Anelastic approximation

Kiyoshi Maruyama

Department of Earth and Ocean Sciences, National Defense Academy,

Yokosuka, Kanagawa 239-8686, Japan

March 11, 2021

Abstract

This paper reconstructs the anelastic approximation in such a manner that it can be applied to any kind of fluid. As a result, it turns out that the work done by the buoyancy force in the approximation corresponds to the conversion between kinetic and internal energy. Also, the conditions for the applicability of the approximation are clarified. It is demonstrated, in addition, that the Boussinesq approximation is not a limiting case of the anelastic approximation.

1. Introduction

The anelastic approximation was devised by Ogura & Phillips (1962) to describe the motion of a thermally stratified fluid. It partly takes into account the compressibility of a fluid by allowing the density of the fluid to vary with height; nevertheless, it excludes sound waves from the solutions of the system of governing equations.

However, the approximation, as it stands, is applicable only to ideal gases. Thus the purpose of this paper is to reconstruct the approximation in such a manner that it can be applied to any kind of fluid. As a result, it becomes apparent that the work done by the buoyancy force in the approximation corresponds to the conversion between kinetic and internal energy. The conditions for the applicability of the approximation and the relation to the Boussinesq approximation are also discussed.

2. Anelastic approximation

We consider the motion of an inviscid fluid in a uniform gravitational field. The fluid is assumed to be contained in a fixed finite domain Ω. In the domain, the z-axis is taken vertically upwards: the unit vector in the positive z-direction is denoted by $\boldsymbol{k}$.

2.1. Equation of motion

Denoting by $\boldsymbol{u}$ the velocity of the fluid, we can express the equation of motion as

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla p/\rho - g\boldsymbol{k}, \tag{2.1}$$

in which D/Dt stands for the material derivative; p and ρ are, respectively, the pressure and the density of the fluid, and g is the acceleration due to gravity.

Let the specific enthalpy of the fluid be denoted by h. We then have the relation

$$dh = Tds + vdp, \tag{2.2}$$

where T and s are, respectively, the temperature and the specific entropy of the fluid; $v = 1/\rho$ denotes the specific volume of the fluid. In the following, all thermodynamic quantities are regarded as known functions of h and s.

By virtue of (2.2), we can rewrite $\nabla p/\rho$ as follows:

$$\nabla p/\rho = \nabla h - T\nabla s. \tag{2.3}$$

Substituting this into (2.1), we obtain

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla h + T\nabla s - g\boldsymbol{k}. \tag{2.4}$$

Now, let h and s be decomposed as follows:

$$h = h_0 + h', \quad s = s_0 + s'. \tag{2.5}$$

Here h_0 and s_0 are given by

$$h_0 = -gz + c_1, \quad s_0 = c_2, \tag{2.6}$$

with c_1 and c_2 being constants. Then, in terms of h' and s', (2.4) is written as

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla h' + T\nabla s'. \tag{2.7}$$

However, regarded as a function of h and s, T can also be decomposed as follows:

$$T = T_0 + T', \tag{2.8}$$

where T_0 is defined by

$$T_0 = T(h_0, s_0). \tag{2.9}$$

We introduce here the following assumption:

$$|T'/T_0| \ll 1. \tag{2.10}$$

Then (2.7) may be approximated by

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla h' + T_0\nabla s'. \tag{2.11}$$

Using the identity $T_0\nabla s' = \nabla(T_0 s') - s'\nabla T_0$, we finally obtain

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla(h' - T_0 s') - s'\nabla T_0. \tag{2.12}$$

This is the equation of motion under the anelastic approximation: as demonstrated in the appendix, this equation of motion reduces to the one derived by Ogura & Phillips (1962) when the fluid is an ideal gas.

Before closing this subsection, we wish to rewrite the last term of (2.12) in a different form. To this end, we first note the following thermodynamic relations:

$$(\partial T/\partial h)_s = \beta T/c_p, \quad (\partial T/\partial s)_h = T(1-\beta T)/c_p, \tag{2.13}$$

where $\beta = v^{-1}(\partial v/\partial T)_p$ is the thermal expansion coefficient, and c_p the specific heat at constant pressure. Then, in view of (2.6) and (2.9), we find

$$\nabla T_0 = (\partial T/\partial h)_s|_{(h_0,s_0)}\nabla h_0 + (\partial T/\partial s)_h|_{(h_0,s_0)}\nabla s_0 = -(\beta_0 T_0 g/c_{p0})\boldsymbol{k}, \tag{2.14}$$

in which we have introduced the notation

$$\beta_0 = \beta(h_0, s_0), \quad c_{p0} = c_p(h_0, s_0). \tag{2.15}$$

As a result, the last term of (2.12) can be rewritten as follows:

$$-s'\nabla T_0 = (\beta_0 T_0 s' g/c_{p0})\boldsymbol{k}. \tag{2.16}$$

The force represented by this term is called the buoyancy force.

2.2. Equation of continuity

In view of (2.5), we can express the density ρ of the fluid as

$$\rho = \rho_0 + \rho', \tag{2.17}$$

where ρ_0 is given by

$$\rho_0 = \rho(h_0, s_0). \tag{2.18}$$

We now introduce the following assumption:

$$|\rho'/\rho_0| \ll 1. \tag{2.19}$$

Then we can approximate ρ as follows:

$$\rho = \rho_0. \tag{2.20}$$

This approximation is the essence of the anelastic approximation. The substitution of (2.20) into the equation of continuity $\partial\rho/\partial t + \nabla\cdot(\rho\boldsymbol{u}) = 0$ yields

$$\nabla\cdot(\rho_0\boldsymbol{u}) = 0. \tag{2.21}$$

This is the equation of continuity under the anelastic approximation.

2.3. Adiabatic equation

When the conduction of heat is neglected, the general equation of heat transfer (see Landau & Lifshitz 1987, § 49) can be written, considering (2.5) and (2.6), as follows:

$$\rho_0 T \frac{Ds'}{Dt} = 0. \tag{2.22}$$

Here the approximation (2.20) has been used. On the assumption (2.10), this equation may be approximated by

$$\rho_0 T_0 \frac{Ds'}{Dt} = 0. \tag{2.23}$$

This adiabatic equation, together with the equation of motion (2.12) and the equation of continuity (2.21), completes the formulation of the anelastic approximation.

3. Energetics of the anelastic approximation

Let us next study, under the anelastic approximation, the energy balance of the fluid considered in the previous section. We first examine the internal energy of the fluid.

3.1. Internal energy

The specific internal energy e of the fluid is related to h by the formula

$$e = h - p/\rho. \tag{3.1}$$

Under the anelastic approximation, h and s are decomposed as (2.5). Correspondingly, p also is decomposed as follows:

$$p = p_0 + p', \tag{3.2}$$

where p_0 is defined by

$$p_0 = p(h_0, s_0). \tag{3.3}$$

Thus we obtain the following expression for e:

$$e = (h_0 + h') - (p_0 + p')/\rho_0 = (h_0 - p_0/\rho_0) + (h' - p'/\rho_0), \tag{3.4}$$

in which the approximation (2.20) has been used.

On the other hand, we can find from (2.2) the thermodynamic relations

$$(\partial p/\partial h)_s = \rho, \quad (\partial p/\partial s)_h = -\rho T. \tag{3.5}$$

Hence p' can be expressed, to the first order of h' and s', as follows:

$$p' = (\partial p/\partial h)_s|_{(h_0,s_0)} h' + (\partial p/\partial s)_h|_{(h_0,s_0)} s' = \rho_0 h' - \rho_0 T_0 s'. \tag{3.6}$$

This enables us to write

$$e = (h_0 - p_0/\rho_0) + T_0 s'. \tag{3.7}$$

Now, taking the material derivative of (3.7), we obtain

$$\frac{De}{Dt} = \boldsymbol{u} \cdot \nabla(h_0 - p_0/\rho_0) + s'\boldsymbol{u} \cdot \nabla T_0 + T_0 \frac{Ds'}{Dt}. \tag{3.8}$$

The multiplication of (3.8) by ρ_0 yields

$$\rho_0 \frac{De}{Dt} = \nabla \cdot \{\rho_0(h_0 - p_0/\rho_0)\boldsymbol{u}\} + \rho_0 s'\boldsymbol{u} \cdot \nabla T_0. \tag{3.9}$$

Here (2.21) and (2.23) have been used. However, using (2.18) and (2.21), we can write

$$\rho_0 \frac{De}{Dt} = \rho_0 \frac{\partial e}{\partial t} + \rho_0 \boldsymbol{u} \cdot \nabla e = \frac{\partial}{\partial t}(\rho_0 e) + \nabla \cdot (\rho_0 e \boldsymbol{u}). \tag{3.10}$$

Thus (3.9) can be rewritten in the following form:

$$\frac{\partial}{\partial t}(\rho_0 e) + \nabla \cdot (\rho_0 e \boldsymbol{u}) = \nabla \cdot \{\rho_0(h_0 - p_0/\rho_0)\boldsymbol{u}\} + \rho_0 s'\boldsymbol{u} \cdot \nabla T_0. \tag{3.11}$$

In light of (3.7), furthermore, we observe that (3.11) reduces to

$$\frac{\partial}{\partial t}(\rho_0 e) + \nabla \cdot (\rho_0 T_0 s' \boldsymbol{u}) = \rho_0 s'\boldsymbol{u} \cdot \nabla T_0. \tag{3.12}$$

Finally, integrating (3.12) over the domain Ω containing the fluid, we get

$$\frac{d}{dt}\int_\Omega \rho_0 e \, dV = \int_\Omega \rho_0 s'\boldsymbol{u} \cdot \nabla T_0 \, dV. \tag{3.13}$$

Here we have assumed that the normal component of $\boldsymbol{u}$ vanishes on the boundary of Ω. This is the equation for the rate of change of the internal energy of the fluid.

3.2. Potential energy

The equation for the rate of change of the potential energy of the fluid is, on account of the approximation (2.20), quite simple under the anelastic approximation:

$$\frac{d}{dt}\int_\Omega \rho_0 g z \, dV = 0. \tag{3.14}$$

This equation states that the potential energy of the fluid is invariable.

3.3. Kinetic energy

In order to find the equation for the rate of change of the kinetic energy of the fluid, we first rewrite the equation of motion (2.12), using (3.6), as follows:

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla(p'/\rho_0) - s'\nabla T_0. \tag{3.15}$$

Taking the inner product of this equation with $\rho_0\boldsymbol{u}$, we get, after some manipulation,

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho_0|\boldsymbol{u}|^2) + \nabla\cdot\left\{\rho_0(\tfrac{1}{2}|\boldsymbol{u}|^2 + p'/\rho_0)\boldsymbol{u}\right\} = -\rho_0 s'\boldsymbol{u}\cdot\nabla T_0. \tag{3.16}$$

When the normal component of $\boldsymbol{u}$ vanishes on the boundary of Ω, (3.16) yields

$$\frac{d}{dt}\int_\Omega \tfrac{1}{2}\rho_0|\boldsymbol{u}|^2 dV = -\int_\Omega \rho_0 s'\boldsymbol{u}\cdot\nabla T_0\, dV. \tag{3.17}$$

This is the desired equation for the rate of change of the kinetic energy of the fluid: the term on the right-hand side represents the work done by the buoyancy force.

3.4. Total energy

Adding all the energy equations (3.13), (3.14), and (3.17), we obtain

$$\frac{d}{dt}\int_\Omega \rho_0\left(\tfrac{1}{2}|\boldsymbol{u}|^2 + gz + e\right) dV = 0. \tag{3.18}$$

This equation shows that the total energy of the fluid is conserved. We see, therefore, that the anelastic approximation is consistent with the conservation law of energy.

We also recognize, comparing (3.13) and (3.17), that the work done by the buoyancy force in the anelastic approximation corresponds to the conversion between kinetic and internal energy; this conclusion is the same as that arrived at by Maruyama (2014) as regards the buoyancy force in the Boussinesq approximation.

4. Applicability of the anelastic approximation

In § 2, the anelastic approximation was derived on the assumptions (2.10) and (2.19); the objective of this section is to find out under what conditions these two assumptions are justifiable. The setting and the notation are the same as in the preceding sections.

4.1. Conditions for the applicability of the anelastic approximation

To begin with, we focus attention on (2.10). With the help of (2.13), T' in (2.10) can be expressed, to the first order of h' and s', as follows:

$$\begin{aligned} T' &= (\partial T/\partial h)_s|_{(h_0,s_0)}h' + (\partial T/\partial s)_h|_{(h_0,s_0)}s' \\ &= (\beta_0 T_0/c_{p0})h' + \{T_0(1-\beta_0 T_0)/c_{p0}\}\, s'. \end{aligned} \tag{4.1}$$

Let $\Delta h'$ and $\Delta s'$ denote, respectively, the characteristic scales of h' and s'. Then, if H denotes the vertical extent of the domain Ω containing the fluid, we have

$$|T'/T_0| = O\{(\Gamma_0 H/T_0)(\Delta h'/gH)\} + O\{(\Gamma_0 H/T_0)(T_0\Delta s'/gH)\} + O(\Delta s'/c_{p0}), \tag{4.2}$$

where $\Gamma_0 = \beta_0 T_0 g / c_{p0}$ is the adiabatic lapse rate. Now, after Ogura & Phillips (1962), we assume that the following condition applies:

$$\Gamma_0 H / T_0 \leq O(1). \tag{4.3}$$

Then, it is seen from (4.2) that (2.10) holds under the following three conditions:

$$\Delta h' / gH \ll 1, \tag{4.4}$$

$$T_0 \Delta s' / gH \ll 1, \tag{4.5}$$

$$\Delta s' / c_{p0} \ll 1. \tag{4.6}$$

We next note the following thermodynamic relations:

$$(\partial \rho / \partial h)_s = \rho / a^2, \quad (\partial \rho / \partial s)_h = -\rho T (\beta / c_p + 1/a^2), \tag{4.7}$$

in which a is the speed of sound. These relations enable ρ' in (2.19) to be expressed, to the first order of h' and s', as follows:

$$\begin{aligned} \rho' &= (\partial \rho / \partial h)_s |_{(h_0, s_0)} h' + (\partial \rho / \partial s)_h |_{(h_0, s_0)} s' \\ &= (\rho_0 / a_0^2) h' - \rho_0 T_0 (\beta_0 / c_{p0} + 1/a_0^2) s', \end{aligned} \tag{4.8}$$

where we have introduced the notation

$$a_0 = a(h_0, s_0). \tag{4.9}$$

In view of (4.8), we can write

$$\begin{aligned} |\rho' / \rho_0| = O\{(gH/a_0^2)(\Delta h' / gH)\} + O\{(gH/a_0^2)(T_0 \Delta s' / gH)\} \\ + O\{(\Gamma_0 H / T_0)(T_0 \Delta s' / gH)\}. \end{aligned} \tag{4.10}$$

Accordingly, (2.19) is satisfied when the condition

$$(gH)^{1/2} / a_0 \leq O(1) \tag{4.11}$$

applies together with (4.3), (4.4), and (4.5).

On the other hand, the following inequality can be obtained from (2.7):

$$|\nabla h'| \leq |\partial \boldsymbol{u} / \partial t| + |(\boldsymbol{u} \cdot \nabla) \boldsymbol{u}| + |T \nabla s'|. \tag{4.12}$$

Denoting by L the length scale characteristic of the motion of the fluid, we can write

$$|\nabla h'| = O(\Delta h' / L), \quad |\nabla s'| = O(\Delta s' / L). \tag{4.13}$$

Also, if U denotes the velocity scale characteristic of the motion, we obtain

$$|\partial \boldsymbol{u} / \partial t| = O(U / \tau), \quad |(\boldsymbol{u} \cdot \nabla) \boldsymbol{u}| = O(U^2 / L), \tag{4.14}$$

with τ being the time scale characteristic of the motion. Now we assume that c_1 and c_2 in (2.6) can be chosen so that the following condition applies:

$$T/T_0 \leq O(1). \tag{4.15}$$

Then it follows from (4.12), (4.13), and (4.14) that, when the conditions

$$U/(gH)^{1/2} \ll 1, \quad (L/\tau)/(gH)^{1/2} \ll 1 \tag{4.16}$$

hold together with (4.5), the condition (4.4) is satisfied.

The above results lead us to the following conclusion: the anelastic approximation is applicable under the conditions (4.3), (4.5), (4.6), (4.11), (4.15), and (4.16).

4.2. Relation between the conditions (4.5) and (4.6)

Of the conditions for the applicability of the anelastic approximation, (4.5) and (4.6) are both conditions on $\Delta s'$; the relation between them needs to be clarified.

To this end, it is sufficient to note that (4.5) can be written as

$$\Delta s'/c_{p0} \ll (\Gamma_0 H/T_0)/(\beta_0 T_0). \tag{4.17}$$

Hence, when $(\Gamma_0 H/T_0)/(\beta_0 T_0) \geq O(1)$, (4.5) is superfluous; (4.5) is satisfied so long as (4.6) holds. When $(\Gamma_0 H/T_0)/(\beta_0 T_0) \ll 1$, in contrast, (4.5) places a far more stringent constraint on $\Delta s'$ than (4.6); (4.6) becomes redundant in this case.

5. Summary and discussion

The anelastic approximation has been reconstructed in such a manner that it can be applied to any kind of fluid. As a result, it has become apparent that the work done by the buoyancy force in the approximation corresponds to the conversion between kinetic and internal energy. The conditions for the applicability of the approximation has also been elucidated: they are given by (4.3), (4.5), (4.6), (4.11), (4.15), and (4.16).

5.1. Density of a fluid under the anelastic approximation

In § 3.3, the equation of motion (2.12) was rewritten in the form (3.15). As explained below, some additional manipulation brings (3.15) into yet another interesting form.

We begin by rewriting the first term on the right-hand side of (3.15) as follows:

$$-\nabla(p'/\rho_0) = -\nabla p'/\rho_0 + (p'/\rho_0)(\nabla\rho_0/\rho_0). \tag{5.1}$$

However, from (2.6) and (2.18), the following expression for $\nabla\rho_0$ is obtained:

$$\nabla\rho_0 = (\partial\rho/\partial h)_s|_{(h_0,s_0)}\nabla h_0 + (\partial\rho/\partial s)_h|_{(h_0,s_0)}\nabla s_0 = -(\partial\rho/\partial h)_s|_{(h_0,s_0)} g\boldsymbol{k}. \tag{5.2}$$

Thus, recalling that (3.6) gives $p'/\rho_0 = h' - T_0 s'$, we have

$$-\nabla(p'/\rho_0) = -\nabla p'/\rho_0 - \{(\partial\rho/\partial h)_s|_{(h_0,s_0)} h' - T_0(\partial\rho/\partial h)_s|_{(h_0,s_0)} s'\}(g/\rho_0)\boldsymbol{k}. \tag{5.3}$$

The second term on the right-hand side of (3.15) can also be rewritten as follows:

$$-s'\nabla T_0 = \rho_0(\partial T/\partial h)_s|_{(h_0,s_0)}s'(g/\rho_0)\boldsymbol{k}. \tag{5.4}$$

The substitution of (5.3) and (5.4) into (3.15) yields

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla p'/\rho_0 - \left[(\partial\rho/\partial h)_s|_{(h_0,s_0)}h' - \{\partial(\rho T)/\partial h\}_s|_{(h_0,s_0)}s'\right](g/\rho_0)\boldsymbol{k}. \tag{5.5}$$

However, we observe from (3.5) that

$$\{\partial(\rho T)/\partial h\}_s = -(\partial\rho/\partial s)_h. \tag{5.6}$$

This enables us to rewrite (5.5) as follows:

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla p'/\rho_0 - \left\{(\partial\rho/\partial h)_s|_{(h_0,s_0)}h' + (\partial\rho/\partial s)_h|_{(h_0,s_0)}s'\right\}(g/\rho_0)\boldsymbol{k}. \tag{5.7}$$

As a result, in view of (4.8), we obtain the equation of motion expressed in the form

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla p'/\rho_0 - (\rho' g/\rho_0)\boldsymbol{k}. \tag{5.8}$$

This expression contains ρ' defined by (4.8) in addition to ρ_0. However, it should be emphasized that, under the anelastic approximation, the density of a fluid is not given by $\rho_0 + \rho'$; as is evident from the discussion in the preceding sections, it is given by ρ_0.

5.2. Anelastic approximation for a shallow layer of fluid

Let us consider again the fluid of § 2. Under the anelastic approximation, its density $\rho = \rho_0$ varies with height: using (2.6), (2.18), and (4.7), we can write

$$\nabla\rho_0 = (\partial\rho/\partial h)_s|_{(h_0,s_0)}\nabla h_0 + (\partial\rho/\partial s)_h|_{(h_0,s_0)}\nabla s_0 = -(\rho_0 g/a_0^2)\boldsymbol{k}. \tag{5.9}$$

However, ρ_0 may be regarded as constant when the following condition is fulfilled:

$$\Delta\rho_0/\rho_0 \ll 1. \tag{5.10}$$

Here $\Delta\rho_0$ denotes the variation scale of ρ_0 over the vertical extent H of the domain Ω containing the fluid. Since it is reasonable to put

$$\Delta\rho_0 = |\nabla\rho_0|H = \rho_0 gH/a_0^2, \tag{5.11}$$

we realize that ρ_0 may be regarded as constant under the condition

$$(gH)^{1/2}/a_0 \ll 1. \tag{5.12}$$

In this case, the equation of continuity $\partial\rho/\partial t + \nabla\cdot(\rho\boldsymbol{u}) = 0$ reduces to

$$\nabla\cdot\boldsymbol{u} = 0. \tag{5.13}$$

This equation is the same as that applying under the Boussinesq approximation.

As for the equation of motion (3.15), it becomes, when ρ_0 is regarded as constant,

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla p'/\rho_0 + (\beta_0 T_0 s' g/c_{p0})\boldsymbol{k}. \tag{5.14}$$

Here (2.16) has been used. On the other hand, from (4.1), we have

$$T'/T_0 = (\Gamma_0 H/T_0)(h'/gH - T_0 s'/gH) + s'/c_{p0}. \tag{5.15}$$

Suppose now that, in addition to (5.12), the following condition applies:

$$\Gamma_0 H/T_0 \ll 1. \tag{5.16}$$

Then it seems reasonable to approximate (5.15) as follows:

$$T'/T_0 = s'/c_{p0}. \tag{5.17}$$

Substituting this into (5.14), we get

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla p'/\rho_0 + \beta_0 T' g\boldsymbol{k}. \tag{5.18}$$

We recognize that this equation of motion has the same form as that obtained under the Boussinesq approximation. We should note, however, that T' in (5.18) denotes the deviation of temperature from an isentropic distribution; the corresponding quantity in the Boussinesq approximation is that from an isothermal distribution.

More importantly, however, it should be noted that (5.15) can be written as

$$T'/T_0 = (\Gamma_0 H/T_0)(h'/gH - T_0 s'/gH) + \{(\Gamma_0 H/T_0)/(\beta_0 T_0)\}\,(T_0 s'/gH). \tag{5.19}$$

Hence, under the condition (5.16), we obtain, unless $(\Gamma_0 H/T_0)/(\beta_0 T_0) \geq O(1)$,

$$T'/T_0 = 0. \tag{5.20}$$

It is seen, therefore, that the argument leading to (5.18) does not apply in general.

From these results, we can conclude as follows: the Boussinesq approximation is not a limiting case of the anelastic approximation.

Appendix. Anelastic approximation for an ideal gas

We consider the same physical situation as that in § 2, but the fluid is now assumed to be an ideal gas: the specific heat at constant pressure c_p is taken to be a constant.

To begin with, we introduce a function $\theta(s)$ of the specific entropy s: it is called the potential temperature, and is related to s by the formula

$$d\theta/ds = \theta/c_p. \tag{A.1}$$

In view of the decomposition (2.5), θ can be expressed as follows:

$$\theta = \theta_0 + \theta', \quad \theta_0 = \theta(s_0). \tag{A.2}$$

Here θ' is given, to the first order of s', as follows:

$$\theta' = (d\theta/ds)|_{s=s_0} s' = (\theta_0/c_p)s'. \tag{A.3}$$

We next introduce the function Π, which is called the Exner function, defined by

$$\Pi = T/\theta. \tag{A.4}$$

However, since the fluid is an ideal gas, the specific enthalpy h is written as

$$h = c_p T; \tag{A.5}$$

this enables us to write

$$\Pi = h/c_p\theta. \tag{A.6}$$

Now, Π can also be decomposed as follows:

$$\Pi = \Pi_0 + \Pi', \quad \Pi_0 = \Pi(h_0, s_0), \tag{A.7}$$

where, to the first order of h' and s', Π' is given by

$$\Pi' = (\partial\Pi/\partial h)_s|_{(h_0,s_0)} h' + (\partial\Pi/\partial s)_h|_{(h_0,s_0)} s' = h'/c_p\theta_0 - (h_0/c_p\theta_0^2)(\theta_0/c_p)s'. \tag{A.8}$$

On the other hand, we observe from (A.5) that

$$T_0 = h_0/c_p. \tag{A.9}$$

Hence, taking account of (2.6), we can write

$$\nabla T_0 = \nabla h_0/c_p = -(g/c_p)\boldsymbol{k}. \tag{A.10}$$

Furthermore, from (A.3), (A.8), and (A.9), we obtain

$$s' = c_p(\theta'/\theta_0), \quad h' - T_0 s' = c_p\theta_0\Pi'. \tag{A.11}$$

The substitution of (A.10) and (A.11) into (2.12) yields, since c_p and θ_0 are constant,

$$\frac{D\boldsymbol{u}}{Dt} = -c_p\theta_0\nabla\Pi' + (g\theta'/\theta_0)\boldsymbol{k}. \tag{A.12}$$

This is the equation of motion derived by Ogura & Phillips (1962).

References

[1] Landau, L. D. & Lifshitz, E. M. 1987 *Fluid Mechanics,* 2nd ed. Butterworth-Heinemann.

[2] Maruyama, K. 2014 Energetics of a fluid under the Boussinesq approximation. `arXiv:1405.1921 [physics.flu-dyn]`.

[3] Ogura, Y. & Phillips, N. A. 1962 Scale analysis of deep and shallow convection in the atmosphere. *J. Atmos. Sci.* **19**, 173–179.