Soundproof approximation for a deep layer of ideal gas

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Abstract

This paper deduces the conditions necessary for sound waves to be ignorable in a deep fluid layer. On the basis of these conditions, it is shown that there exists only one physically meaningful soundproof approximation for a deep layer of ideal gas: the original anelastic approximation of Ogura & Phillips (1962).

1. Introduction

The anelastic approximation is an approximation devised by Ogura & Phillips (1962) in order to study the deep convection of an ideal gas. It is a *soundproof* approximation: it excludes sound waves from the solutions of the governing equations, though it takes into account the compressibility of a gas by allowing the density to vary with height.

The anelastic approximation assumes an isentropic state as its base state. Maruyama (2021b), by assuming an isothermal state as the base state, constructed a variant of the anelastic approximation in such a way that it can be applied to any kind of fluid. This isothermal anelastic approximation, however, turned out to be inapplicable to an ideal gas, although it also is a soundproof approximation.

The purpose of this paper is to show that the only physically meaningful soundproof approximation applicable to a deep layer of ideal gas is the anelastic approximation of Ogura & Phillips. This can be done on the basis of the conditions necessary for sound waves to be ignorable in a deep fluid layer: they are elucidated in the next section.

2. Conditions necessary for sound waves to be ignorable in a deep fluid layer

We consider the motion in a uniform gravitational field of a layer of inviscid fluid in a fixed finite domain Ω. In the domain, the z-axis is taken vertically upwards: the unit vector in the positive z-direction is denoted by $\boldsymbol{k}$.

Let s and p denote, respectively, the specific entropy and the pressure of the fluid: all thermodynamic quantities, such as the density ρ and the temperature T of the fluid, are regarded as known functions of s and p. Now, let us decompose s and p as follows:

$$s = s_0(z) + s', \quad p = p_0(z) + p'. \tag{2.1}$$

Here $s_0(z)$ is an arbitrary function of z, and $p_0(z)$ is a function satisfying the equation

$$dp_0/dz = -\rho_0 g, \tag{2.2}$$

where $\rho_0 = \rho(s_0, p_0)$, and g is the acceleration due to gravity. Then, when the velocity of the fluid is denoted by $\boldsymbol{u}$, the equation of motion takes the form

$$(\rho_0 + \rho')D\boldsymbol{u}/Dt = -\nabla p' - \rho' g\boldsymbol{k}, \tag{2.3}$$

in which $\rho' = \rho - \rho_0$, and D/Dt stands for the material derivative. On the other hand, the equation of continuity can be written in the form

$$\partial\rho'/\partial t + \nabla\cdot\big\{(\rho_0 + \rho')\boldsymbol{u}\big\} = 0. \tag{2.4}$$

Also, when the conduction of heat is neglected, we have the following equation:

$$D(s_0 + s')/Dt = 0. \tag{2.5}$$

2.1. Assumption on the depth of the layer

Let H denote the depth of the fluid layer, and let Δp_0 be defined by

$$\Delta p_0 = |dp_0/dz|H = \rho_0 g H. \tag{2.6}$$

We may regard Δp_0 as a measure of the variation of p_0. In terms of Δp_0, we can define a measure $\Delta\rho_0$ of the variation of ρ_0 caused by that of p_0 as follows:

$$\Delta\rho_0 = (\partial\rho/\partial p)_s|_{(s_0,p_0)}\Delta p_0. \tag{2.7}$$

Note here that the following thermodynamic relations hold:

$$(\partial\rho/\partial s)_p = -\rho\beta T/c_p, \quad (\partial\rho/\partial p)_s = 1/a^2, \tag{2.8}$$

in which β stands for the thermal expansion coefficient, c_p the specific heat at constant pressure, and a the speed of sound. Thus $\Delta\rho_0$ can be written as

$$\Delta\rho_0 = \rho_0 g H/a_0^2, \tag{2.9}$$

where $a_0 = a(s_0, p_0)$. In the following, the fluid layer is assumed to be *deep* in the sense that $\Delta\rho_0$ is of the same order of magnitude as ρ_0. That is to say, we assume that

$$gH/a_0^2 = O(1). \tag{2.10}$$

2.2. Conditions for sound waves to be ignorable

Let us proceed to formulate the conditions necessary for sound waves to be ignorable in the fluid layer. Using (2.8), we have, to the first order of s' and p',

$$\rho' = -(\rho_0 \beta_0 T_0 / c_{p0}) s' + (1/a_0^2) p', \tag{2.11}$$

in which $\beta_0 = \beta(s_0, p_0)$, $T_0 = T(s_0, p_0)$, and $c_{p0} = c_p(s_0, p_0)$. The second term on the right-hand side represents the fluctuation in density due to that in pressure; hence, for sound waves to be ignorable in the fluid layer, this term needs to be negligible.

Now, let $\Delta s'$ and $\Delta p'$ denote respectively the characteristic scales of s' and p'. Then the following estimate can be obtained from (2.11):

$$|\rho'/\rho_0| = O\left\{\beta_0 T_0 (\Delta s'/c_{p0})\right\} + O\left\{(gH/a_0^2)(\Delta p'/\rho_0 gH)\right\}. \tag{2.12}$$

From this estimate and (2.10), we observe that the second term on the right-hand side of (2.11) is negligible, in comparison with ρ_0, when the following condition applies:

$$\Delta p'/\rho_0 gH \ll 1. \tag{2.13}$$

This gives a condition necessary for sound waves to be ignorable in the fluid layer.

When the second term on the right-hand side of (2.11) is negligible, however, the first term must also be negligible. This can be seen from the following argument.

Suppose that the second term on the right-hand side of (2.11) is negligible. Then the total mass M of the fluid layer is given by

$$M = \int_\Omega \rho_0 dV - \int_\Omega (\rho_0 \beta_0 T_0 / c_{p0}) s' dV. \tag{2.14}$$

The first term on the right is constant, but the second term varies when the fluid layer is heated or cooled. Accordingly, the conservation law of mass requires that the second term be absent; the first term on the right-hand side of (2.11) must be negligible.

As a result, we see from (2.12) that the condition

$$\beta_0 T_0 (\Delta s'/c_{p0}) \ll 1 \tag{2.15}$$

is necessary, in addition to (2.13), for sound waves to be ignorable in the fluid layer.

2.3. A requirement on the basic distribution of entropy

If we introduce the scale Δs_0 characterizing the variation of s_0, the condition (2.15) can be rewritten as

$$\beta_0 T_0 (\Delta s_0 / c_{p0})(\Delta s'/\Delta s_0) \ll 1. \tag{2.16}$$

Here $\Delta s'/\Delta s_0$, the magnitude of $\Delta s'$ relative to Δs_0, can be estimated as follows.

Let us now consider an arbitrary fluid particle in the fluid layer: its position and its z-coordinate are denoted, respectively, by $\boldsymbol{r}(t)$ and $z(t)$. Then, from (2.5), we have

$$s'[\boldsymbol{r}(t), t] = s'(\boldsymbol{r}_0, t_0) + \{s_0(z_0) - s_0[z(t)]\}, \tag{2.17}$$

in which $\boldsymbol{r}_0$ and z_0 are, respectively, the position and the z-coordinate of the particle at an initial instant $t = t_0$. Since the particle can move over the whole depth of the layer, the second term on the right is expected to be of the same order of magnitude as Δs_0; the first term may be large or small depending on the initial distribution of s'. Thus, in view of the fact that the particle is arbitrary, we can get from (2.17) the estimate

$$\Delta s'/\Delta s_0 \geq O(1). \tag{2.18}$$

As a result, the following condition on the variation scale of s_0 is obtained:

$$\beta_0 T_0(\Delta s_0/c_{p0}) \ll 1. \tag{2.19}$$

This provides a requirement for the existence of a soundproof approximation for a deep fluid layer: the basic distribution of entropy in the layer must satisfy the requirement in order for a physically meaningful soundproof approximation for the layer to exist.

3. Soundproof approximation for a deep layer of ideal gas

We next consider, in particular, a deep layer of ideal gas: its specific heat at constant pressure c_p is taken to be constant. Applying the results obtained so far, we can prove that essentially only one soundproof approximation exists for this layer of ideal gas.

3.1. Basic distribution of entropy

First of all, it should be noted that the following relation holds for an ideal gas:

$$\beta T = 1. \tag{3.1}$$

Hence, applied to the layer of ideal gas, the requirement (2.19) reduces to

$$\Delta s_0/c_p \ll 1. \tag{3.2}$$

We discuss below the physical significance of (3.2).

To this end, it is convenient to introduce a function $\theta(s)$ of the specific entropy s: it is known as the potential temperature, and is related to s by the formula

$$d\theta/ds = \theta/c_p. \tag{3.3}$$

Now, let θ_0 denote $\theta(s_0)$: θ_0 represents, in place of s_0, the basic distribution of entropy in the layer. Let us also define, in terms of Δs_0, the following quantity:

$$\Delta\theta_0 = (d\theta/ds)|_{s=s_0}\Delta s_0 = (\theta_0/c_p)\Delta s_0. \tag{3.4}$$

This provides a measure of the variation of θ_0. Then (3.2) can be written as

$$\Delta\theta_0/\theta_0 \ll 1. \tag{3.5}$$

From this expression, we see that the basic distribution of entropy in the layer must be virtually uniform for a soundproof approximation for the layer to exist:

$$s_0 = \text{constant}. \tag{3.6}$$

3.2. Conditions on temperature and density

Let us next apply (2.13) and (2.15) to the layer of ideal gas: the result is

$$\Delta p'/\rho_0 g H \ll 1, \quad \Delta s'/c_p \ll 1. \tag{3.7}$$

We consider below the temperature and the density of the gas under these conditions.

To begin with, let T_0 and T' be defined as follows:

$$T_0 = T(s_0, p_0), \quad T' = T - T_0. \tag{3.8}$$

We note here the following thermodynamic relations:

$$(\partial T/\partial s)_p = T/c_p, \quad (\partial T/\partial p)_s = \beta T/\rho c_p. \tag{3.9}$$

These relations and (3.1) enable us to get, to the first order of s' and p',

$$T' = (T_0/c_p)s' + (1/\rho_0 c_p)p'. \tag{3.10}$$

From this expression for T', the following estimate is obtained:

$$|T'/T_0| = O(\Delta s'/c_p) + O\left\{(\Gamma H/T_0)(\Delta p'/\rho_0 g H)\right\}, \tag{3.11}$$

where $\Gamma = g/c_p$ is the adiabatic lapse rate. We also have, for an ideal gas,

$$\Gamma H/T_0 = (\gamma - 1)gH/a_0^2. \tag{3.12}$$

Here γ is the ratio of specific heats, and $\gamma - 1 < 1$ for an ideal gas. Thus, considering (2.10), we observe that the following condition applies under the conditions (3.7):

$$|T'/T_0| \ll 1. \tag{3.13}$$

As for the density of the gas, the following estimate is obtained from (2.12):

$$|\rho'/\rho_0| = O(\Delta s'/c_p) + O\left\{(gH/a_0^2)(\Delta p'/\rho_0 g H)\right\}. \tag{3.14}$$

Hence, under the conditions (3.7), we have

$$|\rho'/\rho_0| \ll 1. \tag{3.15}$$

3.3. Anelastic approximation of Ogura & Phillips (1962)

Now, suppose that a soundproof approximation has been found for the layer of ideal gas. Then, in view of the above argument, the approximation, so long as it is physically meaningful, must have the properties (3.6), (3.13), and (3.15). As shown by Maruyama (2021a), however, an approximation with the properties is the anelastic approximation of Ogura & Phillips (1962). It therefore follows that the soundproof approximation is, so long as it is physically meaningful, this anelastic approximation.

From this result, we recognize that the anelastic approximation of Ogura & Phillips is the only physically meaningful soundproof approximation for a deep layer of ideal gas.

4. Conclusion

For a physically meaningful soundproof approximation to exit for a fluid layer, sound waves must be ignorable in the layer. In the present study, we considered a fluid layer which is deep in the sense that the condition (2.10) holds, and obtained the conditions (2.13) and (2.15) necessary for sound waves to be ignorable in the layer; as regards the basic distribution of entropy in the layer, we further found the requirement (2.19).

We then applied the results to a deep layer of ideal gas, and arrived at the conclusion that there exists one and only one physically meaningful soundproof approximation for a deep layer of ideal gas: the anelastic approximation of Ogura & Phillips (1962).

References

[1] Maruyama, K. 2021a Anelastic approximation. `https://doi.org/10.5281/zenodo.4596697`.

[2] Maruyama, K. 2021b Isothermal anelastic approximation. `https://doi.org/10.5281/zenodo.4656529`.

[3] Ogura, Y. & Phillips, N. A. 1962 Scale analysis of deep and shallow convection in the atmosphere. *J. Atmos. Sci.* **19**, 173–179.