Thermal and non-thermal internal gravity waves

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April 19, 2020

Abstract

This note deals with internal gravity waves in a single-component fluid and those in a two-component fluid in a physically reasonable manner. As a result, it becomes apparent that internal gravity waves can be classified, on the basis of their source of kinetic energy, into two categories: thermal and non-thermal internal gravity waves. The source of the kinetic energy of a thermal internal gravity wave is, in spite of its being called an internal "gravity" wave, the internal energy of the fluid. In contrast, that of a non-thermal internal gravity wave is the gravitational potential energy of the fluid: waves in this category can exist only in multi-component fluids.

1. Introduction

Internal gravity waves can propagate inside a continuously stratified fluid. Standard textbooks deal with waves of this kind on the assumption that the density of the fluid is a function only of the specific entropy of the fluid (see e.g. Landau & Lifshitz 1987; Gill 1982). This assumption, however, is physically unacceptable as explained below.

Suppose that a fluid is contained in a vessel with a fixed volume. If the density of the fluid is a function only of the specific entropy of the fluid, then we can change the mass of the fluid by heating or cooling the fluid; this conclusion is obviously absurd.

This study deals with internal gravity waves in a single-component fluid and those in a two-component fluid in a physically reasonable manner. As a result, it turns out that internal gravity waves can be classified, according to their source of kinetic energy, into two categories: thermal and non-thermal internal gravity waves.

2. Internal gravity waves in a single-component fluid

2.1. Physical situation

Let us consider a stationary layer of single-component fluid in a uniform gravitational field. We set up in the fluid a system of rectangular coordinates (x_1, x_2, x_3) with the x_3-axis taken vertically upwards. The unit vectors in the positive x_1-, x_2-, and x_3-directions are denoted, respectively, by $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$. Latin indices are used in the following to represent the numbers 1, 2, and 3; Greek indices, on the other hand, take on the values 1 and 2. The summation convention is implied throughout.

We first assume that the thermal expansion coefficient β of the fluid is nonzero:

$$\beta = v^{-1}(\partial v/\partial T)_p \neq 0. \tag{2.1}$$

Here v is the specific volume of the fluid; T and p are the temperature and the pressure of the fluid, respectively. This assumption will usually be satisfied in practice.

Now, let H denote the thickness of the fluid layer. We assume that H satisfies

$$(gH)^{1/2}/a \ll 1, \tag{2.2}$$

where g denotes the acceleration due to gravity, and a the speed of sound in the fluid.

We next require that the characteristic scale ΔT of the temperature variation in the fluid should be small in the following sense:

$$\beta \Delta T \ll 1. \tag{2.3}$$

Furthermore, we assume that the temperature of the fluid varies only slightly from some constant reference temperature T_0. We require, accordingly, that

$$\Delta T/T_0 \ll 1. \tag{2.4}$$

Suppose now that an internal gravity wave is present in the fluid layer: its wavelength λ is assumed to fulfill the condition $\lambda/H \ll 1$. Let U denote the scale characterizing the magnitude of the fluid velocity associated with the wave. We assume that

$$U/(g\lambda)^{1/2} \ll 1. \tag{2.5}$$

We also make the following assumption on the phase velocity V_p of the wave:

$$V_p/(g\lambda)^{1/2} \ll 1. \tag{2.6}$$

Finally, it is assumed that the viscosity of the fluid and the thermal conduction in the fluid may be ignored so far as the fundamental properties of the wave are concerned.

2.2. Basic system of equations

When the above conditions are met, the internal gravity wave can be dealt with under the Boussinesq approximation (Maruyama 2019a). In this subsection, we formulate the equations for the physical quantities necessary for describing the wave.

We first need to recognize that, under the Boussinesq approximation, the density ρ of the fluid must be regarded as constant (see Maruyama 2019a):

$$\rho = \rho_0. \tag{2.7}$$

The temperature T of the fluid, on the other hand, is expressed in the form

$$T = T_0 + T', \tag{2.8}$$

where T' denotes the small deviation from T_0. We can also write the pressure p of the fluid, denoting by p' the small perturbation pressure, as follows:

$$p = p_0 + p'. \tag{2.9}$$

Here the hydrostatic pressure p_0 is defined by

$$p_0 = -\rho_0 g x_3 + \text{constant}. \tag{2.10}$$

Now, on account of (2.7), the equation of continuity reduces to

$$\nabla \cdot \boldsymbol{u} = 0, \tag{2.11}$$

where $\boldsymbol{u} = u_i \boldsymbol{e}_i$ denotes the velocity of the fluid.

As for the equation of motion, it takes the following form:

$$\rho_0 \frac{D\boldsymbol{u}}{Dt} = -\nabla p' + \rho_0 \beta_0 T' g \boldsymbol{e}_3. \tag{2.12}$$

Here D/Dt denotes the material derivative, and β_0 is β at $T = T_0$ and $p = p_0$:

$$\beta_0 = \beta(T_0, p_0). \tag{2.13}$$

The last term of (2.12) represents the buoyancy force due to changes in temperature: it is required for consistency with the conservation law of energy (see Maruyama 2019a).

The final equation is the temperature equation

$$\rho_0 c_{p0} \frac{DT'}{Dt} = -\rho_0 \beta_0 T_0 g u_3. \tag{2.14}$$

Here c_{p0} is the specific heat at constant pressure c_p at $T = T_0$ and $p = p_0$:

$$c_{p0} = c_p(T_0, p_0). \tag{2.15}$$

It should be mentioned that, in (2.14), a few terms small in comparison with the term on the right-hand side have been omitted (see Maruyama 2019a).

2.3. Dispersion relation

Having formulated the basic system of equations, we are now in a position to derive the dispersion relation for the internal gravity wave.

To this end, we first assume that β_0 and c_{p0}, which may in general be functions of x_3 through p_0, change little over distances of the order of the wavelength. This assumption allows us to regard β_0 and c_{p0} as constant.

We also assume that the velocity scale U is much smaller than the phase velocity V_p:

$$U/V_p \ll 1. \tag{2.16}$$

Then the wave can be looked upon as a small amplitude wave.

Now, suppose that T' in (2.12) and (2.14) is given by

$$T' = \tau_0(x_3) + \tau. \tag{2.17}$$

Here $\tau_0(x_3)$, which is assumed to be a linear function of x_3, represents the basic thermal stratification; τ corresponds to the temperature perturbation associated with the wave. It is then reasonable to express p' as follows:

$$p' = \pi_0(x_3) + \pi, \tag{2.18}$$

where $\pi_0(x_3)$ is a function of x_3 satisfying the relation $d\pi_0/dx_3 = \rho_0\beta_0\tau_0 g$.

Substituting (2.17) and (2.18) into (2.12), we get

$$\rho_0 \frac{\partial \boldsymbol{u}}{\partial t} = -\nabla\pi + \rho_0\beta_0\tau g \boldsymbol{e}_3, \tag{2.19}$$

where $D\boldsymbol{u}/Dt$ has been approximated by $\partial\boldsymbol{u}/\partial t$. Similarly, from (2.14), we have

$$\frac{\partial \tau}{\partial t} + \frac{d\tau_0}{dx_3}u_3 = -\Gamma_0 u_3, \tag{2.20}$$

in which $\Gamma_0 = \beta_0 T_0 g/c_{p0}$ is the adiabatic lapse rate.

From these equations and (2.11), we can find the following equation for u_3:

$$\frac{\partial^2}{\partial t^2}\frac{\partial^2 u_3}{\partial x_i \partial x_i} + N_\theta^2 \frac{\partial^2 u_3}{\partial x_\mu \partial x_\mu} = 0. \tag{2.21}$$

Here N_θ, which may be called the thermal buoyancy frequency, is defined as follows:

$$N_\theta^2 = \beta_0 g \left(d\tau_0/dx_3 + \Gamma_0\right). \tag{2.22}$$

Suppose now that the wave is a sinusoidal plane wave. We put, accordingly,

$$u_3 = W \sin(k_i x_i - \omega t), \tag{2.23}$$

where W is a constant; k_i denote the components of the wave vector $k_i\boldsymbol{e}_i$, and ω is the angular frequency. The substitution of (2.23) into (2.21) yields

$$\omega^2 = N_\theta^2 k_\mu k_\mu / k_i k_i. \tag{2.24}$$

This is the dispersion relation for the internal gravity wave.

2.4. Energetics

Now, with respect to the energy of the internal gravity wave, we can find from (2.11), (2.19), and (2.20) the following equation:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_0|\boldsymbol{u}|^2 + P_\theta\right) + \nabla\cdot(\pi\boldsymbol{u}) = 0. \tag{2.25}$$

Here P_θ is the potential energy density associated with the restoring force of the wave:

$$P_\theta = \frac{1}{2}\rho_0 N_\theta^2\left(\frac{\tau}{d\tau_0/dx_3 + \Gamma_0}\right)^2. \tag{2.26}$$

From (2.25), we observe that the total energy density of the wave consists of the kinetic energy density $\rho_0|\boldsymbol{u}|^2/2$ and the potential energy density P_θ; the energy flux density of the wave, on the other hand, is seen to be given by $\pi\boldsymbol{u}$.

It is important to note here that, as is evident from (2.19), the restoring force of the wave is the buoyancy force due to changes in temperature. Thus the work done by the restoring force corresponds to the conversion between kinetic and internal energy (see Maruyama 2014). This implies that the potential energy density P_θ is stored as part of the internal energy of the fluid. We can therefore conclude as follows: the source of the kinetic energy of the wave is not the gravitational potential energy of the fluid but the internal energy of the fluid, despite the wave being called an internal "gravity" wave.

3. Internal gravity waves in a two-component fluid

3.1. Physical situation

In this section, we consider again an internal gravity wave in a layer of fluid, but the fluid is now assumed to be composed of two components $\mathcal{A}$ and $\mathcal{B}$. We denote by c the concentration of component $\mathcal{A}$, i.e. the ratio of the mass of $\mathcal{A}$ to the total mass of the fluid in a given volume element. As regards the scale Δc characterizing the variation of c, we assume that the following condition is satisfied:

$$\beta_c\Delta c \ll 1, \tag{3.1}$$

where $\beta_c = \rho^{-1}(\partial\rho/\partial c)_{T,p}$. Furthermore, Δc itself is taken to be small:

$$\Delta c \ll 1. \tag{3.2}$$

It is also assumed that the change in c caused by diffusion may be ignored.

Except for this difference, the same physical situation as that in § 2 is assumed.

3.2. Basic system of equations

As in § 2, the temperature T and the pressure p of the fluid are respectively written as (2.8) and (2.9). The concentration c is similarly expressed in the form

$$c = c_0 + c', \tag{3.3}$$

where c_0 denotes a constant reference concentration, and c' the small deviation from c_0. The density ρ of the fluid is, in contrast to § 2, not constant: it is taken to be given by

$$\rho = \rho_0 + \rho_0 \beta_{c0} c'. \tag{3.4}$$

Here β_{c0} stands for β_c at $T = T_0$, $p = p_0$, and $c = c_0$:

$$\beta_{c0} = \beta_c(T_0, p_0, c_0). \tag{3.5}$$

It is assumed that β_{c0} may be regarded as constant (see Maruyama 2019b).

When c is written as (3.3), we get the following concentration equation:

$$\frac{Dc'}{Dt} = 0, \tag{3.6}$$

where, as stated in the previous subsection, the effect of diffusion has been neglected.

When ρ is given by (3.4), on the other hand, the equation of motion can be obtained in the following form (see Maruyama 2019b):

$$\rho_0 \frac{D\boldsymbol{u}}{Dt} = -\nabla p' + \rho_0 \beta_0 T' g \boldsymbol{e}_3 - \rho_0 \beta_{c0} c' g \boldsymbol{e}_3. \tag{3.7}$$

Here β_0 is redefined as follows:

$$\beta_0 = \beta(T_0, p_0, c_0). \tag{3.8}$$

The last term of (3.7) represents the buoyancy force due to changes in concentration.

In view of (3.4) and (3.6), the equation of continuity is seen to be given by (2.11) as before. Also, the temperature equation is given by (2.14), with c_{p0} redefined by

$$c_{p0} = c_p(T_0, p_0, c_0). \tag{3.9}$$

3.3. Dispersion relation

In order to find the dispersion relation for the internal gravity wave, we again regard β_0 and c_{p0} as constant and the wave as a small amplitude wave.

We next assume that T' is given by (2.17) and that c' is similarly written as

$$c' = \sigma_0(x_3) + \sigma. \tag{3.10}$$

Here $\sigma_0(x_3)$ is a linear function of x_3. In accord with this assumption, p' is expressed as (2.18) with π_0 satisfying the relation $d\pi_0/dx_3 = \rho_0\beta_0\tau_0 g - \rho_0\beta_{c0}\sigma_0 g$.

The equation of motion (3.7) then reduces to

$$\rho_0 \frac{\partial \boldsymbol{u}}{\partial t} = -\nabla \pi + \rho_0 \beta_0 \tau g \boldsymbol{e}_3 - \rho_0 \beta_{c0} \sigma g \boldsymbol{e}_3, \tag{3.11}$$

while the temperature equation is given by (2.20) as before. The concentration equation (3.6), on the other hand, takes the following form:

$$\frac{\partial \sigma}{\partial t} + \frac{d\sigma_0}{dx_3} u_3 = 0. \tag{3.12}$$

These equations, combined with (2.11), yield the following equation for u_3:

$$\frac{\partial^2}{\partial t^2}\frac{\partial^2 u_3}{\partial x_i \partial x_i} + (N_\theta^2 + N_c^2)\frac{\partial^2 u_3}{\partial x_\mu \partial x_\mu} = 0. \tag{3.13}$$

Here N_c, which we term the non-thermal buoyancy frequency, is defined by

$$N_c^2 = -\beta_{c0} g (d\sigma_0/dx_3). \tag{3.14}$$

From (3.13), we obtain the following dispersion relation for the wave:

$$\omega^2 = (N_\theta^2 + N_c^2) k_\mu k_\mu / k_i k_i. \tag{3.15}$$

When $N_c/N_\theta \ll 1$, (3.15) reduces to (2.24); this implies that the effect of the variation of concentration is negligible. When $N_\theta/N_c \ll 1$, in contrast, (3.15) becomes

$$\omega^2 = N_c^2 k_\mu k_\mu / k_i k_i. \tag{3.16}$$

The thermal effect can be ignored in this case.

3.4. Energetics

The energy equation for the internal gravity wave can now be obtained as follows:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_0 |\boldsymbol{u}|^2 + P_\theta + P_c\right) + \nabla \cdot (\pi \boldsymbol{u}) = 0, \tag{3.17}$$

where P_c is the potential energy density associated with the restoring force of the wave arising from the buoyancy force due to changes in concentration: it is given by

$$P_c = \frac{1}{2}\rho_0 N_c^2 \left(\frac{\sigma}{d\sigma_0/dx_3}\right)^2. \tag{3.18}$$

As observed in § 2.4, the potential energy density P_θ is stored as part of the internal energy of the fluid. It is to be noted that, in contrast, the potential energy density P_c is stored as part of the gravitational potential energy of the fluid. This can be seen from the following fact: P_c is the potential energy density associated with the restoring force arising from the buoyancy force due to changes in concentration; however, the work done by this buoyancy force corresponds to the conversion between kinetic and gravitational potential energy (see Maruyama 2014). It therefore follows that, when $N_\theta/N_c \ll 1$, the source of the kinetic energy of the wave is the gravitational potential energy of the fluid; when $N_c/N_\theta \ll 1$, on the other hand, it is the internal energy of the fluid.

4. Conclusion

Internal gravity waves in a single-component fluid and those in a two-component fluid have been treated in a physically reasonable way, and their fundamental properties have been derived. As a consequence, it has become apparent that internal gravity waves can be classified into two categories on the basis of their source of kinetic energy.

Waves in the first category may be called thermal internal gravity waves; the source of the kinetic energy of a thermal internal gravity wave in a fluid is the internal energy of the fluid. Internal gravity waves in a single-component fluid are all in this category.

Waves in the second category, which are termed non-thermal internal gravity waves, can exist only in multi-component fluids. The source of the kinetic energy of this kind of wave in a multi-component fluid is the gravitational potential energy of the fluid.

It should be noted, however, that these two categories are not mutually exclusive: the source of the kinetic energy of an internal gravity wave in a multi-component fluid is, in general, both the internal and the gravitational potential energy of the fluid. Note also that the dispersion relations for thermal and non-thermal waves are formally the same; these waves are indistinguishable in their kinematic behavior.

References

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