Extension of the anelastic approximation to a two-component fluid

Kiyoshi Maruyama

Department of Earth and Ocean Sciences, National Defense Academy,

Yokosuka, Kanagawa 239-8686, Japan

August 11, 2022

Abstract

This paper extends the anelastic approximation to a two-component fluid. Under the extended approximation, a force arises owing to changes in concentration of one component; it is demonstrated that the work done by this force corresponds to the conversion between kinetic and internal energy. Furthermore, the conditions under which the extended approximation is applicable are formulated.

1. Introduction

The anelastic approximation is an approximation devised by Ogura & Phillips (1962) in order to study convection in a deep layer of ideal gas. It considers the compressibility of a gas by allowing the density of the gas to vary with height; it excludes, nonetheless, sound waves from the solutions of the system of governing equations.

The original anelastic approximation of Ogura & Phillips, however, is applicable only to ideal gases. In order to overcome this drawback, Maruyama (2021) reconstructed the approximation in such a manner that it can be applied to any kind of fluid.

The object of this paper is to extend the anelastic approximation, on the basis of the reconstruction by Maruyama, to a general two-component fluid. The energetics and the applicability of the extended approximation are also discussed.

2. Extended anelastic approximation

We consider the motion of an inviscid fluid in a uniform gravitational field. The fluid consists of two components, $\mathcal{A}$ and $\mathcal{B}$, and is contained in a fixed finite domain Ω. In the domain, the z-axis is taken vertically upwards: the unit vector in the positive z-direction is denoted by $\boldsymbol{k}$. We also denote by c the concentration of component $\mathcal{A}$: the mass of $\mathcal{A}$ in a unit volume of the fluid is given by ρc, with ρ being the density of the fluid.

2.1. Equation of motion

We first express the equation of motion for the fluid as follows:

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla p/\rho - g\boldsymbol{k}, \tag{2.1}$$

where D/Dt stands for the material derivative, and $\boldsymbol{u}$ denotes the velocity of the fluid; p is the pressure of the fluid, and g the acceleration due to gravity.

Let us next introduce the specific enthalpy h of the fluid: it satisfies the relation

$$dh = Tds + vdp + \mu dc, \tag{2.2}$$

in which T and s are the temperature and the specific entropy of the fluid, respectively; $v = 1/\rho$ denotes the specific volume of the fluid, and μ is the chemical potential of the fluid (see Landau & Lifshitz 1987, § 58). In the following, all thermodynamic quantities are regarded as known functions of h, s and c.

The relation (2.2) enables us to rewrite $\nabla p/\rho$ as follows:

$$\nabla p/\rho = \nabla h - T\nabla s - \mu\nabla c. \tag{2.3}$$

Accordingly, the equation of motion (2.1) can be expressed in the form

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla h + T\nabla s + \mu\nabla c - g\boldsymbol{k}. \tag{2.4}$$

Now, suppose that h, s, and c are decomposed as follows:

$$h = h_0 + h', \quad s = s_0 + s', \quad c = c_0 + c'. \tag{2.5}$$

Here h_0, s_0, and c_0 are defined by

$$h_0 = -gz + \alpha_1, \quad s_0 = \alpha_2, \quad c_0 = \alpha_3, \tag{2.6}$$

with α_1, α_2, and α_3 being constants. Then (2.4) is written as

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla h' + T\nabla s' + \mu\nabla c'. \tag{2.7}$$

On the other hand, in view of the decomposition (2.5), we can write

$$T = T_0 + T', \quad \mu = \mu_0 + \mu', \tag{2.8}$$

where T_0 and μ_0 are given by

$$T_0 = T(h_0, s_0, c_0), \quad \mu_0 = \mu(h_0, s_0, c_0). \tag{2.9}$$

We now introduce the following assumptions:

$$|T'/T_0| \ll 1, \quad |\mu'/\mu_0| \ll 1. \tag{2.10}$$

These assumptions allow us to approximate (2.7) as

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla h' + T_0 \nabla s' + \mu_0 \nabla c'. \tag{2.11}$$

Since $T_0 \nabla s' = \nabla(T_0 s') - s' \nabla T_0$ and $\mu_0 \nabla c' = \nabla(\mu_0 c') - c' \nabla \mu_0$, we finally obtain

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla(h' - T_0 s' - \mu_0 c') - s' \nabla T_0 - c' \nabla \mu_0. \tag{2.12}$$

This is the equation of motion under the extended anelastic approximation.

The second term on the right-hand side of (2.12) represents the buoyancy force that arises owing to changes in entropy. Here we note the thermodynamic relations

$$\begin{aligned}
(\partial T/\partial h)_{s,c} &= \beta T/c_p, \\
(\partial T/\partial s)_{h,c} &= T(1 - \beta T)/c_p, \\
(\partial T/\partial c)_{h,s} &= T\{(\partial \mu/\partial T)_{p,c} - \beta \mu\}/c_p,
\end{aligned} \tag{2.13}$$

where $\beta = v^{-1}(\partial v/\partial T)_{p,c}$ is the thermal expansion coefficient, and c_p the specific heat at constant pressure. Hence, considering (2.6) and (2.9), we can write

$$\nabla T_0 = (\partial T/\partial h)_{s,c}|_{(h_0,s_0,c_0)} \nabla h_0 = -(\beta_0 T_0 g/c_{p0})\boldsymbol{k}, \tag{2.14}$$

in which the following notation has been introduced:

$$\beta_0 = \beta(h_0, s_0, c_0), \quad c_{p0} = c_p(h_0, s_0, c_0). \tag{2.15}$$

The second term on the right-hand side of (2.12) can therefore be rewritten as

$$-s' \nabla T_0 = (\beta_0 T_0 s' g/c_{p0})\boldsymbol{k}. \tag{2.16}$$

The third term on the right-hand side of (2.12) represents the buoyancy force due to changes in concentration. This term can also be put into a different form. To this end, we first observe that the following thermodynamic relations hold:

$$\begin{aligned}
(\partial \mu/\partial h)_{s,c} &= -\hat{\beta}_c \\
&= \rho(\partial \mu/\partial p)_{T,c} + (\beta T/c_p)(\partial \mu/\partial T)_{p,c}, \\
(\partial \mu/\partial s)_{h,c} &= (T/c_p)(1 - \beta T)(\partial \mu/\partial T)_{p,c} - \rho T(\partial \mu/\partial p)_{T,c}, \\
(\partial \mu/\partial c)_{h,s} &= (\partial \mu/\partial c)_{T,p} - \rho \mu(\partial \mu/\partial p)_{T,c} \\
&\quad + (T/c_p)\{(\partial \mu/\partial T)_{p,c} - \beta \mu\}(\partial \mu/\partial T)_{p,c},
\end{aligned} \tag{2.17}$$

where $\hat{\beta}_c = \rho^{-1}(\partial \rho/\partial c)_{p,s}$. Thus, similarly to ∇T_0, $\nabla \mu_0$ can be written as

$$\nabla \mu_0 = (\partial \mu/\partial h)_{s,c}|_{(h_0,s_0,c_0)} \nabla h_0 = \hat{\beta}_{c0} g \boldsymbol{k}, \tag{2.18}$$

where $\hat{\beta}_{c0} = \hat{\beta}_c(h_0, s_0, c_0)$. We obtain, as a result, the expression

$$-c' \nabla \mu_0 = -\hat{\beta}_{c0} c' g \boldsymbol{k}. \tag{2.19}$$

2.2. Equation of continuity

The density ρ of the fluid, similarly to T and μ, can be expressed as follows:

$$\rho = \rho_0 + \rho', \tag{2.20}$$

where ρ_0 is given by

$$\rho_0 = \rho(h_0, s_0, c_0). \tag{2.21}$$

Let us now introduce the following assumption:

$$|\rho'/\rho_0| \ll 1. \tag{2.22}$$

This assumption allows ρ to be approximated as follows:

$$\rho = \rho_0. \tag{2.23}$$

Substituting (2.23) into the equation of continuity $\partial\rho/\partial t + \nabla \cdot (\rho \boldsymbol{u}) = 0$, we get

$$\nabla \cdot (\rho_0 \boldsymbol{u}) = 0. \tag{2.24}$$

This is the equation of continuity under the extended anelastic approximation.

2.3. Concentration equation

In the absence of diffusion, the equation for the rate of change of the concentration c of component $\mathcal{A}$ is given by (see Landau & Lifshitz 1987, § 58)

$$\rho \frac{Dc}{Dt} = 0. \tag{2.25}$$

In view of (2.5) and (2.23), we can approximate this equation as follows:

$$\rho_0 \frac{Dc'}{Dt} = 0. \tag{2.26}$$

2.4. General equation of heat transfer

Neglecting the conduction of heat, we can write the general equation of heat transfer (see Landau & Lifshitz 1987, § 58) in the following form:

$$\rho T \frac{Ds}{Dt} + \rho\mu \frac{Dc}{Dt} = 0. \tag{2.27}$$

On the assumptions (2.10), this equation may be approximated by

$$\rho_0 T_0 \frac{Ds'}{Dt} + \rho_0 \mu_0 \frac{Dc'}{Dt} = 0, \tag{2.28}$$

where (2.5) and (2.23) have been used.

2.5. Another form of the equation of motion

We have now completed the formulation of the anelastic approximation extended to a two-component fluid. However, it is helpful for later reference to express the equation of motion (2.12) in a slightly different form.

Note first that, in view of (2.5), the pressure p of the fluid can be written as

$$p = p_0 + p', \tag{2.29}$$

where p_0 is given by

$$p_0 = p(h_0, s_0, c_0). \tag{2.30}$$

We next observe from (2.2) that the following thermodynamic relations hold:

$$(\partial p/\partial h)_{s,c} = \rho, \quad (\partial p/\partial s)_{h,c} = -\rho T, \quad (\partial p/\partial c)_{h,s} = -\rho\mu. \tag{2.31}$$

Thus p' can be expressed, to the first order of h', s', and c', as follows:

$$p' = \rho_0 h' - \rho_0 T_0 s' - \rho_0 \mu_0 c'. \tag{2.32}$$

This expression enables us to rewrite (2.12) in the following form:

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla(p'/\rho_0) - s'\nabla T_0 - c'\nabla\mu_0. \tag{2.33}$$

2.6. Energetics of the extended approximation

Now, let us proceed to examine the energy balance of the fluid. We first consider the specific internal energy e of the fluid: it satisfies the relation

$$e = h - p/\rho. \tag{2.34}$$

Taking account of (2.5), (2.23), and (2.29), we obtain from (2.34) the expression

$$e = (h_0 + h') - (p_0 + p')/\rho_0 = (h_0 - p_0/\rho_0) + (h' - p'/\rho_0). \tag{2.35}$$

In light of (2.32), however, we observe that e can be written as

$$e = (h_0 - p_0/\rho_0) + T_0 s' + \mu_0 c'. \tag{2.36}$$

Let us next take the material derivative of (2.36): the result is

$$\frac{De}{Dt} = \boldsymbol{u}\cdot\nabla(h_0 - p_0/\rho_0) + s'\boldsymbol{u}\cdot\nabla T_0 + c'\boldsymbol{u}\cdot\nabla\mu_0 + T_0\frac{Ds'}{Dt} + \mu_0\frac{Dc'}{Dt}. \tag{2.37}$$

Multiplying (2.37) by ρ_0, and considering (2.24) and (2.28), we obtain

$$\rho_0\frac{De}{Dt} = \nabla\cdot\{\rho_0(h_0 - p_0/\rho_0)\boldsymbol{u}\} + \rho_0 s'\boldsymbol{u}\cdot\nabla T_0 + \rho_0 c'\boldsymbol{u}\cdot\nabla\mu_0. \tag{2.38}$$

However, since $\rho_0 De/Dt = \rho_0 \partial e/\partial t + \rho_0 \boldsymbol{u} \cdot \nabla e = \partial(\rho_0 e)/\partial t + \nabla \cdot (\rho_0 e \boldsymbol{u})$, we can write

$$\frac{\partial}{\partial t}(\rho_0 e) + \nabla \cdot (\rho_0 e \boldsymbol{u}) = \nabla \cdot \{\rho_0 (h_0 - p_0/\rho_0) \boldsymbol{u}\} + \rho_0 s' \boldsymbol{u} \cdot \nabla T_0 + \rho_0 c' \boldsymbol{u} \cdot \nabla \mu_0. \tag{2.39}$$

Furthermore, in view of (2.36), this equation can be put into the following form:

$$\frac{\partial}{\partial t}(\rho_0 e) + \nabla \cdot \left\{\rho_0 (T_0 s' + \mu_0 c') \boldsymbol{u}\right\} = \rho_0 s' \boldsymbol{u} \cdot \nabla T_0 + \rho_0 c' \boldsymbol{u} \cdot \nabla \mu_0. \tag{2.40}$$

The integration of (2.40) over the domain Ω containing the fluid yields

$$\frac{d}{dt} \int_\Omega \rho_0 e \, dV = \int_\Omega \left(\rho_0 s' \boldsymbol{u} \cdot \nabla T_0 + \rho_0 c' \boldsymbol{u} \cdot \nabla \mu_0\right) dV, \tag{2.41}$$

where it has been assumed that the normal component of $\boldsymbol{u}$ vanishes on the boundary of Ω. This is the equation for the rate of change of the internal energy of the fluid.

On the other hand, the equation for the rate of change of the potential energy of the fluid becomes, as a consequence of the approximation (2.23),

$$\frac{d}{dt} \int_\Omega \rho_0 g z \, dV = 0. \tag{2.42}$$

It therefore follows that the potential energy of the fluid is invariable.

In order to find the equation for the rate of change of the kinetic energy of the fluid, we first take the inner product of (2.33) with $\rho_0 \boldsymbol{u}$ to obtain, after some manipulation,

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho_0 |\boldsymbol{u}|^2) + \nabla \cdot \left\{\rho_0 (\tfrac{1}{2}|\boldsymbol{u}|^2 + p'/\rho_0) \boldsymbol{u}\right\} = -\rho_0 s' \boldsymbol{u} \cdot \nabla T_0 - \rho_0 c' \boldsymbol{u} \cdot \nabla \mu_0. \tag{2.43}$$

Integrating (2.43) over the domain Ω, on the assumption that the normal component of $\boldsymbol{u}$ vanishes on the boundary of Ω, we get the following desired equation:

$$\frac{d}{dt} \int_\Omega \tfrac{1}{2}\rho_0 |\boldsymbol{u}|^2 dV = -\int_\Omega \left(\rho_0 s' \boldsymbol{u} \cdot \nabla T_0 + \rho_0 c' \boldsymbol{u} \cdot \nabla \mu_0\right) dV. \tag{2.44}$$

When (2.44) is compared with (2.41), the conclusion reached by Maruyama (2021) is confirmed: the work done by the buoyancy force due to changes in entropy corresponds to the conversion between kinetic and internal energy.

At the same time, we notice the following fact: the work done by the buoyancy force due to changes in concentration also corresponds to the conversion between kinetic and internal energy. This is in contrast with the result obtained by Maruyama (2014, 2019) for the Boussinesq approximation extended to a two-component fluid: the work done by such a force corresponds to the conversion between kinetic and *potential* energy. Here it is to be noted that, since the Boussinesq approximation is not a subset of the anelastic approximation (Maruyama 2021), this contrast is not a contradiction.

Finally, adding all the energy equations (2.41), (2.42), and (2.44), we have

$$\frac{d}{dt} \int_\Omega \rho_0 \left(\tfrac{1}{2}|\boldsymbol{u}|^2 + gz + e\right) dV = 0. \tag{2.45}$$

Thus the total energy of the fluid is conserved; this implies that the extended anelastic approximation is consistent with the conservation law of energy.

2.7. Applicability of the extended approximation

The extended anelastic approximation was formulated on the assumptions (2.10) and (2.22); our aim in the following is to find out under what conditions these assumptions are justifiable. To begin with, we consider the assumption (2.22).

When ρ is regarded as a function of h, s, and c, the following relations hold:

$$\begin{aligned}
&(\partial\rho/\partial h)_{s,c} = \rho/a^2,\\
&(\partial\rho/\partial s)_{h,c} = -\rho T(\beta/c_p + 1/a^2),\\
&(\partial\rho/\partial c)_{h,s} = -\rho^2(\partial\mu/\partial p)_{T,c} - \rho(\beta T/c_p)(\partial\mu/\partial T)_{p,c} - \rho(\mu/a^2),
\end{aligned} \tag{2.46}$$

in which a is the speed of sound. Hence ρ' in (2.22) can be expressed, to the first order of h', s', and c', as follows:

$$\begin{aligned}
\rho' = {}&(\rho_0/a_0^2)h' - \rho_0 T_0(\beta_0/c_{p0} + 1/a_0^2)s'\\
&- \rho_0\left\{\rho_0(\partial\mu/\partial p)_{T,c}|_{(h_0,s_0,c_0)} + (\beta_0 T_0/c_{p0})(\partial\mu/\partial T)_{p,c}|_{(h_0,s_0,c_0)} + (\mu_0/a_0^2)\right\} c'.
\end{aligned} \tag{2.47}$$

Now, let the characteristic scales of h', s', and c' be denoted, respectively, by $\Delta h'$, $\Delta s'$, and $\Delta c'$. Then, denoting by H the vertical extent of the domain Ω containing the fluid, we obtain the following estimate for $|\rho'/\rho_0|$:

$$\begin{aligned}
&|\rho'/\rho_0|\\
&\quad = O\{(gH/a_0^2)(\Delta h'/gH)\}\\
&\qquad + O\{(\Gamma_0 H/T_0)(T_0\Delta s'/gH)\}\\
&\qquad + O\{(gH/a_0^2)(T_0\Delta s'/gH)\}\\
&\qquad + O\{(\rho_0 gH/\mu_0)(\partial\mu/\partial p)_{T,c}|_{(h_0,s_0,c_0)}(\mu_0\Delta c'/gH)\}\\
&\qquad + O\{(\Gamma_0 H/T_0)(T_0/\mu_0)(\partial\mu/\partial T)_{p,c}|_{(h_0,s_0,c_0)}(\mu_0\Delta c'/gH)\}\\
&\qquad + O\{(gH/a_0^2)(\mu_0\Delta c'/gH)\},
\end{aligned} \tag{2.48}$$

where $\Gamma_0 = \beta_0 T_0 g/c_{p0}$ is the adiabatic lapse rate, and a_0 is defined by

$$a_0 = a(h_0, s_0, c_0). \tag{2.49}$$

We assume here that the following conditions on H are satisfied (Maruyama 2021):

$$(gH)^{1/2}/a_0 \le O(1), \quad \Gamma_0 H/T_0 \le O(1). \tag{2.50}$$

Then, it is seen from (2.48) that (2.22) holds when the conditions

$$\Delta h'/gH \ll 1, \tag{2.51}$$

$$T_0\Delta s'/gH \ll 1, \tag{2.52}$$

$$|\mu_0|\Delta c'/gH \ll 1 \tag{2.53}$$

are fulfilled together with

$$\left|(\rho_0 gH/\mu_0)(\partial\mu/\partial p)_{T,c}|_{(h_0,s_0,c_0)}\right| \le O(1), \quad \left|(T_0/\mu_0)(\partial\mu/\partial T)_{p,c}|_{(h_0,s_0,c_0)}\right| \le O(1). \tag{2.54}$$

We next recall the thermodynamic relations (2.13); they enable us to write

$$T' = (\beta_0 T_0/c_{p0})h' + \{T_0(1-\beta_0 T_0)/c_{p0}\}\, s' \\ + (T_0/c_{p0})\left\{(\partial\mu/\partial T)_{p,c}|_{(h_0,s_0,c_0)} - \beta_0\mu_0\right\} c'. \tag{2.55}$$

From this expression, we can find the following estimate for $|T'/T_0|$:

$$\begin{aligned} &|T'/T_0| \\ &= O\{(\Gamma_0 H/T_0)(\Delta h'/gH)\} \\ &\quad + O(\Delta s'/c_{p0}) \\ &\quad + O\{(\Gamma_0 H/T_0)(T_0\Delta s'/gH)\} \\ &\quad + O\{(T_0/\mu_0)(\partial\mu/\partial T)_{p,c}|_{(h_0,s_0,c_0)}(\mu_0\Delta c'/c_{p0}T_0)\} \\ &\quad + O\{(\Gamma_0 H/T_0)(\mu_0\Delta c'/gH)\}. \end{aligned} \tag{2.56}$$

Hence the first assumption of (2.10) is justifiable when the conditions

$$\Delta s'/c_{p0} \ll 1, \tag{2.57}$$

$$|\mu_0|\Delta c'/c_{p0}T_0 \ll 1 \tag{2.58}$$

hold together with (2.50) to (2.54).

On the other hand, using the thermodynamic relations (2.17), we can write

$$\begin{aligned} \mu' &= \{\rho_0(\partial\mu/\partial p)_{T,c}|_{(h_0,s_0,c_0)} + (\beta_0 T_0/c_{p0})(\partial\mu/\partial T)_{p,c}|_{(h_0,s_0,c_0)}\}h' \\ &\quad + \{(T_0/c_{p0})(1-\beta_0 T_0)(\partial\mu/\partial T)_{p,c}|_{(h_0,s_0,c_0)} - \rho_0 T_0(\partial\mu/\partial p)_{T,c}|_{(h_0,s_0,c_0)}\}s' \\ &\quad + \big[(\partial\mu/\partial c)_{T,p}|_{(h_0,s_0,c_0)} - \rho_0\mu_0(\partial\mu/\partial p)_{T,c}|_{(h_0,s_0,c_0)} \\ &\quad + (T_0/c_{p0})\{(\partial\mu/\partial T)_{p,c}|_{(h_0,s_0,c_0)} - \beta_0\mu_0\}(\partial\mu/\partial T)_{p,c}|_{(h_0,s_0,c_0)}\big]\, c'. \end{aligned} \tag{2.59}$$

Accordingly, $|\mu'/\mu_0|$ can be estimated as follows:

$$\begin{aligned} &|\mu'/\mu_0| \\ &= O\{(\rho_0 gH/\mu_0)(\partial\mu/\partial p)_{T,c}|_{(h_0,s_0,c_0)}(\Delta h'/gH)\} \\ &\quad + O\{(\Gamma_0 H/T_0)(T_0/\mu_0)(\partial\mu/\partial T)_{p,c}|_{(h_0,s_0,c_0)}(\Delta h'/gH)\} \\ &\quad + O\{(T_0/\mu_0)(\partial\mu/\partial T)_{p,c}|_{(h_0,s_0,c_0)}(\Delta s'/c_{p0})\} \\ &\quad + O\{(\Gamma_0 H/T_0)(T_0/\mu_0)(\partial\mu/\partial T)_{p,c}|_{(h_0,s_0,c_0)}(T_0\Delta s'/gH)\} \\ &\quad + O\{(\rho_0 gH/\mu_0)(\partial\mu/\partial p)_{T,c}|_{(h_0,s_0,c_0)}(T_0\Delta s'/gH)\} \\ &\quad + O\{(gH\mu_0^{-1}/\mu_0)(\partial\mu/\partial c)_{T,p}|_{(h_0,s_0,c_0)}(\mu_0\Delta c'/gH)\} \\ &\quad + O\{(\rho_0 gH/\mu_0)(\partial\mu/\partial p)_{T,c}|_{(h_0,s_0,c_0)}(\mu_0\Delta c'/gH)\} \\ &\quad + O[\{(T_0/\mu_0)(\partial\mu/\partial T)_{p,c}|_{(h_0,s_0,c_0)}\}^2(\mu_0\Delta c'/c_{p0}T_0)] \\ &\quad + O\{(\Gamma_0 H/T_0)(T_0/\mu_0)(\partial\mu/\partial T)_{p,c}|_{(h_0,s_0,c_0)}(\mu_0\Delta c'/gH)\}. \end{aligned} \tag{2.60}$$

The second assumption of (2.10) is therefore justifiable when the condition

$$\left|(gH\mu_0^{-1}/\mu_0)(\partial\mu/\partial c)_{T,p}|_{(h_0,s_0,c_0)}\right| \le O(1) \tag{2.61}$$

is satisfied in addition to the above conditions.

We observe here, however, that the following inequality is obtained from (2.7):

$$|\nabla h'| \le |\partial \boldsymbol{u}/\partial t| + |(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}| + |T\nabla s'| + |\mu\nabla c'|. \tag{2.62}$$

Denoting by L the length scale characteristic of the motion of the fluid, we have

$$|\nabla h'| = O(\Delta h'/L), \quad |\nabla s'| = O(\Delta s'/L), \quad |\nabla c'| = O(\Delta c'/L). \tag{2.63}$$

Also, denoting by U the velocity scale characteristic of the motion, we get

$$|\partial \boldsymbol{u}/\partial t| = O(U/\tau), \quad |(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}| = O(U^2/L), \tag{2.64}$$

where τ stands for the time scale characteristic of the motion. Now, let us assume that α_1, α_2, and α_3 in (2.6) can be chosen so that the following conditions are satisfied:

$$T/T_0 \le O(1), \quad |\mu/\mu_0| \le O(1). \tag{2.65}$$

Then it follows from (2.62), (2.63), and (2.64) that, when the conditions

$$U/(gH)^{1/2} \ll 1, \quad (L/\tau)/(gH)^{1/2} \ll 1 \tag{2.66}$$

hold together with (2.52) and (2.53), the condition (2.51) is fulfilled.

In conclusion, the extended anelastic approximation is applicable under the following conditions: (2.50), (2.52), (2.53), (2.54), (2.57), (2.58), (2.61), (2.65), and (2.66).

3. Summary and Discussion

The anelastic approximation has been extended to a two-component fluid. Under the extended approximation, changes in concentration cause a force; the work done by this force corresponds to the conversion between kinetic and internal energy. The conditions under which the extended approximation is applicable have also been elucidated.

3.1. On the conditions (2.52) and (2.53)

Let us now focus attention on the conditions (2.52) and (2.53) for the applicability of the extended anelastic approximation. We readily see that they can be written as

$$\Delta s'/c_{p0} \ll (\Gamma_0 H/T_0)/(\beta_0 T_0), \quad |\mu_0|\Delta c'/c_{p0}T_0 \ll (\Gamma_0 H/T_0)/(\beta_0 T_0). \tag{3.1}$$

Thus, unless $(\Gamma_0 H/T_0)/(\beta_0 T_0) \ll 1$, (2.52) and (2.53) are harmless: they are satisfied so long as (2.57) and (2.58) are fulfilled. When $(\Gamma_0 H/T_0)/(\beta_0 T_0) \ll 1$, however, (2.52) and (2.53) place highly restrictive constraints on $\Delta s'$ and $\Delta c'$. This should be kept in mind when the approximation is applied to a shallow layer of fluid.

3.2. Yet another form of the equation of motion

Let us next turn our attention to the equation of motion (2.33). In terms of ρ' given by (2.47), this equation can be expressed as follows:

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla p'/\rho_0 - (\rho' g/\rho_0)\boldsymbol{k}. \tag{3.2}$$

Our aim in the following is to demonstrate this fact.

We begin by rewriting the first term on the right-hand side of (2.33) as follows:

$$-\nabla(p'/\rho_0) = -\nabla p'/\rho_0 + (p'/\rho_0)(\nabla\rho_0/\rho_0). \tag{3.3}$$

The substitution of (3.3) into (2.33) yields

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla p'/\rho_0 + (p'/\rho_0)(\nabla\rho_0/\rho_0) - s'\nabla T_0 - c'\nabla\mu_0. \tag{3.4}$$

However, from (2.6) and (2.21), the following expression for $\nabla\rho_0$ is obtained:

$$\nabla\rho_0 = (\partial\rho/\partial h)_{s,c}|_{(h_0,s_0,c_0)}\nabla h_0 = -(\partial\rho/\partial h)_{s,c}|_{(h_0,s_0,c_0)} g\boldsymbol{k}. \tag{3.5}$$

Thus, noting that (2.32) gives $p'/\rho_0 = h' - T_0 s' - \mu_0 c'$, we have

$$\begin{aligned}(p'/\rho_0)(\nabla\rho_0/\rho_0) = -\{&(\partial\rho/\partial h)_{s,c}|_{(h_0,s_0,c_0)}h' \\ &- T_0(\partial\rho/\partial h)_{s,c}|_{(h_0,s_0,c_0)}s' \\ &- \mu_0(\partial\rho/\partial h)_{s,c}|_{(h_0,s_0,c_0)}c'\}(g/\rho_0)\boldsymbol{k}.\end{aligned} \tag{3.6}$$

The terms $-s'\nabla T_0$ and $-c'\nabla\mu_0$ in (3.4) can also be rewritten as follows:

$$\begin{aligned}-s'\nabla T_0 &= -s'(\partial T/\partial h)_{s,c}|_{(h_0,s_0,c_0)}\nabla h_0 = \rho_0(\partial T/\partial h)_{s,c}|_{(h_0,s_0,c_0)}s'(g/\rho_0)\boldsymbol{k}, \\ -c'\nabla\mu_0 &= -c'(\partial\mu/\partial h)_{s,c}|_{(h_0,s_0,c_0)}\nabla h_0 = \rho_0(\partial\mu/\partial h)_{s,c}|_{(h_0,s_0,c_0)}c'(g/\rho_0)\boldsymbol{k}.\end{aligned} \tag{3.7}$$

Hence, from (3.6) and (3.7), we obtain

$$\begin{aligned}&(p'/\rho_0)(\nabla\rho_0/\rho_0) - s'\nabla T_0 - c'\nabla\mu_0 \\ &\quad= -[(\partial\rho/\partial h)_{s,c}|_{(h_0,s_0,c_0)}h' \\ &\qquad - \{\partial(\rho T)/\partial h\}_{s,c}|_{(h_0,s_0,c_0)}s' \\ &\qquad - \{\partial(\rho\mu)/\partial h\}_{s,c}|_{(h_0,s_0,c_0)}c'](g/\rho_0)\boldsymbol{k}.\end{aligned} \tag{3.8}$$

However, we observe from (2.31) that the following thermodynamic relations hold:

$$\{\partial(\rho T)/\partial h\}_{s,c} = -(\partial\rho/\partial s)_{h,c}, \quad \{\partial(\rho\mu)/\partial h\}_{s,c} = -(\partial\rho/\partial c)_{h,s}. \tag{3.9}$$

These relations enable us to write (3.8) as follows:

$$\begin{aligned}&(p'/\rho_0)(\nabla\rho_0/\rho_0) - s'\nabla T_0 - c'\nabla\mu_0 \\ &\quad= -\{(\partial\rho/\partial h)_{s,c}|_{(h_0,s_0,c_0)}h' \\ &\qquad + (\partial\rho/\partial s)_{h,c}|_{(h_0,s_0,c_0)}s' \\ &\qquad + (\partial\rho/\partial c)_{h,s}|_{(h_0,s_0,c_0)}c'\}(g/\rho_0)\boldsymbol{k}.\end{aligned} \tag{3.10}$$

Accordingly, in view of (2.46) and (2.47), we find that

$$(p'/\rho_0)(\nabla\rho_0/\rho_0) - s'\nabla T_0 - c'\nabla\mu_0 = -(\rho' g/\rho_0)\boldsymbol{k}. \tag{3.11}$$

The substitution of (3.11) into (3.4) leads to (3.2).

Finally, it should be noted that, despite the above result, the density of a fluid under the extended anelastic approximation is given by ρ_0, not by $\rho_0 + \rho'$.

References

[1] Landau, L. D. & Lifshitz, E. M. 1987 *Fluid Mechanics,* 2nd ed. Butterworth-Heinemann.

[2] Maruyama, K. 2014 Energetics of a fluid under the Boussinesq approximation. `arXiv:1405.1921 [physics.flu-dyn]`.

[3] Maruyama, K. 2019 Rational extension of the Boussinesq approximation to a two-component fluid. `https://doi.org/10.5281/zenodo.3757258`.

[4] Maruyama, K. 2021 Anelastic approximation. `https://doi.org/10.5281/zenodo.4596697`.

[5] Ogura, Y. & Phillips, N. A. 1962 Scale analysis of deep and shallow convection in the atmosphere. *J. Atmos. Sci.* **19**, 173–179.