Rational extension of the Boussinesq approximation to a two-component fluid

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Abstract

This note extends the Boussinesq approximation to a two-component fluid, in a manner consistent with the conservation law of mass. It is shown that the governing equations for a two-component fluid can systematically be formulated, with the aid of the conservation law of energy, on the following assumption: the density of the fluid is a function solely of the concentration of one component, but the thermal expansion coefficient of the fluid does not vanish. It is also shown that the velocity of the fluid cannot in general be solenoidal under the extended approximation.

1. Introduction

The Boussinesq approximation is originally an approximation for a single-component fluid with a nonuniform temperature distribution in a uniform gravitational field. It is, however, extended in geophysical fluid dynamics so that the motion of a two-component fluid such as seawater can be dealt with (see e.g. Pedlosky 1987, § 1.4; Cushman-Roisin & Beckers 2011, § 3.7). Though this extended approximation is now widely used, it has the fatal defect that the equation of continuity does not hold; this implies not only that the conservation law of mass is violated, but also that the velocity of the fluid is not the momentum per unit mass of the fluid (see Maruyama 2014).

Hence the object of this note is to extend the Boussinesq approximation in a manner consistent with the conservation law of mass, i.e. in such a manner that the equation of continuity is satisfied. It is shown that, with the aid of the conservation law of energy, we can systematically formulate the governing equations for a two-component fluid.

2. Governing equations for a two-component fluid

We consider the motion in a uniform gravitational field of a fluid consisting of two components, $\mathcal{A}$ and $\mathcal{B}$, and occupying a fixed domain Ω. In this domain, we set up a

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system of rectangular coordinates (x_1, x_2, x_3) with the x_3-axis taken vertically upwards. The unit vectors in the positive x_1-, x_2-, and x_3-directions are respectively denoted by $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$. Latin indices are used in the following to represent the numbers 1, 2, and 3; the summation convention is also implied.

2.1. Basic assumptions

We first assume that the temperature T of the fluid can be written in the form

$$T = T_0 + T', \tag{2.1}$$

where T_0 is a constant reference temperature, and T' the small deviation from T_0. We also write the pressure p of the fluid as follows:

$$p = p_0 + p'. \tag{2.2}$$

Here p' denotes the small perturbation from the hydrostatic pressure p_0 defined by

$$p_0 = -\rho_0 g x_3 + \text{constant}, \tag{2.3}$$

in which ρ_0 is a constant reference density, and g the acceleration due to gravity.

Let c denote the concentration of component $\mathcal{A}$, i.e. the ratio of the mass of $\mathcal{A}$ to the total mass of the fluid in a given volume element. We assume that c can be written as

$$c = c_0 + c', \tag{2.4}$$

where c_0 is a constant reference concentration, and c' the small deviation from c_0.

Next, let ρ denote the density of the fluid and let $\beta_c = \rho^{-1}(\partial\rho/\partial c)_{T,p}$. It is supposed that ρ is given by

$$\rho = \rho_0 + \rho_0 \beta_{c0} c', \tag{2.5}$$

where $\beta_{c0} = \beta_c(T_0, p_0, c_0)$. We assume that β_{c0} may be regarded as constant.

According to the above expression, ρ is a function only of c'. It is assumed, however, that the thermal expansion coefficient β of the fluid does not vanish:

$$\beta = v^{-1}(\partial v/\partial T)_{p,c} \neq 0, \tag{2.6}$$

where $v = \rho^{-1}$ denotes the specific volume of the fluid.

2.2. Concentration equation

The equation for the rate of change of the concentration c of component $\mathcal{A}$ is given by (see Landau & Lifshitz 1987, § 58)

$$\rho Dc/Dt = -\nabla \cdot \boldsymbol{i}. \tag{2.7}$$

Here the symbol D/Dt denotes the material derivative, and $\boldsymbol{i}$ the diffusion flux density of component $\mathcal{A}$. It is to be noted that the total flux density of component $\mathcal{A}$ and that

of component $\mathcal{B}$ are given by $\rho c \boldsymbol{u} + \boldsymbol{i}$ and $\rho(1 - c)\boldsymbol{u} - \boldsymbol{i}$, respectively. Substituting (2.4) and (2.5) into the left-hand side of (2.7), we obtain

$$\rho_0 Dc'/Dt = -\nabla \cdot \boldsymbol{i}, \tag{2.8}$$

where a term of the second order in c' has been omitted.

On the other hand, denoting by κ_c the diffusion coefficient of component $\mathcal{A}$ in $\mathcal{B}$, we can write $\boldsymbol{i}$ in the following form (see Landau & Lifshitz 1987, § 59):

$$\boldsymbol{i} = -\rho \kappa_c \left\{ \nabla c + (\nu_T/T)\nabla T + (\nu_p/p)\nabla p \right\}. \tag{2.9}$$

Here ν_T and ν_p are respectively the thermodiffusion and barodiffusion ratios; they are known to vanish as $c \to 0$ or $c \to 1$. Hence, for simplicity, we assume that

$$c_0 \ll 1. \tag{2.10}$$

The last two terms in the braces in (2.9) can then be omitted; the substitution of (2.4) and (2.5) into (2.9) yields, to the first order of small quantities,

$$\boldsymbol{i} = -\rho_0 \kappa_{c0} \nabla c'. \tag{2.11}$$

Here κ_{c0} denotes κ_c evaluated at $T = T_0$, $p = p_0$, and $c = c_0$ in view of the smallness of T', p', and c'; terms above the first order in primed variables have thus been ignored. It is to be noted that κ_{c0} may in general depend upon x_3 through p_0.

Finally, from (2.8) and (2.11), we obtain

$$Dc'/Dt = \nabla \cdot (\kappa_{c0} \nabla c'). \tag{2.12}$$

This is the concentration equation for the two-component fluid.

2.3. Equation of continuity

Now, the velocity $\boldsymbol{u}$ of the fluid satisfies the equation of continuity

$$\rho^{-1} D\rho/Dt + \nabla \cdot \boldsymbol{u} = 0. \tag{2.13}$$

Using (2.5) and neglecting terms above the first order in c', we can rewrite (2.13) as

$$\nabla \cdot \boldsymbol{u} = -\beta_{c0} Dc'/Dt. \tag{2.14}$$

The substitution of (2.12) allows us to further rewrite (2.14) as

$$\nabla \cdot \boldsymbol{u} = -\beta_{c0} \nabla \cdot (\kappa_{c0} \nabla c'). \tag{2.15}$$

This is the equation of continuity for the two-component fluid: this equation guarantees that $\boldsymbol{u}$ is the momentum per unit mass of the fluid (see Maruyama 2014).

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2.4. Equation of motion

When the pressure is given by (2.2), the equation of motion takes the form

$$\rho\frac{D\boldsymbol{u}}{Dt} = -\nabla p' + \frac{\partial\tau_{ij}}{\partial x_j}\boldsymbol{e}_i - (\rho - \rho_0)g\boldsymbol{e}_3, \tag{2.16}$$

where τ_{ij} are the components of the viscous stress tensor. If we simply substitute (2.5) into the last term of (2.16), the following equation is obtained:

$$\rho\frac{D\boldsymbol{u}}{Dt} = -\nabla p' + \frac{\partial\tau_{ij}}{\partial x_j}\boldsymbol{e}_i - \rho_0\beta_{c0}c'g\boldsymbol{e}_3. \tag{2.17}$$

As explained below, however, this equation must contain an additional term in order to be consistent with the conservation law of energy.

Let e denote the specific internal energy of the fluid. We assume, for simplicity, that the component of $\boldsymbol{u}$ normal to the boundary Σ of the domain Ω containing the fluid is zero. The conservation law of energy for the fluid is then expressed as follows:

$$\frac{d}{dt}\int_\Omega \rho\left(\tfrac{1}{2}|\boldsymbol{u}|^2 + gx_3 + e\right)dV = \int_\Sigma u_i\tau_{ij}n_j dS - \int_\Sigma \boldsymbol{q}\cdot\boldsymbol{n}dS, \tag{2.18}$$

where $\boldsymbol{n}$ denotes the unit outward normal on Σ; u_i and n_j are the components of $\boldsymbol{u}$ and $\boldsymbol{n}$, respectively; $\boldsymbol{q}$ is the heat flux density due to thermal conduction. This expression states that the total energy of the fluid changes owing to the work done by the viscous force acting on Σ and owing to the heat transfer across Σ.

On the other hand, concerning the potential energy of the fluid, we have

$$\frac{d}{dt}\int_\Omega \rho gx_3 dV = \int_\Omega(\rho_0 gu_3 + \rho_0\beta_{c0}c'gu_3)dV, \tag{2.19}$$

where (2.5) has been used on the right-hand side.

We can also find the equation for the rate of change of the internal energy of the fluid from the general equation of heat transfer (see Landau & Lifshitz 1987, §58)

$$\rho T\frac{Ds}{Dt} = \tau_{ij}\frac{\partial u_i}{\partial x_j} - \nabla\cdot\boldsymbol{q} + \mu\nabla\cdot\boldsymbol{i}, \tag{2.20}$$

in which s and μ are respectively the specific entropy and the chemical potential of the fluid; μ is so defined that it satisfies the thermodynamic relation

$$de = Tds - pdv + \mu dc. \tag{2.21}$$

In the following, we derive the desired equation after Maruyama (2014).

When s is regarded as a function of T, p, and c, we can express Ds/Dt as follows:

$$\frac{Ds}{Dt} = \left(\frac{\partial s}{\partial T}\right)_{p,c}\frac{DT}{Dt} + \left(\frac{\partial s}{\partial p}\right)_{T,c}\frac{Dp}{Dt} + \left(\frac{\partial s}{\partial c}\right)_{T,p}\frac{Dc}{Dt}. \tag{2.22}$$

Here, using thermodynamic relations, the coefficients can be written as

$$\left(\frac{\partial s}{\partial T}\right)_{p,c} = \frac{c_p}{T}, \quad \left(\frac{\partial s}{\partial p}\right)_{T,c} = -\left(\frac{\partial v}{\partial T}\right)_{p,c} = -v\beta, \quad \left(\frac{\partial s}{\partial c}\right)_{T,p} = -\left(\frac{\partial \mu}{\partial T}\right)_{p,c}, \quad (2.23)$$

where c_p is the specific heat at constant pressure, and $(\partial \mu/\partial T)_{p,c}$ is henceforth denoted by Λ for simplicity of notation. Since T, p and c deviate only slightly from T_0, p_0, and c_0, these coefficients may be evaluated at $T = T_0$, $p = p_0$, and $c = c_0$. We then obtain, using (2.1), (2.2), and (2.4),

$$\frac{Ds}{Dt} = \frac{c_{p0}}{T_0}\frac{DT'}{Dt} - v_0\beta_0\left(\frac{Dp_0}{Dt} + \frac{Dp'}{Dt}\right) - \Lambda_0\frac{Dc'}{Dt}, \quad (2.24)$$

in which $v_0 = \rho_0^{-1}$; we have also introduced the notation

$$c_{p0} = c_p(T_0, p_0, c_0), \quad \beta_0 = \beta(T_0, p_0, c_0), \quad \Lambda_0 = \Lambda(T_0, p_0, c_0). \quad (2.25)$$

In a similar fashion, when s is regarded as a function of e, v, and c, we get

$$\frac{Ds}{Dt} = \frac{1}{T_0}\frac{De}{Dt} + \frac{p_0}{T_0}\frac{Dv}{Dt} - \frac{\mu_0}{T_0}\frac{Dc}{Dt}, \quad (2.26)$$

where μ_0 stands for μ evaluated at $T = T_0$, $p = p_0$, and $c = c_0$:

$$\mu_0 = \mu(T_0, p_0, c_0). \quad (2.27)$$

Note that μ_0, and also c_{p0}, β_0, and Λ_0, may in general depend upon x_3 through p_0.

We can now rewrite the term on the left-hand side of (2.20) as follows:

$$\begin{aligned}
\rho T\frac{Ds}{Dt} &= \rho T_0\frac{Ds}{Dt} + \rho T'\frac{Ds}{Dt} \\
&= \rho\left\{\frac{De}{Dt} + p_0\frac{Dv}{Dt} - \mu_0\frac{Dc}{Dt}\right\} \\
&\quad + \rho T'\left\{\frac{c_{p0}}{T_0}\frac{DT'}{Dt} - v_0\beta_0\left(\frac{Dp_0}{Dt} + \frac{Dp'}{Dt}\right) - \Lambda_0\frac{Dc'}{Dt}\right\}. \quad (2.28)
\end{aligned}$$

However, since $\rho Dv/Dt = \nabla \cdot \boldsymbol{u}$, $\rho Dc/Dt = -\nabla \cdot \boldsymbol{i}$, and $Dp_0/Dt = -\rho_0 g u_3$, we have

$$\rho T\frac{Ds}{Dt} = \rho\frac{De}{Dt} + p_0\nabla \cdot \boldsymbol{u} + \mu_0\nabla \cdot \boldsymbol{i} + \rho_0\beta_0 T' g u_3, \quad (2.29)$$

where terms above the first order in primed variables have been neglected. Substituting this into (2.20), we get, to the first order of primed variables,

$$\rho\frac{De}{Dt} = \tau_{ij}\frac{\partial u_i}{\partial x_j} - \nabla \cdot \boldsymbol{q} - p_0\nabla \cdot \boldsymbol{u} - \rho_0\beta_0 T' g u_3. \quad (2.30)$$

We note here that $\rho De/Dt = \partial(\rho e)/\partial t + \nabla \cdot (\rho e \boldsymbol{u})$ and that

$$-p_0\nabla \cdot \boldsymbol{u} = -\nabla \cdot (p_0\boldsymbol{u}) + \boldsymbol{u} \cdot \nabla p_0 = -\nabla \cdot (p_0\boldsymbol{u}) - \rho_0 g u_3. \quad (2.31)$$

Thus, integrating (2.30) over the domain Ω, we finally find

$$\frac{d}{dt}\int_\Omega \rho e\,dV = \int_\Omega \tau_{ij}\frac{\partial u_i}{\partial x_j}dV - \int_\Sigma \boldsymbol{q}\cdot\boldsymbol{n}\,dS - \int_\Omega (\rho_0 g u_3 + \rho_0\beta_0 T' g u_3)dV. \qquad (2.32)$$

This is the desired equation for the rate of change of the internal energy of the fluid.

As a consequence, subtracting (2.19) and (2.32) from (2.18), we obtain the following equation for the rate of change of the kinetic energy of the fluid:

$$\frac{d}{dt}\int_\Omega \tfrac{1}{2}\rho|\boldsymbol{u}|^2 dV = \int_\Sigma u_i\tau_{ij}n_j\,dS - \int_\Omega \tau_{ij}\frac{\partial u_i}{\partial x_j}dV + \int_\Omega (\rho_0\beta_0 T'g u_3 - \rho_0\beta_{c0}c'g u_3)dV. \quad (2.33)$$

This equation must also be derived from the equation of motion if terms above the first order in primed variables are ignored in the derivation; this requires that (2.17) should contain the term

$$\rho_0\beta_0 T'g\boldsymbol{e}_3. \qquad (2.34)$$

This term represents the buoyancy force due to changes in temperature: the force arises as a result of the conversion between kinetic and internal energy. In contrast, the term $-\rho_0\beta_{c0}c'g\boldsymbol{e}_3$ in (2.17) represents the buoyancy force due to changes in concentration: the force is responsible for the conversion between kinetic and potential energy.

We have thus obtained the following equation:

$$\rho\frac{D\boldsymbol{u}}{Dt} = -\nabla p' + \frac{\partial\tau_{ij}}{\partial x_j}\boldsymbol{e}_i + \rho_0\beta_0 T'g\boldsymbol{e}_3 - \rho_0\beta_{c0}c'g\boldsymbol{e}_3. \qquad (2.35)$$

This equation, to the first order of primed variables, is equivalent to

$$\rho_0\frac{D\boldsymbol{u}}{Dt} = -\nabla p' + (1 - \beta_{c0}c')\frac{\partial\tau_{ij}}{\partial x_j}\boldsymbol{e}_i + \rho_0\beta_0 T'g\boldsymbol{e}_3 - \rho_0\beta_{c0}c'g\boldsymbol{e}_3. \qquad (2.36)$$

Here the factor $(1 - \beta_{c0}c')$ in the viscous term may usually be identified with unity. We take (2.36) as the equation of motion for the two-component fluid. In this equation, τ_{ij} are given, for example, by (see Landau & Lifshitz 1987, §15)

$$\tau_{ij} = \eta\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right) + \zeta\delta_{ij}\frac{\partial u_k}{\partial x_k}, \qquad (2.37)$$

in which the symbol δ_{ij} denotes the Kronecker delta; η and ζ are the dynamic viscosity and the second viscosity, respectively.

2.5. Temperature equation

Comparing (2.24) and (2.26), we observe that, to the first order of primed variables,

$$\rho\frac{De}{Dt} = \rho_0 c_{p0}\frac{DT'}{Dt} - \left\{(1 + \beta_{c0}c')\beta_0 T_0\frac{Dp_0}{Dt} + \beta_0 T_0\frac{Dp'}{Dt}\right\}$$
$$+ (T_0\Lambda_0 - \mu_0)\nabla\cdot\boldsymbol{i} - p_0\nabla\cdot\boldsymbol{u}, \qquad (2.38)$$

where $\rho Dv/Dt = \nabla \cdot \boldsymbol{u}$ and $\rho Dc/Dt = \rho_0 Dc'/Dt = -\nabla \cdot \boldsymbol{i}$ have been used. This allows us to rewrite (2.30) as follows:

$$\rho_0 c_{p0} \frac{DT'}{Dt} = \tau_{ij} \frac{\partial u_i}{\partial x_j} - \nabla \cdot \boldsymbol{q} + \left\{ (1 + \beta_{c0} c') \beta_0 T_0 \frac{Dp_0}{Dt} + \beta_0 T_0 \frac{Dp'}{Dt} - \rho_0 \beta_0 T' g u_3 \right\}$$
$$- (T_0 \Lambda_0 - \mu_0) \nabla \cdot \boldsymbol{i}. \tag{2.39}$$

On the other hand, when k denotes the thermal conductivity, the heat flux density $\boldsymbol{q}$ can be expressed in the following form (see Landau & Lifshitz 1987, § 59):

$$\boldsymbol{q} = \{ \nu_T (\partial \mu / \partial c)_{T,p} - (T \Lambda - \mu) \} \boldsymbol{i} - k \nabla T. \tag{2.40}$$

The first term in the braces, however, may be omitted on the assumption (2.10). Thus, to the first order of small quantities, we obtain

$$\boldsymbol{q} = -(T_0 \Lambda_0 - \mu_0) \boldsymbol{i} - k_0 \nabla T', \tag{2.41}$$

where k_0 stands for k evaluated at $T = T_0$, $p = p_0$, and $c = c_0$.

Now, let us substitute (2.41) into (2.39). Noting that Λ_0 and μ_0, and also k_0, may in general be dependent upon x_3 through p_0, we get the following result:

$$\rho_0 c_{p0} \frac{DT'}{Dt} = \tau_{ij} \frac{\partial u_i}{\partial x_j} + \nabla \cdot (k_0 \nabla T') + \left\{ (1 + \beta_{c0} c') \beta_0 T_0 \frac{Dp_0}{Dt} + \beta_0 T_0 \frac{Dp'}{Dt} - \rho_0 \beta_0 T' g u_3 \right\}$$
$$+ \boldsymbol{i} \cdot \boldsymbol{e}_3 \frac{dp_0}{dx_3} \frac{d}{dp_0} (T_0 \Lambda_0 - \mu_0). \tag{2.42}$$

However, since $\Lambda_0 = (\partial \mu / \partial T)|_{(T_0, p_0, c_0)}$, we see from thermodynamic relations that

$$\frac{d\Lambda_0}{dp_0} = \frac{\partial^2 \mu}{\partial p \partial T} \bigg|_{(T_0, p_0, c_0)}$$
$$= \frac{\partial^2 v}{\partial c \partial T} \bigg|_{(T_0, p_0, c_0)} = \frac{\partial (v\beta)}{\partial c} \bigg|_{(T_0, p_0, c_0)} = v_0 \frac{\partial \beta}{\partial c} \bigg|_{(T_0, p_0, c_0)} - v_0 \beta_{c0} \beta_0. \tag{2.43}$$

It also follows from thermodynamic relations that

$$\frac{d\mu_0}{dp_0} = \frac{\partial \mu}{\partial p} \bigg|_{(T_0, p_0, c_0)} = \frac{\partial v}{\partial c} \bigg|_{(T_0, p_0, c_0)} = -v_0 \beta_{c0}. \tag{2.44}$$

Accordingly, since $\boldsymbol{i}$ is given by (2.11) and $dp_0/dx_3 = -\rho_0 g$, we obtain

$$\rho_0 c_{p0} \frac{DT'}{Dt} = \tau_{ij} \frac{\partial u_i}{\partial x_j} + \nabla \cdot (k_0 \nabla T') + \left\{ (1 + \beta_{c0} c') \beta_0 T_0 \frac{Dp_0}{Dt} + \beta_0 T_0 \frac{Dp'}{Dt} - \rho_0 \beta_0 T' g u_3 \right\}$$
$$+ (\rho_0 \kappa_{c0} \frac{\partial c'}{\partial x_3}) \left\{ T_0 \frac{\partial \beta}{\partial c} \bigg|_{(T_0, p_0, c_0)} - (\beta_0 T_0 - 1) \beta_{c0} \right\} g. \tag{2.45}$$

To the first order of primed variables, however, this equation is equivalent to

$$\rho_0 c_{p0} \frac{DT'}{Dt} = (1 - \beta_{c0}c')\tau_{ij}\frac{\partial u_i}{\partial x_j} + \nabla \cdot (k_0 \nabla T') + \left\{ \beta_0 T_0 \left(\frac{Dp_0}{Dt} + \frac{Dp'}{Dt} \right) - \rho_0 \beta_0 T' g u_3 \right\}$$

$$+ (\rho_0 \kappa_{c0}\frac{\partial c'}{\partial x_3}) \left\{ T_0 \left. \frac{\partial \beta}{\partial c} \right|_{(T_0, p_0, c_0)} - (\beta_0 T_0 - 1)\beta_{c0} \right\} g. \tag{2.46}$$

We take (2.46) as the temperature equation for the two-component fluid.

3. Summary and discussion

The Boussinesq approximation has been extended to a two-component fluid in a way consistent with the conservation law of mass; the governing equations are (2.12), (2.15), (2.36), and (2.46). The equations have been obtained on the following assumption: the density of the fluid is a function only of the concentration of one component, but the thermal expansion coefficient of the fluid does not vanish. The equation of motion, in particular, has been formulated on the basis of the conservation law of energy. Also, it has turned out that the velocity of the fluid cannot in general be solenoidal.

3.1. Further approximations to the temperature equation

The temperature equation (2.46) is consistent with the conservation law of energy. It may usually be approximated, however, by simpler equations as discussed below.

We first consider the last term of (2.46). If c' changes over a vertical distance Δh by an amount $\Delta c'$, and if the motion to be analyzed has a characteristic time scale τ, this term may be omitted when the following condition is satisfied:

$$|\Gamma_{c0}| \left(\kappa_{c0}\frac{\Delta c'}{\Delta h}\tau \right) \frac{1}{\Delta T'} \ll 1, \quad \Gamma_{c0} = \left\{ T_0 \left. \frac{\partial \beta}{\partial c} \right|_{(T_0, p_0, c_0)} - (\beta_0 T_0 - 1)\beta_{c0} \right\} \frac{g}{c_{p0}}, \tag{3.1}$$

where $\Delta T'$ denotes the scale characterizing the change in T'. In fact, the term vanishes identically for a mixture of ideal gases; in that case, $\beta = T^{-1}$, so that $\Gamma_{c0} = 0$. We can also introduce, by ignoring terms containing primed variables, the approximation

$$\left\{ \beta_0 T_0 \left(\frac{Dp_0}{Dt} + \frac{Dp'}{Dt} \right) - \rho_0 \beta_0 T' g u_3 \right\} \approx \beta_0 T_0 \frac{Dp_0}{Dt} = -\rho_0 \beta_0 T_0 g u_3. \tag{3.2}$$

The viscous heating term $(1 - \beta_{c0}c')\tau_{ij}\partial u_i/\partial x_j$ may also be ignored in comparison with $\rho_0 \beta_0 T_0 g u_3$ when the following condition applies:

$$\left(\frac{1}{\beta_0 T_0} \right) \left\{ \frac{\nu}{(gL)^{1/2}L} \right\} \left\{ \frac{U}{(gL)^{1/2}} \right\} \ll 1, \tag{3.3}$$

where $\nu = \eta/\rho_0$ is the kinematic viscosity; U and L denote respectively the velocity and length scales characteristic of the motion to be analyzed. Then (2.46) takes the form

$$\rho_0 c_{p0} \frac{DT'}{Dt} = \nabla \cdot (k_0 \nabla T') - \rho_0 \beta_0 T_0 g u_3. \tag{3.4}$$

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The last term, however, can further be neglected under the following condition:

$$\Gamma_0 H/\Delta T' \ll 1, \quad \Gamma_0 = \beta_0 T_0 g/c_{p0}, \tag{3.5}$$

in which H denotes the vertical extent of the fluid. We then have

$$\rho_0 c_{p0} \frac{DT'}{Dt} = \nabla \cdot (k_0 \nabla T'). \tag{3.6}$$

This is essentially the same equation as that in the original Boussinesq approximation.

3.2. Potential temperature

When the temperature equation is approximated by (3.4), it is possible to employ, in place of T', the potential temperature θ defined by

$$\theta = \theta_0 + T', \quad \theta_0 = T_0 + \int_a^{x_3} \Gamma_0(x_3')dx_3', \tag{3.7}$$

where the integral is taken from a reference level $x_3' = a$. Then (3.4) can be written as

$$\rho_0 c_{p0} \frac{D\theta}{Dt} = \nabla \cdot (k_0 \nabla \theta) - \frac{d(k_0 \Gamma_0)}{dx_3}. \tag{3.8}$$

The equation of motion (2.36) can also be written in the form

$$\rho_0 \frac{D\boldsymbol{u}}{Dt} = -\nabla \hat{p}' + (1 - \beta_{c0}c')\frac{\partial \tau_{ij}}{\partial x_j}\boldsymbol{e}_i + \rho_0 \beta_0 \theta g \boldsymbol{e}_3 - \rho_0 \beta_{c0} c' g \boldsymbol{e}_3, \tag{3.9}$$

where the modified perturbation pressure $\hat{p}'$ is defined by

$$\hat{p}' = p' + \rho_0 g \int_a^{x_3} \beta_0(x_3')\theta_0(x_3')dx_3'. \tag{3.10}$$

These equations, together with (2.12) and (2.15), form a closed system of equations.

The boundary conditions for θ can be obtained from those for T'. Let us consider, as an example, the following boundary conditions for T':

$$T'|_{x_3=a} = 0, \quad T'|_{x_3=b} = 0. \tag{3.11}$$

In view of (3.7), these conditions can be rewritten as follows:

$$\theta|_{x_3=a} = T_0, \quad \theta|_{x_3=b} = T_0 + \int_a^b \Gamma_0(x_3')dx_3'. \tag{3.12}$$

We next consider the following boundary conditions for T':

$$(\partial T'/\partial x_3)|_{x_3=a} = 0, \quad (\partial T'/\partial x_3)|_{x_3=b} = 0. \tag{3.13}$$

In this case, the following conditions for θ are obtained:

$$(\partial\theta/\partial x_3)|_{x_3=a} = \Gamma_0(a), \quad (\partial\theta/\partial x_3)|_{x_3=b} = \Gamma_0(b). \tag{3.14}$$

As is apparent from these examples, although θ plays in the governing equations almost the same role as T', the boundary conditions for θ are different from those for T'.

We note, furthermore, that the heat flux density $\boldsymbol{q}$ is given in terms of θ as follows:

$$\boldsymbol{q} = -(T_0\Lambda_0 - \mu_0)\boldsymbol{i} - k_0\nabla\theta + k_0\Gamma_0\boldsymbol{e}_3. \tag{3.15}$$

Hence $\boldsymbol{q}$ is not proportional to the gradient of θ even when $\boldsymbol{i} = 0$.

3.3. On the coefficient of the concentration perturbation in (2.5)

The density of a fluid is assumed to be constant under the Boussinesq approximation (see Maruyama 2014). Thus we assumed in (2.5) that the coefficient $\beta_{c0} = \beta_c(T_0, p_0, c_0)$ may be regarded, though it may in general be a function of x_3 through p_0, as constant; ρ then takes a constant value when c' is uniform throughout the fluid. This assumption is justifiable when the following condition applies:

$$\left|(\beta_{c0} - \beta_{c0}|_{x_3=a})(\beta_{c0}|_{x_3=a})^{-1}\right| \ll 1. \tag{3.16}$$

If β_c is independent of the pressure, however, β_{c0} is indeed constant. This is the case when the fluid considered is a mixture of ideal gases; β_c is then given by

$$\beta_c = \frac{k_B T}{p}\left(\frac{c}{m_a} + \frac{1-c}{m_b}\right)\frac{\partial}{\partial c}\left\{\frac{k_B T}{p}\left(\frac{c}{m_a} + \frac{1-c}{m_b}\right)\right\}^{-1} = \frac{m_a - m_b}{m_a - (m_a - m_b)c}. \tag{3.17}$$

Here k_B stands for the Boltzmann constant; m_a and m_b are the mass of a molecule of component $\mathcal{A}$ and that of component $\mathcal{B}$, respectively.

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