Adiabatic heating and cooling of an incompressible fluid element due to vertical displacement

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Abstract

This note analyzes the process of displacing a fluid element vertically in a layer of fluid. It is shown that, when the thickness of the fluid layer is very small compared with the height over which the density of the fluid element changes significantly, the fluid element may be regarded as incompressible. However, it also turns out that the temperature of the element changes, no matter how thin the fluid layer may be, at the adiabatic lapse rate; it becomes apparent, consequently, that an incompressible fluid, as well as a compressible fluid, undergoes adiabatic heating and cooling.

1. Introduction

The Boussinesq approximation is an approximation often used to study the motion of a fluid with a non-uniform temperature distribution. Under the approximation, the work done by the buoyancy force due to changes in temperature corresponds to the conversion between kinetic and internal energy (Maruyama 2014). Conversely, the necessity of the buoyancy force in the approximation can be inferred on the basis of the consideration of the conversion between kinetic and internal energy (Maruyama 2019).

This energy conversion under the Boussinesq approximation occurs on account of the adiabatic heating and cooling of fluid elements displaced vertically (Maruyama 2019). However, under the approximation, fluid elements are taken to be incompressible. Thus it may seem somewhat paradoxical that incompressible fluid elements undergo adiabatic heating and cooling; incompressible fluid elements neither expand nor contract.

Hence the object of this note is to show, on the basis of a simple thought experiment, that incompressible fluid elements certainly undergo adiabatic heating and cooling.

2. Vertical displacement of a fluid element

In this section, we consider a stationary layer of fluid in a uniform gravitational field: the z-axis is taken vertically upwards in this layer.

2.1. Adiabatic expansion and contraction

Suppose now that a fluid element taken in the fluid layer is displaced quasistatically and adiabatically in the vertical direction by an infinitesimal distance dz. Then, since the element experiences a change in pressure, its density changes accordingly. Denoting this density change by $d\rho_a$, we can write

$$d\rho_a = (\partial\rho/\partial p)_s(dp/dz)dz, \tag{2.1}$$

in which ρ and p are respectively the density and the pressure of the fluid; s stands for the specific entropy of the fluid. It is well known, however, that

$$(\partial\rho/\partial p)_s = 1/a^2, \tag{2.2}$$

where a is the speed of sound. Furthermore, the pressure p satisfies the relation

$$dp/dz = -\rho g, \tag{2.3}$$

where g is the acceleration due to gravity. From (2.1), (2.2), and (2.3), we obtain

$$d\rho_a = -(\rho g/a^2)dz. \tag{2.4}$$

2.2. Adiabatic heating and cooling

Let us next examine the change in temperature of the fluid element. To this end, we first note the following thermodynamic relation:

$$(\partial T/\partial\rho)_s = \beta T a^2/\rho c_p, \tag{2.5}$$

where T is the temperature of the fluid, c_p the specific heat at constant pressure, and β the thermal expansion coefficient. We also note that the temperature change dT_a of the fluid element can be written, in terms of the density change $d\rho_a$, as follows:

$$dT_a = (\partial T/\partial\rho)_s d\rho_a. \tag{2.6}$$

The substitution of (2.4) and (2.5) into (2.6) leads to

$$dT_a = -(\beta T g/c_p)dz. \tag{2.7}$$

2.3. Condition for incompressibility

Now, let us rewrite (2.4) in the following dimensionless form:

$$-(d\rho_a/dz)/\Gamma_\rho = 1, \tag{2.8}$$

where $\Gamma_\rho = \rho g/a^2$. This relation states that the density of the fluid element changes at the rate Γ_ρ. On the basis of this rate of change in density, the height H_ρ over which the density of the fluid element changes significantly can be determined as follows:

$$H_\rho = \rho/\Gamma_\rho = a^2/g. \tag{2.9}$$

The ratio of the thickness H of the fluid layer to this height is then given by

$$H/H_\rho = \chi^2, \tag{2.10}$$

where the dimensionless parameter χ is defined by

$$\chi = (gH)^{1/2}/a. \tag{2.11}$$

Thus, when $\chi \ll 1$, we may regard the fluid element as incompressible.

On the other hand, from (2.7), we obtain the following dimensionless relation:

$$-(dT_a/dz)/\Gamma = 1, \tag{2.12}$$

where $\Gamma = \beta Tg/c_p$ is the adiabatic lapse rate. Since this relation is independent of the parameter χ, the temperature of the fluid element changes, no matter how small χ may be, at the adiabatic lapse rate. It therefore follows that an incompressible fluid, as well as a compressible fluid, undergoes adiabatic heating and cooling.

3. Conclusion

We have investigated the quasistatic adiabatic process of displacing vertically a fluid element taken in a layer of fluid. It has been shown that, when the thickness of the fluid layer is very small compared with the height over which the density of the fluid element changes significantly, the fluid element may be regarded as incompressible. Nonetheless, it has also turned out that the temperature of the element changes, no matter how thin the fluid layer may be, at the adiabatic lapse rate. This implies that an incompressible fluid, as well as a compressible fluid, undergoes adiabatic heating and cooling.

As stated in § 1, adiabatic heating and cooling in an incompressible fluid is of crucial importance in the Boussinesq approximation. This does not mean, nevertheless, that it exerts a discernible influence on the temperature distribution in the fluid. Indeed, under the original Boussinesq approximation, adiabatic heating and cooling is entirely ignored in the determination of the temperature distribution (see Maruyama 2019).

References

[1] Maruyama, K. 2014 Energetics of a fluid under the Boussinesq approximation. arXiv:1405.1921 [physics.flu-dyn].

[2] Maruyama, K. 2019 Rational derivation of the Boussinesq approximation.