Front conditions for a gravity current in a background flow

Kiyoshi Maruyama

Department of Earth and Ocean Sciences, National Defense Academy,

Yokosuka, Kanagawa 239-8686, Japan

August 12, 2016

Abstract

The motion of a gravity current in a channel can be studied within the framework of the one-dimensional two-layer shallow-water equations if the front conditions, i.e. the boundary conditions to be imposed at the front of the gravity current, are known in advance. This paper presents the front conditions for a gravity current advancing in a background flow along a no-slip boundary. The conditions take different forms depending on whether or not the background flow is a "headwind". It is also shown that the conditions are directly applicable to a stationary gravity current.

1. Introduction

The motion of a gravity current in a channel can be studied within the framework of the one-dimensional two-layer shallow-water equations (Rottman & Simpson 1983; Klemp, Rotunno & Skamarock 1994). To this end, however, it is necessary to know in advance the boundary conditions to be imposed at the front of the gravity current.

Maruyama (2014) showed that these front conditions can be obtained from the four basic physical laws that the equations represent; he formulated, in particular, the front conditions for a gravity current advancing along a no-slip boundary.

The front conditions were derived, however, on the assumption that no separation of the boundary layer occurs on the no-slip boundary. The validity of this assumption is doubtful when the gravity current is in a background flow: the separation of the flow may take place near the front of the gravity current, since the velocity of the flow, and hence the pressure, changes rapidly there.

Thus the aim of this paper is to formulate the front conditions for a gravity current in a background flow without excluding the possibility of boundary layer separation. It turns out that the front conditions derived by Maruyama need modification when the background flow is a "headwind".

2. Circulation flux from the lower boundary layer

We consider a gravity current of a fluid with a constant density ρ_1 advancing along the lower boundary of a horizontal channel of uniform rectangular cross-section. This channel has a rigid upper boundary and is occupied by a flowing fluid with a smaller density ρ_2. The fluids are assumed to be incompressible and Newtonian; it is assumed, however, that they have very small viscosity.

We now concentrate our attention on the vertical plane containing the centerline of the lower boundary of the channel. This vertical plane is assumed to lie well away from the boundary layers on the side boundaries of the channel; it is also assumed that the fluid velocity on this plane is parallel to the side boundaries. We set up on this plane a system of rectangular coordinates (x_*, z_*) by taking the x_*-axis along the centerline of the lower boundary in the direction of advance of the gravity current and the z_*-axis vertically upward. We also define the unit normal $\boldsymbol{n}$ to the plane by $\boldsymbol{n} = \boldsymbol{i} \times \boldsymbol{k}$, where $\boldsymbol{i}$ and $\boldsymbol{k}$ are the unit vectors in the x_*- and z_*-directions, respectively.

It is expected that, in this plane, boundary layers are formed on the upper and lower boundaries of the channel and around the interface between the fluids: the fluid motion outside the boundary layers is assumed to be irrotational. The purpose of this section is to evaluate the rate of transfer of circulation from the boundary layer on the lower boundary to that around the interface. To this end, we first need to discuss the nature of the lower boundary layer well ahead of the gravity current.

2.1. Lower boundary layer well ahead of the gravity current

We consider the boundary layer on the lower boundary of the channel in the region satisfying the condition

$$\{x_* - x_{f*}(t_*)\}/H \gg 1, \tag{2.1}$$

where $x_{f*}(t_*)$ denotes the x_*-coordinate at time t_* of the foremost point of the gravity current, and H the depth of the channel.

Now let L denote the horizontal length scale characteristic of the fluid motion in this region: it is assumed that $L \gg H$. Then, to describe the motion in the boundary layer appropriately, it is reasonable to introduce the following dimensionless coordinates:

$$\varXi = \{x_* - x_{f*}(t_*)\}/L, \quad \zeta = z_*/\delta_l, \tag{2.2}$$

where $\delta_l \ll H$ denotes the scale of the thickness of the boundary layer. Together with these dimensionless coordinates, we use the dimensionless time t defined by

$$t = t_*(\beta g H)^{\frac{1}{2}}/L, \tag{2.3}$$

in which $\beta = (\rho_1 - \rho_2)/\rho_1$ and g is the acceleration due to gravity.

In the boundary layer, we denote the velocities in the x_*- and z_*-directions by u_* and w_*, respectively; we then define the dimensionless velocities u and w by

$$u = u_*/(\beta g H)^{\frac{1}{2}}, \quad w = (L/\delta_l)w_*/(\beta g H)^{\frac{1}{2}}. \tag{2.4}$$

In addition, the following dimensionless speed $U_f(t)$ is introduced:

$$U_f = U_{f*}/(\beta g H)^{\frac{1}{2}}, \tag{2.5}$$

where $U_{f*} = dx_{f*}/dt_*$ gives the rate of advance of the gravity current. The pressure p_* in the boundary layer is also nondimensionalized as follows:

$$p = \{p_* - \rho_2 g(H - z_*)\}/\rho_2 \beta g H. \tag{2.6}$$

In terms of the above dimensionless variables, the equations governing the motion in the boundary layer can be expressed as follows (see e.g. Prandtl & Tietjens 1957b):

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial \Xi} + w\frac{\partial v}{\partial \zeta} = -\frac{\partial p}{\partial \Xi} - \frac{dU_f}{dt} + \mathcal{F}, \tag{2.7}$$

$$\frac{\partial p}{\partial \zeta} = 0, \tag{2.8}$$

$$\frac{\partial v}{\partial \Xi} + \frac{\partial w}{\partial \zeta} = 0, \tag{2.9}$$

where $v = u - U_f$; $\mathcal{F}$ is the dimensionless x_*-component of the viscous force per unit mass, and is assumed to be $O(1)$.

We first observe from (2.8) that p is independent of ζ. Also, from (2.9), we see that v and w can be written as follows:

$$v = -\frac{\partial \psi}{\partial \zeta}, \quad w = \frac{\partial \psi}{\partial \Xi}, \tag{2.10}$$

where the stream function ψ is defined by

$$\psi = -\int_0^{\zeta} v d\zeta'. \tag{2.11}$$

It is assumed that ψ is a monotone increasing function of ζ for $\zeta \geq 0$: this implies that $v \leq 0$ for $\zeta \geq 0$. In terms of ψ, (2.7) can be rewritten in the form

$$\frac{\partial v}{\partial t} - \frac{\partial \psi}{\partial \zeta}\frac{\partial v}{\partial \Xi} + \frac{\partial \psi}{\partial \Xi}\frac{\partial v}{\partial \zeta} = -\frac{\partial p}{\partial \Xi} - \frac{dU_f}{dt} + \mathcal{F}. \tag{2.12}$$

On the basis of this equation, we examine in the following the nature of the boundary layer near the front of the gravity current.

2.2. Lower boundary layer near the front of the gravity current

In the vicinity of the front of the gravity current, it is expected that the fluid motion possesses a horizontal length scale l_f much smaller than L. This length scale will be of the same order of magnitude as the depth of the channel; we assume that $l_f/H = O(1)$. It is then natural to use, in place of Ξ, the dimensionless coordinate

$$\xi = (L/l_f)\Xi, \tag{2.13}$$

where we put the restriction $\xi \geq 0$. Rewriting (2.12) in terms of ξ and ζ, we get

$$\frac{l_f}{L}\frac{\partial v}{\partial t} - \frac{\partial \psi}{\partial \zeta}\frac{\partial v}{\partial \xi} + \frac{\partial \psi}{\partial \xi}\frac{\partial v}{\partial \zeta} = -\frac{\partial p}{\partial \xi} - \frac{l_f}{L}\frac{dU_f}{dt} + \frac{l_f}{L}\mathcal{F}. \tag{2.14}$$

Since $l_f/L \ll 1$, we can approximate (2.14) by neglecting the terms containing l_f/L. It should be noticed, however, that a secondary boundary layer must be formed on the lower boundary owing to the no-slip condition. In our dimensionless coordinate system, this secondary boundary layer may be identified with a vortex sheet on $\zeta = 0$; the last term of (2.14), which represents the effect of viscosity, cannot be neglected on $\zeta = 0$.

Now let us consider (2.14) on $\zeta = 0$. Since $v = -U_f(t)$ and $\psi = 0$ there, it reduces to

$$0 = -\partial p/\partial \xi + (l_f/L)\mathcal{F}. \tag{2.15}$$

Integrating (2.15) with respect to ξ from 0 to ∞, we have

$$\int_0^\infty (l_f/L)\mathcal{F} d\xi = -\left(p|_{\xi=0} - p|_{\xi=\infty}\right). \tag{2.16}$$

The left-hand side of (2.16), being a line integral of the dimensionless viscous force per unit mass, gives the dimensionless diffusion flux of circulation from the lower boundary near the front of the gravity current (see Maruyama 2014, § 2).

Let us next consider (2.14) in the region $\zeta > 0$. It is convenient to use ψ in place of ζ as an independent variable, i.e. to apply the von Mises transformation (see e.g. Meyer 1971); neglecting the terms containing l_f/L, we obtain

$$v(\xi, \psi, t)\left\{\partial v(\xi, \psi, t)/\partial \xi\right\} = -\partial p/\partial \xi, \tag{2.17}$$

where $\psi > 0$. The integration of (2.17) with respect to ξ from 0 to ∞ yields

$$p|_{\xi=0} - p|_{\xi=\infty} = \tfrac{1}{2}v^2|_{\xi=\infty,\psi>0} - \tfrac{1}{2}v^2|_{\xi=0,\psi>0}. \tag{2.18}$$

As explained below, the left-hand side of this equation, and thus the right-hand side of (2.16), can be evaluated from this equation on the basis of a simple argument.

2.3. Pressure rise near the front of the gravity current

We first observe from (2.18) that, since $\frac{1}{2}v^2|_{\xi=0,\psi>0} \geq 0$,

$$p|_{\xi=0} - p|_{\xi=\infty} \leq \tfrac{1}{2}v^2|_{\xi=\infty,\psi>0}. \tag{2.19}$$

Now let the infimum of $\frac{1}{2}v^2|_{\xi=\infty,\psi>0}$ over ψ be denoted by $\frac{1}{2}v^2{}_{\mathrm{inf}}$. Since (2.19) applies for all $\psi > 0$, we see that

$$p|_{\xi=0} - p|_{\xi=\infty} \leq \tfrac{1}{2}v^2{}_{\mathrm{inf}}. \tag{2.20}$$

We next observe the following fact (Simpson & Britter 1980): because of the no-slip condition at the lower boundary, at least part of the fluid in the lower boundary layer must be left behind by the foremost point of the gravity current beneath the following heavier fluid; the lighter fluid approaching the front of the gravity current bifurcates at the point, and thus the point must be a stagnation point. Let ψ_0 denote the value of ψ on the streamline ending at the point: note that $\psi_0 > 0$, for the point cannot lie on the lower boundary where $v = -U_f \neq 0$. Evaluating (2.18) at $\psi = \psi_0$, we get

$$p|_{\xi=0} - p|_{\xi=\infty} = \tfrac{1}{2}v^2|_{\xi=\infty,\psi=\psi_0}, \tag{2.21}$$

since $\frac{1}{2}v^2|_{\xi=0,\psi=\psi_0} = 0$. However, since $\frac{1}{2}v^2|_{\xi=\infty,\psi=\psi_0} \geq \frac{1}{2}{v^2}_{\mathrm{inf}}$, it follows that

$$p|_{\xi=0} - p|_{\xi=\infty} \geq \tfrac{1}{2}{v^2}_{\mathrm{inf}}. \tag{2.22}$$

As a consequence, comparing (2.20) and (2.22), we have

$$p|_{\xi=0} - p|_{\xi=\infty} = \tfrac{1}{2}{v^2}_{\mathrm{inf}}. \tag{2.23}$$

This determines the rise in pressure near the front of the gravity current.

2.4. Inflow of circulation to the interfacial boundary layer

As has already been explained, the lighter fluid approaching the front of the gravity current bifurcates at the foremost point of the gravity current. The fluid flowing above the streamline ending at the foremost point continues to flow along the interface with the heavier fluid below; a boundary layer thus develops around the interface. On the other hand, the fluid flowing below the streamline is left behind by the foremost point beneath the following heavier fluid. However, as elucidated by Maruyama (2014), this fluid also is absorbed into the interfacial boundary layer through convective instability. Thus all the circulation carried by the lighter fluid approaching the front of the gravity current must be transferred to the interfacial boundary layer. We are now in a position to evaluate the rate of this transfer of circulation.

We first calculate, at $\xi = \infty$, the rate of inflow of circulation to the lower boundary layer near the front of the gravity current. This rate, which is denoted by Λ_{I*}, is given as follows (see Maruyama 2014, § 2):

$$\Lambda_{I*} = -\beta g H \int_0^\infty \omega_n v d\zeta \bigg|_{\xi=\infty}, \tag{2.24}$$

where ω_n denotes the dimensionless component in the direction of $\boldsymbol{n}$ of the vorticity in the boundary layer. Since ω_n may be approximated by

$$\omega_n = -\frac{\partial v}{\partial \zeta}, \tag{2.25}$$

we obtain, by calculating the right-hand side of (2.24),

$$\Lambda_{I*} = \beta g H \left(\tfrac{1}{2}v^2|_{\xi=\infty,\zeta=\infty} - \tfrac{1}{2}U_f^2\right). \tag{2.26}$$

Here the no-slip condition $v = -U_f$ at $\zeta = 0$ has been used.

Let us next consider the diffusion flux of circulation from the lower boundary to the boundary layer near the front of the gravity current. This flux $\varLambda_{D*}$ is given by

$$\varLambda_{D*} = \beta g H \int_0^\infty (l_f/L)\mathcal{F} d\xi. \tag{2.27}$$

Accordingly, from (2.16) and (2.23), we can write $\varLambda_{D*}$ as follows:

$$\varLambda_{D*} = -\beta g H \tfrac{1}{2} {v^2}_{\mathrm{inf}}. \tag{2.28}$$

The circulation carried inside the lower boundary layer per unit time to the front of the gravity current is obtained as the sum of $\varLambda_{I*}$ and $\varLambda_{D*}$: no circulation is carried to the front by the fluid outside the boundary layer. As stated above, all the circulation carried to the front is transferred to the interfacial boundary layer. Hence the rate of inflow of circulation $\varLambda_*$ at the front to the interfacial boundary layer is given by

$$\varLambda_* = \beta g H \left(\tfrac{1}{2} v^2|_{\xi=\infty,\zeta=\infty} - \tfrac{1}{2} U_f^2 - \tfrac{1}{2} {v^2}_{\mathrm{inf}} \right). \tag{2.29}$$

The purpose of this section has now been accomplished.

3. Front conditions

Now let us proceed to discuss, within the framework of the one-dimensional two-layer shallow-water equations, the front conditions for the gravity current considered in § 2. Note that, in the discussion below, a zero following a plus or a minus is used to denote an infinitesimal positive number for convenience of notation.

3.1. Conditions based on the conservation laws of mass and of momentum

In this section, we employ the following dimensionless coordinates:

$$x = x_*/L, \quad z = z_*/H. \tag{3.1}$$

The front of the gravity current is represented, in this dimensionless coordinate system, by a wall of fluid located at $x = x_f (= x_{f*}/L)$. The boundary layers, on the other hand, are identified with vortex sheets coincident with the boundaries and the interface.

We first focus our attention on the region behind the front of the gravity current, i.e. the region for which $x < x_f$. Let $z = h(x,t)$ give the position of the interface between the fluids. The pressure p_* can then be expressed as follows (see Maruyama 2014, § 3):

$$p_* = \begin{cases} \rho_2 \beta g H \eta(x,t) + \rho_2 g H (1-h) + \rho_1 g H (h-z), & 0 \le z \le h, \\ \rho_2 \beta g H \eta(x,t) + \rho_2 g H (1-z), & h \le z \le 1. \end{cases} \tag{3.2}$$

Here $\rho_2 \beta g H \eta$ gives the pressure at the upper boundary. The horizontal component u_* of the fluid velocity outside the boundary layers can also be written as

$$u_* = \begin{cases} (\beta g H)^{\frac{1}{2}} u_1(x,t), & 0 < z < h, \\ (\beta g H)^{\frac{1}{2}} u_2(x,t), & h < z < 1. \end{cases} \tag{3.3}$$

The fluid motion in the region $x < x_f$ can thus be described in terms of u_1, u_2, h, and η. These variables satisfy the one-dimensional two-layer shallow-water equations:

$$\left.\begin{aligned} \frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x} = -\frac{\partial h}{\partial x} - (1-\beta)\frac{\partial \eta}{\partial x}, \quad \frac{\partial u_2}{\partial t} + u_2\frac{\partial u_2}{\partial x} = -\frac{\partial \eta}{\partial x}, \\ \frac{\partial h}{\partial t} = -\frac{\partial}{\partial x}(hu_1), \quad \frac{\partial}{\partial t}(1-h) = -\frac{\partial}{\partial x}\left\{(1-h)u_2\right\}. \end{aligned}\right\} \tag{3.4}$$

Let us next turn our attention to the region $x > x_f$. Since the heavier fluid is absent in this region, the pressure distribution can be written as

$$p_* = \rho_2\beta g H\eta(x,t) + \rho_2 g H(1-z), \quad 0 \le z \le 1, \tag{3.5}$$

in place of (3.2); we also obtain, in place of (3.3),

$$u_* = (\beta g H)^{\frac{1}{2}} u_2(x,t), \quad 0 < z < 1. \tag{3.6}$$

The variables u_2 and η, which specify the fluid motion in this region, are governed by

$$\frac{\partial u_2}{\partial x} = 0, \quad \frac{\partial \eta}{\partial x} = -\frac{\partial u_2}{\partial t} - u_2\frac{\partial u_2}{\partial x}. \tag{3.7}$$

In order to solve (3.4) simultaneously with (3.7), four conditions must be specified at $x = x_f$ (see Maruyama 2014, § 5). Of these four front conditions, the first three are

$$\left.\begin{aligned} hv_1|_{x=x_f-0} = 0, \quad (1-\beta)(1-h)v_2|_{x=x_f-0} - (1-\beta)v_2|_{x=x_f+0} = 0, \\ \left\{hv_1^2 + (1-\beta)(1-h)v_2^2 + \tfrac{1}{2}h^2 + (1-\beta)\eta\right\}\big|_{x=x_f-0} \\ - \left\{(1-\beta)v_2^2 + (1-\beta)\eta\right\}\big|_{x=x_f+0} = 0. \end{aligned}\right\} \tag{3.8}$$

Here $v_i = u_i - U_f$ $(i = 1, 2)$. These front conditions are obtained from the conservation laws of mass and of momentum. Our remaining task in this section is to formulate the fourth condition that follows from the conservation law of circulation.

3.2. Condition based on the conservation law of circulation

Applying the discussion of Maruyama (2014) without modification, we obtain, on the basis of the conservation law of circulation, the following front condition:

$$(\tfrac{1}{2}v_1^2 - \tfrac{1}{2}v_2^2 + h - \beta\eta)|_{x=x_f-0} + \beta\eta|_{x=x_f+0} + \beta\tfrac{1}{2}v_2^2|_{x=x_f+0} - \tfrac{1}{2}U_f^2 = 0. \tag{3.9}$$

Here the third term on the left-hand side represents the rise in pressure near the front of the gravity current, and the last term the rate of inflow of circulation at the front to the interfacial boundary layer. However, it can be seen from (2.23) and (2.29) that this condition needs to be modified as follows:

$$(\tfrac{1}{2}v_1^2 - \tfrac{1}{2}v_2^2 + h - \beta\eta)|_{x=x_f-0} + \beta\eta|_{x=x_f+0} + \beta\tfrac{1}{2}v^2{}_{\mathrm{inf}} + \Lambda = 0, \tag{3.10}$$

where $\mathit{\Lambda} = \mathit{\Lambda}_*/\beta g H$. As shown in the following, the last two terms on the left-hand side of this modified condition can be expressed in terms of $v_2|_{x=x_f+0}$ and U_f.

Let us again focus our attention on the fluid motion in the lower boundary layer. We first note that, since $\psi > 0$ is mapped onto $\zeta > 0$, $\frac{1}{2}{v^2}_{\text{inf}}$ can be written as

$$\tfrac{1}{2}{v^2}_{\text{inf}} = \inf\left\{\tfrac{1}{2}v^2|_{\xi=\infty,\psi>0}\right\} = \inf\left\{\tfrac{1}{2}v^2|_{\xi=\infty,\zeta>0}\right\}. \tag{3.11}$$

Suppose now that v at $\xi = \infty$ is a monotone function of ζ: this implies that ω_n defined by (2.25) does not change sign in the lower boundary layer well ahead of the front of the gravity current. Then, since we have assumed that $v \le 0$ in the boundary layer, $\frac{1}{2}v^2$ at $\xi = \infty$ is also a monotone function of ζ. Hence it can be seen from (3.11) that

$$\tfrac{1}{2}{v^2}_{\text{inf}} = \min\left\{\tfrac{1}{2}v^2|_{\xi=\infty,\zeta=+0},\ \tfrac{1}{2}v^2|_{\xi=\infty,\zeta=\infty}\right\}. \tag{3.12}$$

Though a secondary boundary layer develops near the front of the gravity current, it is absent at $\xi = \infty$; hence v is continuous at $\xi = \infty$ for $\zeta \ge 0$. As a result, taking account of the no-slip condition $v = -U_f$ at $\zeta = 0$, we have

$$\tfrac{1}{2}v^2|_{\xi=\infty,\zeta=+0} = \tfrac{1}{2}U_f^2. \tag{3.13}$$

On the other hand, it follows from (2.13) that, when $l_f/L = +0$, $\xi = \infty$ corresponds to $\mathit{\Xi} = +0$ (see e.g. Meyer 1971, § 21). Hence we obtain the matching condition

$$\tfrac{1}{2}v^2|_{\xi=\infty,\zeta=\infty} = \tfrac{1}{2}v^2|_{\mathit{\Xi}=+0,\zeta=\infty}. \tag{3.14}$$

We also have the following matching condition at $\zeta = \infty$:

$$\tfrac{1}{2}v^2|_{\mathit{\Xi}=+0,\zeta=\infty} = \tfrac{1}{2}v_2^2|_{x=x_f+0}, \tag{3.15}$$

where the fact that $\mathit{\Xi} = +0$ corresponds to $x = x_f + 0$ has been used. In consequence, we can rewrite (3.12) as follows:

$$\tfrac{1}{2}{v^2}_{\text{inf}} = \min\left\{\tfrac{1}{2}U_f^2,\ \tfrac{1}{2}v_2^2|_{x=x_f+0}\right\}. \tag{3.16}$$

In addition, using the above matching conditions, we find from (2.29) the expression

$$\mathit{\Lambda} = \tfrac{1}{2}v_2^2|_{x=x_f+0} - \tfrac{1}{2}U_f^2 - \tfrac{1}{2}{v^2}_{\text{inf}}. \tag{3.17}$$

We recall here that $v_2 = u_2 - U_f$. Accordingly, in view of (3.7), we observe that

$$v_2|_{x=x_f+0} = U_\infty - U_f, \tag{3.18}$$

where $U_\infty = u_2|_{x=\infty}$ is the velocity of the background flow.

Now, suppose that $U_\infty < 0$, i.e. that the background flow is a “headwind”. Then we see from (3.18) that $v_2|_{x=x_f+0} < -U_f < 0$, and obtain from (3.16) and (3.17)

$$\tfrac{1}{2}{v^2}_{\text{inf}} = \tfrac{1}{2}U_f^2, \quad \mathit{\Lambda} = \tfrac{1}{2}v_2^2|_{x=x_f+0} - U_f^2. \tag{3.19}$$

When $0 \le U_\infty < U_f$, i.e. when the background flow is absent or is a “tailwind”, we get

$$\tfrac{1}{2}{v^2}_{\text{inf}} = \tfrac{1}{2}v_2^2|_{x=x_f+0}, \quad \mathit{\Lambda} = -\tfrac{1}{2}U_f^2. \tag{3.20}$$

The substitution of (3.19) or (3.20) into (3.10) yields the desired front condition.

It should be noted here that, when (3.20) is substituted into (3.10), (3.9) is obtained. This proves that the front conditions formulated by Maruyama (2014) are valid so long as the background flow is not a headwind.

4. Summary and discussion

The front conditions for a gravity current in a background flow have been formulated within the framework of the one-dimensional two-layer shallow-water equations: (3.8) and (3.10). In (3.10), the last two terms on the left-hand side are obtained from (3.19) when the background flow is a headwind and from (3.20) otherwise.

In deriving the front conditions, it was assumed that the gravity current is advancing relative to the lower boundary. However, as explained below, the conditions apply also to a gravity current brought to rest relative to a no-slip boundary by a headwind.

Now let us consider such a gravity current and examine the rise in pressure near its front. Since $U_f = 0$, it can be seen from (3.16) that

$$\tfrac{1}{2}v^2{}_{\mathrm{inf}} = 0. \tag{4.1}$$

Substituting this into (2.20), we obtain

$$p|_{\xi=0} - p|_{\xi=\infty} \leq 0. \tag{4.2}$$

On the other hand, (2.22) cannot be used in the present case, for it is dependent on the assumption that $U_f \neq 0$. However, evaluating (2.18) at $\psi = \infty$, we have

$$p|_{\xi=0} - p|_{\xi=\infty} = \tfrac{1}{2}v^2|_{\xi=\infty,\psi=\infty} - \tfrac{1}{2}v^2|_{\xi=0,\psi=\infty}. \tag{4.3}$$

We should note here that, ahead of the front of the gravity current, the flow above the lower boundary layer behaves as a potential flow upstream of a concave corner. Hence the flow decelerates downstream (see e.g. Prandtl & Tietjens 1957a, § 77):

$$\tfrac{1}{2}v^2|_{\xi=\infty,\psi=\infty} \geq \tfrac{1}{2}v^2|_{\xi=0,\psi=\infty}. \tag{4.4}$$

From (4.3) and (4.4), it follows that

$$p|_{\xi=0} - p|_{\xi=\infty} \geq 0. \tag{4.5}$$

Comparing this with (4.2), we get

$$p|_{\xi=0} - p|_{\xi=\infty} = 0. \tag{4.6}$$

However, in view of (4.1), we can rewrite (4.6) as follows:

$$p|_{\xi=0} - p|_{\xi=\infty} = \tfrac{1}{2}v^2{}_{\mathrm{inf}}. \tag{4.7}$$

This equation is identical to (2.23).

On the basis of this result, we again obtain the expression (2.29) for Λ_*. On account of (4.1), and since $U_f = 0$, it reduces in the present case to

$$\mathit{\Lambda}_* = \beta g H \tfrac{1}{2}v^2|_{\xi=\infty,\zeta=\infty}. \tag{4.8}$$

It can now readily be seen that the front conditions (3.8) and (3.10) again hold: the last two terms on the left-hand side of (3.10) are obtained from

$$\tfrac{1}{2}{v^2}_{\mathrm{inf}} = 0, \quad \mathit{\Lambda} = \tfrac{1}{2}v_2^2|_{x=x_f+0}. \tag{4.9}$$

However, (4.9) follows from (3.19) when $U_f = 0$. Thus we have reached the conclusion that the present theory applies also to a stationary gravity current in a headwind.

Finally, it should be noted that a gravity current needs to satisfy, in addition to the front conditions, two supplementary conditions. These conditions, the energy condition and the evolutionary condition, are given in Maruyama (2014).

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