Potential energy density of an internal gravity wave

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Abstract

The potential energy density of an internal gravity wave in a thermally stratified fluid is stored, at least partly, in the internal energy of the fluid. This note verifies this fact on the basis of a simple thought experiment.

1. Introduction

Internal gravity waves can be classified into two categories: thermal and non-thermal waves (Maruyama 2020). The classification is based on the following fact: the potential energy density of a thermal wave is stored in the internal energy of the fluid, while that of a non-thermal wave is stored in the gravitational potential energy of the fluid.

It may seem somewhat paradoxical, however, that the potential energy density of an internal "gravity" wave is stored in the internal energy of the fluid. Thus the object of this note is to verify this seemingly curious fact through a simple thought experiment.

2. Thought experiment

Let us consider a layer of fluid in a uniform gravitational field. The fluid is assumed, in particular, to consist of two components $\mathcal{C}_1$ and $\mathcal{C}_2$. In this layer, the z-axis is taken vertically upwards: the unit vector in the positive z-direction is denoted by $\boldsymbol{k}$.

2.1. Preliminaries

We first assume that the applicability of the Boussinesq approximation is guaranteed (see Maruyama 2020). Then the temperature T of the fluid can be written as

$$T = T_0 + T', \tag{2.1}$$

where T_0 is a constant reference temperature, and T' the small deviation from T_0. The pressure p of the fluid can also be written in the following form:

$$p = p_0 + p'. \tag{2.2}$$

Here p' denotes the small perturbation from the hydrostatic pressure p_0 defined by

$$p_0 = -\rho_0 g z + \text{constant}, \tag{2.3}$$

in which ρ_0 is a constant reference density, and g the acceleration due to gravity.

Let c denote the concentration of component $\mathcal{C}_1$, i.e. the ratio of the mass of $\mathcal{C}_1$ to the total mass of the fluid in a given volume element. We assume that c is written as

$$c = c_0 + c', \tag{2.4}$$

where c_0 is a constant reference concentration, and c' the small deviation from c_0. The density ρ of the fluid is then assumed to be given, in terms of c', as follows:

$$\rho = \rho_0 + \rho_0 \beta_c c'. \tag{2.5}$$

Here the coefficient β_c is taken to be constant. Though the explicit dependence of ρ on T and p is ignored, the thermal expansion coefficient β is assumed not to vanish:

$$\beta = -\rho^{-1}(\partial\rho/\partial T)_{p,c} \neq 0. \tag{2.6}$$

This coefficient also is regarded as constant.

2.2. Interchange of the positions of two fluid elements

Now, suppose that the fluid is homogeneous and that it is stationary:

$$T = T_0, \quad c = c_0, \quad \rho = \rho_0, \quad p = p_0. \tag{2.7}$$

We next consider a fluid element A with an infinitesimal volume δV lying at $z = \zeta$: its temperature T_a, concentration c_a, and density ρ_a are assumed to be given by

$$T_a = T_0 + T'_a, \quad c_a = c_0 + c'_a, \quad \rho_a = \rho_0 + \rho_0 \beta_c c'_a. \tag{2.8}$$

Let $\boldsymbol{F}_a$ denote the buoyancy force acting on A: it is given by (see Maruyama 2020)

$$\boldsymbol{F}_a = \delta V \rho_0 \beta T'_a g \boldsymbol{k} - \delta V \rho_0 \beta_c c'_a g \boldsymbol{k}. \tag{2.9}$$

We consider below the quasistatic process of displacing A vertically by interchanging its position with that of a fluid element B with the same infinitesimal volume δV lying at $z = \zeta + d\zeta$, $d\zeta$ being an infinitesimal distance: B is assumed to have the same physical properties as the ambient fluid, so that no buoyancy force acts on B.

The process is carried out adiabatically and without change in concentration. Then, since the specific entropy s is invariable, the temperature of A changes by the amount

$$dT_a = (\partial T/\partial p)_{s,c} dp_a. \tag{2.10}$$

Here dp_a denotes the change in pressure experienced by A, and is given by

$$dp_a = -\rho_0 g d\zeta. \tag{2.11}$$

The thermodynamic coefficient on the right-hand side of (2.10) can also be written as

$$(\partial T/\partial p)_{s,c} = \beta T_a/\rho_a c_p. \tag{2.12}$$

Here c_p denotes the specific heat at constant pressure: it is regarded as constant. Thus we see that the temperature of A becomes, after the process,

$$T_a - (\rho_0 \beta T_a g/\rho_a c_p)d\zeta. \tag{2.13}$$

On the other hand, since B possesses the same physical properties as the ambient fluid, the temperature of B after the process is more simply given by

$$T_0 + (\beta T_0 g/c_p)d\zeta. \tag{2.14}$$

Now, let us examine the work done on the system consisting of A and B through the above process. We have already seen that the only buoyancy force acting on the system is $\boldsymbol{F}_a$ given by (2.9). Thus the work W_a done on the system against $\boldsymbol{F}_a$ is

$$W_a = -\delta V \rho_0 \beta T_a' g d\zeta + \delta V \rho_0 \beta_c c_a' g d\zeta. \tag{2.15}$$

This work is composed of two components $W_{T'}$ and $W_{c'}$:

$$W_{T'} = -\delta V \rho_0 \beta T_a' g d\zeta, \quad W_{c'} = \delta V \rho_0 \beta_c c_a' g d\zeta. \tag{2.16}$$

Of these two components, $W_{c'}$ is stored as part of the gravitational potential energy of the system. This can be seen from the following fact: the gravitational potential energy of A increases through the process by $\delta V \rho_a g d\zeta$, while that of B increases by $-\delta V \rho_0 g d\zeta$; the sum of these increases is equal to $W_{c'}$. On the other hand, $W_{T'}$ is stored as part of the internal energy of the system. We confirm this, in the following, by comparing $W_{T'}$ with the increase ΔU in the internal energy of the system.

In order to calculate ΔU, we now consider another quasistatic process connecting the initial and the final state of the above process. In this process, before interchanging the positions of A and B, we first add to A the following amount of heat:

$$Q_1 = \delta V \rho_a c_p (T_0 - T_a). \tag{2.17}$$

Then the temperature of A becomes T_0, and the buoyancy force $\boldsymbol{F}_a$ reduces to

$$\boldsymbol{F}_a = -\delta V \rho_0 \beta_c c_a' g \boldsymbol{k}. \tag{2.18}$$

After the positions of A and B are interchanged, the temperature of A becomes

$$T_0 - (\rho_0 \beta T_0 g/\rho_a c_p)d\zeta. \tag{2.19}$$

The temperature of B, on the other hand, becomes (2.14) as before. Finally, we add to A the following amount of heat:

$$Q_2 = \delta V \rho_a c_p \{(T_a - T_0) - (\rho_0 \beta T_a g/\rho_a c_p - \rho_0 \beta T_0 g/\rho_a c_p) d\zeta\}. \tag{2.20}$$

Then the temperature of A becomes (2.13); we have reached the final state.

With respect to this process, we can obtain the following energy equation:

$$Q_1 + Q_2 + W_{c'} = \Delta U + \Delta E_g, \tag{2.21}$$

where $W_{c'}$ is the work done against $\boldsymbol{F}_a$ given by (2.18), and ΔE_g the increase through the process in the gravitational potential energy of the system. We have already seen, however, that $W_{c'} = \Delta E_g$. As a result, we obtain

$$\Delta U = Q_1 + Q_2 = -\delta V \rho_0 \beta T'_a g d\zeta. \tag{2.22}$$

This proves that, as expected, ΔU is equal to $W_{T'}$ given in (2.16).

On the basis of these observations, we can conclude as follows: the work done against the buoyancy force due to changes in concentration is stored as part of the gravitational potential energy of the fluid, while that done against the buoyancy force due to changes in temperature is stored as part of the internal energy of the fluid.

3. Work done on a fluid element in uniform stratification

Next, suppose that the fluid layer is uniformly stratified and that an internal gravity wave is present in the fluid. Correspondingly, we write T' in (2.1) and c' in (2.4) as

$$T' = \tau_0(z) + \tau, \quad c' = \sigma_0(z) + \sigma. \tag{3.1}$$

Here $\tau_0(z)$ and $\sigma_0(z)$ are linear functions of z, and represent the uniform stratification; τ and σ denote the perturbations due to the wave. We also write p' in (2.2) as follows:

$$p' = \pi_0(z) + \pi, \tag{3.2}$$

where $\pi_0(z)$ is a function of z satisfying the relation $d\pi_0/dz = \rho_0 \beta \tau_0 g - \rho_0 \beta_c \sigma_0 g$.

We assume here, after Maruyama (2020), that T' satisfies the equation

$$DT'/Dt = -\Gamma_0 w, \tag{3.3}$$

where D/Dt denotes the material derivative, $\Gamma_0 = \beta T_0 g/c_p$, and w is the z-component of the fluid velocity. In terms of τ, this equation can be rewritten as

$$D\tau/Dt + (d\tau_0/dz + \Gamma_0)w = 0. \tag{3.4}$$

Since $w = Dz/Dt$, we see from (3.4) that, for each fluid element, the condition

$$z + \tau/(d\tau_0/dz + \Gamma_0) = \text{constant} \tag{3.5}$$

holds. Here we have used the fact that $d\tau_0/dz + \Gamma_0$ is a constant.

On the other hand, ignoring the change in concentration due to diffusion, we have

$$Dc'/Dt = 0. \tag{3.6}$$

When rewritten in terms of σ, this equation becomes

$$D\sigma/Dt + (d\sigma_0/dz)w = 0. \tag{3.7}$$

Thus we observe that each fluid element satisfies, in addition to (3.5), the condition

$$z + \sigma/(d\sigma_0/dz) = \text{constant}. \tag{3.8}$$

Let us now focus our attention on an arbitrary fluid element X with an infinitesimal volume δV. We denote by z_0 the z-coordinate of the rest position of X: when X lies at the position, both τ and σ vanish. Then, from (3.5) and (3.8), we obtain the following expressions for the z-coordinate of X at an arbitrary instant:

$$z - z_0 = -\tau/(d\tau_0/dz + \Gamma_0) = -\sigma/(d\sigma_0/dz). \tag{3.9}$$

Note here that the buoyancy force $\boldsymbol{F}$ acting on X is given by (see Maruyama 2020)

$$\boldsymbol{F} = \delta V \rho_0 \beta \tau g \boldsymbol{k} - \delta V \rho_0 \beta_c \sigma g \boldsymbol{k}. \tag{3.10}$$

Using (3.9), we can rewrite $\boldsymbol{F}$ in the following form:

$$\boldsymbol{F} = -\delta V \rho_0 N_\theta^2 (z - z_0)\boldsymbol{k} - \delta V \rho_0 N_c^2 (z - z_0)\boldsymbol{k}. \tag{3.11}$$

Here N_θ and N_c are respectively the thermal and the non-thermal buoyancy frequency defined as follows (see Maruyama 2020):

$$N_\theta^2 = \beta g \left(d\tau_0/dz + \Gamma_0\right), \quad N_c^2 = -\beta_c g (d\sigma_0/dz). \tag{3.12}$$

It is seen from (3.11) that the work W done against the buoyancy force $\boldsymbol{F}$ to displace X from its rest position is given by

$$W = \delta V \rho_0 N_\theta^2 (z - z_0)^2/2 + \delta V \rho_0 N_c^2 (z - z_0)^2/2. \tag{3.13}$$

This work, similarly to W_a in § 2.2, consists of two components W_τ and W_σ:

$$W_\tau = \delta V \rho_0 N_\theta^2 (z - z_0)^2/2, \quad W_\sigma = \delta V \rho_0 N_c^2 (z - z_0)^2/2. \tag{3.14}$$

In light of the conclusion of § 2.2, we recognize that W_τ is stored as part of the internal energy of the fluid, since it represents the work done against the buoyancy force due to changes in temperature. In contrast, W_σ is stored as part of the gravitational potential energy of the fluid. This follows from the fact that it represents the work done against the buoyancy force due to changes in concentration.

4. Conclusion

We finally define, in terms of W_τ and W_σ, the following quantities:

$$P_\theta = W_\tau/\delta V, \quad P_c = W_\sigma/\delta V. \tag{4.1}$$

These quantities, with the aid of (3.9), can be expressed as follows:

$$P_\theta = \frac{1}{2}\rho_0 N_\theta^2 \left(\frac{\tau}{d\tau_0/dz + \Gamma_0}\right)^2, \quad P_c = \frac{1}{2}\rho_0 N_c^2 \left(\frac{\sigma}{d\sigma_0/dz}\right)^2. \tag{4.2}$$

Now, written in terms of P_θ and P_c, (3.13) becomes

$$W = \delta V(P_\theta + P_c). \tag{4.3}$$

Thus $P_\theta + P_c$ is the work, per unit volume, done against the buoyancy force to displace a fluid element from its rest position. This implies that $P_\theta + P_c$ is the potential energy density of the internal gravity wave (see Maruyama 2020).

However, as is evident from the derivation, P_θ is stored as part of the internal energy of the fluid. This proves that the potential energy density of an internal "gravity" wave can really be stored in the internal energy of the fluid.

References

[1] Maruyama, K. 2020 Thermal and non-thermal internal gravity waves.