Misinterpretations of the Boussinesq approximation reflected in a theory on the energy balance of the thermohaline circulation

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1. Kinetic energy source under the Boussinesq approximation

Under the Boussinesq approximation, the system of governing equations of a fluid in a uniform gravitational field is, when the viscosity of the fluid is ignored,

$$\rho_0\{\partial \boldsymbol{u}/\partial t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\} = -\nabla p' + \rho_0\beta T' g\boldsymbol{k}, \qquad (1)$$

$$\partial T'/\partial t + \boldsymbol{u}\cdot\nabla T' = \kappa\nabla^2 T', \qquad (2)$$

$$\nabla\cdot\boldsymbol{u} = 0. \qquad (3)$$

Suppose that the fluid occupies a fixed domain Ω. The above system then yields the following equation for the rate of change of the kinetic energy of the fluid:

$$\frac{d}{dt}\int_\Omega \tfrac{1}{2}\rho_0|\boldsymbol{u}|^2 dV = \int_\Omega \rho_0\beta T' gw\, dV. \qquad (4)$$

It should be noted that the term on the right-hand side represents the rate of work done by the buoyancy force in the Boussinesq approximation; we see that the work is the only source of the kinetic energy.

2. Potential energy crisis and the mixing-driven thermohaline circulation theory

The energy balance of the thermohaline circulation in the oceans can be discussed on the basis of (4). We first note, taking viscous dissipation into consideration, that the kinetic energy of the circulation is irreversibly converted to internal energy. On the other hand, the work done by the buoyancy force, which is the only source of the kinetic energy, is customarily interpreted as corresponding to the conversion from potential to kinetic energy. It is then concluded that the potential energy of the oceans is sooner or later exhausted if the thermohaline circulation continues to revolve.

Some physical oceanographers considered as follows: since the circulation is actually continuing to revolve, there must be mechanisms supplying potential energy to the oceans. They identified winds and tides as chief sources of potential energy [1]: they asserted that winds and tides can supply potential energy to the oceans by mixing the seawater. The mixing-driven thermohaline circulation theory has thus come into existence.

3. Density under the Boussinesq approximation

Here let us recall the reason why the work done by the buoyancy force in the Boussinesq approximation is considered in general to correspond to the conversion from potential to kinetic energy. This is because the density of a fluid is believed to be given by

$$\rho = \rho_0 - \rho_0\beta T' \qquad (5)$$

under the approximation. However, it is evident that, if (5) is assumed, the equation of continuity

$$\partial\rho/\partial t + \nabla\cdot(\rho\boldsymbol{u}) = 0 \qquad (6)$$

does not hold. This implies that the conservation law of mass does not hold and that the fluid velocity $\boldsymbol{u}$ is no longer the momentum per unit mass. In fact, the density of a fluid under the Boussinesq approximation must be regarded as a constant [2]:

$$\rho = \rho_0. \qquad (7)$$

4. Work done by the buoyancy force

In order to examine the work done by the buoyancy force under the condition (7), we now consider two fluid elements of unit volume A and B: the temperatures of A and B are respectively T'_a and T'_b. The positions of the elements are then interchanged quasistatically and adiabatically. If the initial position of A is dz lower than that of B, the work done by the buoyancy force in the interchange of positions is given by

$$\rho_0\beta(T'_a - T'_b)gdz. \qquad (8)$$

After the interchange, since the adiabatic lapse rate is $\beta(T_0+T')g/c_p$, the temperatures of A and B become

$$T'_a - \beta(T_0+T'_a)gdz/c_p, \quad T'_b + \beta(T_0+T'_b)gdz/c_p, \qquad (9)$$

respectively. We next add to A the amount of heat

$$\rho_0 c_p(T'_b - T'_a) + \rho_0\beta(T_0+T'_a)gdz, \qquad (10)$$

and to B the amount of heat

$$\rho_0 c_p(T'_a - T'_b) - \rho_0\beta(T_0+T'_b)gdz. \qquad (11)$$

The initial state is then restored: a fluid element with the temperature T'_a lies dz lower than one with the temperature T'_b. This implies that the interchange of positions reduced the internal energy of the system of the fluid elements by the sum of (10) and (11). Since this sum is equal to (8), we see that the work done by the buoyancy force corresponds to the conversion from internal to kinetic energy.

5. Conclusion

The above result on the work done by the buoyancy force should have been anticipated in view of the fact that the buoyancy force is actually a pressure gradient force. The potential energy crisis is merely an illusion, and the mixing-driven thermohaline circulation theory is groundless. The thermohaline circulation is a heat engine driven mainly by the difference in temperature between the equator and the polar regions: this is in accord with our intuition about the circulation.

References

1) Munk W., Wunsch C. 1998: *Deep-Sea Res. I*, **45**, 1977-2010
2) Maruyama K. 2014: `arXiv:1405.1921`