kR^3} \\
\overline{\Phi}_{xy} &\doteqdot -\frac{(x-x')(y-y')}{kR^3(R-\overline{x-x'})} + \frac{(y-y')}{kR^2(R-\overline{x-x'})} \doteqdot \frac{y-y'}{2kR^3}\left(1 - \frac{x-x'}{R}\right) \doteqdot \frac{y-y'}{kR^3}, \\
\overline{\Phi}_{xz} &\doteqdot -\frac{(x-x')(z-z')}{kR^3(R-\overline{x-x'})} - \frac{(z-z')}{kR^2(R-\overline{x-x'})} \doteqdot \frac{z-z'}{kR^3}, \\
\overline{\Phi}_{yy} &\doteqdot \frac{R^2-(y-y')^2}{kR^3(R-\overline{x-x'})} - \frac{(y-y')^2}{kR^2(R-\overline{x-x'})^2} \doteqdot \frac{1}{2kR} - \frac{3(y-y')^2}{4kR^4} \\
\overline{\Phi}_{yz} &\doteqdot -\frac{(y-y')(z-z')}{kR^3(R-\overline{x-x'})}\left[1 + \frac{R}{R-x}\right] \doteqdot -\frac{3(y-y')(z-z')}{4kR^4} \\
\overline{\Phi}_{zz} &\doteqdot \frac{1}{2kR} - \frac{3(z-z')^2}{4kR^4} \\
\Delta\overline{\Phi} &\doteqdot 0
\end{aligned}\right\} \quad (A\cdot 16)$$

$x \gg x'$ では wake が残る。

附録B　可逆定理

任意の速度場 u_j と $\tilde{u}_j$ に関する次の積分

$$E(u_j, \tilde{u}_j) = \frac{\mu}{2} \iiint_D \left[\sum_i \sum_j \gamma_{ij} \tilde{\gamma}_{ij} \right] dx\,dy\,dz, \quad \cdots (B.1)$$

は可逆的，つまり u_j と $\tilde{u}_j$ を入れかえても値は変らない。

$$E(u_j, \tilde{u}_j) = E(\tilde{u}_j, u_j), \quad \cdots (B.2)$$

部分積分して (1.1), (1.2) を使うと，

$$E = -\rho U \iiint_D \left[\sum_j \tilde{u}_j \frac{\partial u_j}{\partial x} \right] dx\,dy\,dz - \iint_S \left[\sum_j \tilde{u}_j \tau_j \right] dS, \quad (B.3)$$

あるいは (1.4), (1.5) を使うと

$$E = \rho U \iiint_D \left[\sum_j u_j \frac{\partial \tilde{u}_j}{\partial x} \right] dx\,dy\,dz - \iint_S \left[\sum_j u_j \tilde{\tau}_j \right] dS, \quad (B.4)$$

それ故 (B.2) により

$$\iint_S \left[\sum_j (\tilde{u}_j \tau_j - u_j \tilde{\tau}_j) \right] dS = -\rho U \iiint_D \sum_j \left(\tilde{u}_j \frac{\partial u_j}{\partial x} + u_j \frac{\partial \tilde{u}_j}{\partial x} \right) dx\,dy\,dz = -\rho U \iint_S \left(\sum_j u_j \tilde{u}_j \right) \frac{\partial x}{\partial n} dS, \quad (B.5)$$

特に (1.6)(1.7) の境界条件を満たす流れで，物体が閉じていれば右辺は消えて

$$U \iint_S \tau_1\, dS = -U \iint_S \tilde{\tau}_1\, dS, \quad \cdots (B.6)$$

となって抗力は逆流れの，元の流れのものに等しい。

(B・3), (B・4) において、

$$\bar{F}(u_j, \tilde{u}_j) = \rho U \iiint_D \sum u_j \frac{\partial \tilde{u}_j}{\partial x} \, dx\,dy\,dz, \qquad (B・7)$$

なる積分を導入すると

$$E(u_j, \tilde{u}_j) = \bar{F}(u_j, \tilde{u}_j) + \iint_S \sum u_j \tilde{T}_j \, ds$$

$$= -F(\tilde{u}_j, u_j) + \iint_S \sum \tilde{u}_j T_j \, ds, \qquad (B・8)$$

$$\bar{F}(u_j, \tilde{u}_j) + \bar{F}(\tilde{u}_j, u_j) = \rho U \iint_S \Big(\sum_j u_j \tilde{u}_j \Big) \frac{\partial x}{\partial n} \, dS, \qquad (B・9)$$

sign?