A Patrol Problem in a Building by Search Theory

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***Abstract*—Art gallery problem has been extensively studied by computational geometry, where major issue was to find the minimum number of guards and their locations to watch inside an art gallery or a facility. In this paper, we are concerned with the dynamic and game-theoretic aspects of a security problem, where a thief tries to invade the gallery while watchmen try to prevent it. We consider the following problems: an invasion scheduling problem and an invasion route problem on thief's side, a selection problem of patrol routes and a distribution problem of watching effort for the guards. We solve the first and the second problems by a dynamic programming formulation, and the third and the fourth problems by game theory and search theory. By the proposed methodology, we can evaluate the vulnerability of patrol routes and thus recommend better strategies for the security of a building or a facility.**

I. Introduction

So-called Art Gallery Problem (AGP) [24] is famous in the field of computational geometry and it started from a question by Mr. Klee in 1973. He asked how many watchmen are required to see the whole interior area of an art gallery, having the form of polygon with n nodes and n arcs [15]. Chvatal [6] gave an answer that $\lfloor n/3 \rfloor$ watchmen are sufficient at most. Fisk [10] proposed a numerical algorithm of order $O(n \log n)$ to determine the deployment of watchmen. During about 20 years since then, there have been many researches on the AGP.

Simple polygon is originally assumed for the AGP but the AGP theory is extended further to orthogonal polygon [25], [18], [7], pyramid [3] and figure with curved walls [19]. The AGP has other extensions. Some consider obstacles like door in the interior of the art gallery to intercept the line of sight of watchman [20], [16], or the change of the range of watchman's vision from 360 degrees to 180 [28] or an acute angle [29]. The AGP is also applied to the patrol on borderline [4], [5], which is regarded as an edge covering problem. Some consider the constraints that each watchman must be stationed in sight of other co-worker [23]. The algorithms proposed for the AGP are also evaluated from the standpoint of computational complexity [22], [11], [9]. Although they made use of computational geometry in almost all cases to approach the AGP, some apply genetic algorithm to it [26], that is a rare case.

We know that computational geometry literally handles problems by a computer in a geometrical and stationary manner. It is good at cognizing shapes of objects and a spatial relation among them in a geometric space, but it is not good at dealing with dynamic problems depending on time, such as scheduling problem. On the dynamic patrol or search problem, there are many researches by search theory, e.g. an optimal search problem along a given route [14] and a border surveillance problem [27]. They focus on the detection of targets or infiltrators to defeat adversaries or prevent the illegal border-crossing by smugglers or terrorists.

In this paper, we consider four decision-making problems of security guards in general facility by mathematical programming methods, such as operations research or search theory. We aim that the methods proposed in this paper also support the automation by robots or closed-circuit televisions (CCTVs) and meet some needs in the field of security.

In the next section, we explain a general situation under which we are going to tackle the four problems. In Section III, we consider an optimal invasion scheduling problem of the invader against guards on patrol by dynamic programming. Taking account of the worst invasion schedule for the guards, we discuss an optimal selection of patrol routes from the game-theoretical point of view in Section IV. Our agenda of Section V is what is the best payment of attention for guards or CCTVs of security robots on patrol. We derive the best payment by search theory. In Section VI, we find an optimal invasion route of the invader by dynamic programming and network theory.

Game theory provides us a wider and deeper understanding of adversary behaviors or strategies that can expose potential vulnerability or resilience of the security or defense system. Red teaming [1], [2] is such a concept, normally used on the defense side, which refers to studying problems from the adversary point of view. Therefore, our methodology proposed here is concerned with the computational red teaming and could be applied to some defense problems such as air surveillance or air defense against opponent intruders.

II. Assumption on Security Situation and Four Problems

We consider the following mathematical model on patrolling an art gallery or a facility.

- A1. Geographical space $\boldsymbol{K}$ is a two-dimensional Euclid space and time space is discrete denoted by a set of time points $\boldsymbol{T} = \{1, \cdots, T\}$. An invader wants to arrive at his destination until time T.
- A2. Watchmen have m scheduled patrol routes. The s-th route is denoted by $p_s = \{p_s(t),\ t \in \boldsymbol{T}\}$, where $p_s(t)$ is the position of the watchmen at time $t \in \boldsymbol{T}$. In the interior of the gallery, there are some obstacles that block the vision of the patrolling watchmen.
- A3. The watchmen estimate n invasion routes of the invader. The j-th route consists of a sequence of L_j waypoints, $\{q(j,l) \in \boldsymbol{K},\ l = 1, \cdots, L_j\}$, where $q(j,l)$ is the l-th

point by which the invader passes. Each point has an indicator on whether it is visible from other places or not. Between these waypoints, the invader is assumed to move at speed u but not stop. He can stop and wait only at the waypoint.

A4. The invader would observe the behavior of the watchmen walking along their patrol route and determine his schedule on his invasion route in an adaptive manner.

A5. Both of the watchman and the invader are interested in the degree of detection of the invader. The degree is defined by $\delta\alpha/d^2$, where δ is the visibility on whether the invader is in sight of the watchmen, d is the distance between them and α is brightness at the invader's position.

Under the mathematical model above, we are going to discuss four problems. The first is to make an optimal schedule of the invader, which gives him the minimum degree of detection along a specified route. The second is how to choose one among the options of patrol routes. The third one is to consider how the watchmen should pay attention while patrolling on their route, taking account of the vulnerability of the patrol route to the invasion routes. The last one is to find an optimal invasion route with the minimum degree of detection for the invader. For the second problem, we use the solution of the first problem and formulate them as a game, where we regard the watchmen and the invader as competitive players.

III. Invasion Scheduling Problem

To formulate the first problem, we consider a scheduled patrol route and an invasion route, and then we use specified notation other than the general assumptions in Section II. The patrol route is represented by $\boldsymbol{p} = \{p(t), t \in \boldsymbol{T}\}$ and the invasion route by $\boldsymbol{q} = \{q^j, j = 1, \cdots, L\}$. $p(t) = (p_x(t), p_y(t))$ is the $x-y$ coordinate of watchmen' position at t. For the invader route, L is the number of waypoints and $q^j = (q^j_x, q^j_y)$ is the two-dimensional coordinate of the j-th waypoint. The j-th waypoint has a visibility indicator $\beta_j \in \{1\ (visible),\ 0\ (invisible)\}$. Because the invader can start from the first waypoint q^1 anytime and the schedule ends on arrival at his destination, we set $\beta_1 = \beta_L = 0$. Generally, we assume that the visibility of invader's position $\boldsymbol{r}$ from the watchmen position on patrol route s at time t is given by a discriminant function $\delta_s(\boldsymbol{r}, t) = \{1\ (visible),\ 0\ (invisible)\}$, which we calculate from the positions of the watchmen, the invader and obstacles, but here we simplify the notation by $\delta(\boldsymbol{r}, t)$ because of just a specified patrol route under consideration.

Visible area from a watchman or visibility graph has been already discussed by computational geometry [24], [12]. El-Gindy and Avis [8] developed an algorithm to construct the visibility polygon in order $O(n)$ for an n-node polygonal art gallery. The algorithm was refined to more effective ones with the same computational complexity by Lee [21] and Joe and Simpson [17]. These researches concern the visibility polygon generated by walls of the art gallery. The dead angles are also generated by obstacles inside the gallery. O'Rourke [24] regards the obstacles as holes in the gallery and proposes an algorithm with the worst-case order $\Omega(n \log n)$ for the visibility region. We can use these algorithms to calculate the visibility $\delta(\boldsymbol{r}, t)$.

It takes $n_j = \left\lfloor \frac{||q^{j+1} - q^j||}{u} \right\rfloor$ time points for the invader to pass from the j-th waypoint to the $j+1$-th. We can estimate the position of the invader at k time points after departure from the j-th waypoint by

$$\begin{aligned} q^j(k) &= q^j + \frac{k}{n_j+1}(q^{j+1} - q^j) \\ &= \left(\frac{n_j+1-k}{n_j+1} q^j_x + \frac{k}{n_j+1} q^{j+1}_x, \right. \\ &\qquad \left. \frac{n_j+1-k}{n_j+1} q^j_y + \frac{k}{n_j+1} q^{j+1}_y \right) \\ &\equiv (q^j_x(k), q^j_y(k)) \end{aligned}$$

using interpolation. From Assumption A5, the degree of detection of position $\boldsymbol{r} = (r_x, r_y)$ at time t is given by

$$\begin{aligned} E(\boldsymbol{r}, t) &= \frac{\delta(\boldsymbol{r}, t)\alpha(\boldsymbol{r})}{||\boldsymbol{r} - p(t)||^2} \\ &= \frac{\delta(\boldsymbol{r}, t)\alpha(\boldsymbol{r})}{(r_x - p_x(t))^2 + (r_y - p_y(t))^2}, \end{aligned} \tag{1}$$

where $\alpha(\boldsymbol{r})$ is brightness at $\boldsymbol{r}$.

We are going to derive an optimal schedule for the invader on the route $\boldsymbol{q}$ against the patrol route $\boldsymbol{p}$. The optimality of the invasion schedule is judged by the total degree of detection. If the invader starts from q^j at time z_j, he reaches q^{j+1} at time $z_j + n_j + 1$.

Because $D^j_k(z_j) = E(q^j(k), z_j + k)$ is the degree of detection of invader's position at k time points after departure from waypoint q^j, we can formulate the invasion scheduling problem as follows.

$$\begin{aligned} \min_{\{z_j\}} \sum_{j=1}^{L-1} \left(\sum_{k=1}^{n_j} D^j_k(z_j) + \beta_{j+1} \sum_{\tau=z_j+n_j+1}^{z_{j+1}} E(q^{j+1}, \tau) \right) \quad (2) \\ s.t.\ \ 1 \le z_1,\ z_i + n_i + 1 \le z_{i+1},\ i = 1, \cdots, L-2, \\ z_{L-1} + n_{L-1} + 1 \le T,\ z_j \in \boldsymbol{Z},\ j = 1, \cdots, L-1. \end{aligned}$$

The objective function is nonlinear for variables $\{z_j\}$ but we can solve the problem by dynamic programming. As a preliminary, we estimate the earliest arrival time at the j-th waypoint, N_j, and the smallest time points M_j for moving from q^j to the destination q^L, as follows.

$$N_j = \sum_{k=1}^{j-1} n_k + j, \quad M_j = \sum_{k=j}^{L-1} n_k + (L - j).$$

Let us define an optimized value $f_j(t)$, which is the minimum degree of detection from the start point to the j-th waypoint provided that the invader departs from the waypoint q^j at time t at latest. The time t when the invader feasibly stays at q^j satisfies $N_j \le t \le T - M_j$. We desire to obtain $f_L(T)$ of the destination q^L. From the objective function (2), we can see that there is the following relation between $f_{j-1}(\cdot)$ and $f_j(\cdot)$, which is a dynamic programming formulation.

$$f_j(t) = \min_{N_{j-1} \le z_{j-1} \le t - n_{j-1} - 1} \left[f_{j-1}(z_{j-1}) + \sum_{k=1}^{n_{j-1}} D_k^{j-1}(z_{j-1}) + \beta_j \sum_{\tau = z_{j-1} + n_{j-1} + 1}^{t} E(q^j, \tau) \right],$$
$$t = N_j, N_j + 1, \cdots, T - M_j, \; j = 2, \cdots, L. \quad (3)$$

It is obvious that the following initial values are valid from $\beta_1 = 0$.

$$f_1(t) = 0, \; t = 1, \cdots, T - M_1. \quad (4)$$

Optimal value $z^*_{j-1} = \arg\min_{z_{j-1}}$ in Problem (3) gives us an optimal departure time from waypoint q^{j-1} conditioned that the invader leaves q^j until t. An optimal arrival time at q^j is $z^*_{j-1} + n_{j-1} + 1$.

If the minimum degree of detection is smaller, it would be harder for the patrolling watchmen to notice the invader on the route $\boldsymbol{q}$. The difference between sequent values $f_j(\cdot)$ and $f_{j+1}(\cdot)$ indicates the risk of the leg between the sequent waypoints on security.

IV. Optimal Selection of Patrol Routes

Here we deals with the selection problem of patrol routes with multiple invasion routes and patrol routes, which are assumed originally in Assumption A2 and A3. We recall that the minimum degree of detection, denoted by $R(p_s, q_j)$ ($s = 1, \cdots, m, \; j = 1, \cdots, n$), is derived by dynamic programming as optimal value $f_L(T)$ from the invasion scheduling problem for a patrol route p_s and an invasion route q_j. The value indicates the quantitative congeniality between p_s and q_j, or the vulnerability of p_s against q_j.

The invader does not know the patrol route that the watchmen would take until he sneaks in the facility and the watchmen also do not know the invader's route. Therefore, we can regard the selection by the invader and the watchmen for their routes as a one-shot matrix game with payoff $R(p_s, q_j)$, as represented by a table below. If an equilibrium point is given by a mixed strategy, we could adopt the equilibrium as the good frequency of taking patrol routes during a period of days. If the value of the game is too small, we'd better plan more efficient patrol routes instead of the present routes.

$$\begin{array}{c} \\ \vdots \\ p_s \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & \cdots & q_j & \cdots & \end{array} \\ \left(\begin{array}{ccc} & \vdots & \\ \cdots & R(p_s, q_j) & \cdots \\ & \vdots & \end{array} \right) \end{array}$$

V. Distribution Problem of Attention

While patrolling, the watchmen would pay attention to all directions knowing weak points of facility on security. Here we are going to discuss a distribution problem of attention by the watchmen on patrol. In search theory, there is already a study on the optimal distribution of search resource to detect a target, called "search allocation game (SAG)" [13]. We modify the SAG model for the optimal distribution problem of the watchmen' attention. The watchmen anticipate n invasion routes, as mentioned in Assumption A3, and know the worst schedule of each invasion route, as discussed in Section III. We denote the j-th scheduled invasion route by $\omega_j = \{\omega_j(t), t \in \boldsymbol{T}\}$, where $\omega_j(t)$ is invader's position at time t on the j-th route. The watchmen are assumed to take a patrol route $\boldsymbol{p} = \{p(t)\}$, as in Section III.

We make some additional assumptions with regard to the distribution strategy of attention and the probability to detect the invader, as follows.

A6. The whole direction $[0, 2\pi]$ is divided into M parts and a direction of attention is defined by $\theta \in \Theta \equiv \{1, \cdots, M\}$. To any direction, the watchmen can pay attention or distribute their attention, which totals to one at each time. We denote the distribution of attention or the amount of attention to direction θ at time t by variable $\varphi(\theta, t)$. On the other hand, the invader chooses one among n invasion routes.

A7. When the invader is at a position $\boldsymbol{r}$ in direction θ from watchmen' position at t, the detection probability of the invader is affected by the product of the amount of attention $\varphi(\theta, t)$ and the detection degree $E(\boldsymbol{r}, t)$ defined by Eq. (1). The total probability of detecting the invader on the j-th invasion route is given by

$$P(\varphi, j) = 1 - \exp\left(-\sum_{t=1}^{T} E(\omega_j(t), t) \varphi(\phi(\omega_j(t), t), t) \right),$$

where $\phi(\boldsymbol{r}, t)$ indicates the direction of position $\boldsymbol{r}$ from watchmen' position $p(t)$.

A8. The watchmen want to maximize the detection probability by their strategy $\{\varphi(\theta, t), \; \theta \in \Theta, \; t \in \boldsymbol{T}\}$ and the invader desires to minimize it.

This problem is a two-person zero-sum game with the detection probability as a payoff. The watchman is a maximizer taking the strategy of the distribution of attention. The invader is a minimizer with the selection strategy of his route, which is defined by $\pi(j)$ as the probability of taking the j-th route. The expected payoff $P(\varphi, \pi) = \sum_j \pi(j) P(\varphi, j)$ is linear for variable π and concave for φ, that implies the coincidence of its minimax value and maximin value, namely, the value of the game.

We trace Hohzaki's work [13] to derive an optimal strategy of the watchmen from the maximin optimization of the game. First we confirm feasible regions Ψ and Π for strategies φ and π, respectively.

$$\Psi = \left\{ \{\varphi(\theta, t), \; \theta \in \Theta, t \in \boldsymbol{T}\} \,\middle|\, \sum_{\theta \in \Theta} \varphi(\theta, t) = 1, t \in \boldsymbol{T}, \right.$$
$$\left. \varphi(\theta, t) \ge 0, \; \theta \in \Theta, \; t \in \boldsymbol{T} \right\} \quad (5)$$

$$\Pi = \left\{ \{\pi(j), \; j = 1, \cdots, n\} \mid \sum_{j=1}^{n} \pi(j) = 1, \right.$$

$$\pi(j) \geq 0,\ j = 1, \cdots, n\} . \tag{6}$$

The maximin optimization problem of the expected payoff is transformed as follows.

$$\max_{\varphi \in \Psi} \min_{\pi \in \Pi} P(\varphi, \pi) = \max_{\varphi \in \Psi} \min_{j} P(\varphi, j) = 1 - \\ - \exp \left(- \max_{\varphi \in \Psi} \min_{j} \sum_{t=1}^{T} E(\omega_j(t), t) \varphi(\phi(\omega_j(t), t), t) \right).$$

As a result, we might execute the maximin optimization just in the shoulder of the exponential function above. Now we have the following linear programming formulation for an optimal distribution of attention φ^*.

$$(P_P) \quad \max_{\varphi,\ \eta} \ \eta$$
$$s.t.$$
$$\sum_{t=1}^{T} E(\omega_j(t), t) \varphi(\phi(\omega_j(t), t), t) \geq \eta, j = 1, \cdots, n, \tag{7}$$
$$\sum_{\theta \in \Theta} \varphi(\theta, t) = 1,\ t \in \boldsymbol{T},\ \varphi(\theta, t) \geq 0,\ \theta \in \Theta,\ t \in \boldsymbol{T}.$$

From the optimal value W of Problem (P_P), we estimate the value of the game with respect to the payoff of detection probability by $1 - \exp(-W)$. We can obtain an optimal mixed strategy of the invader, $\pi^* = \{\pi^*(j)\}$, by solving the following linear problem, which is dual to Problem (P_P).

$$(P_I) \quad \min_{\pi,\ \nu} \sum_{t \in \boldsymbol{T}} \nu(t)$$
$$s.t \sum_{j \in \Omega_{\theta t}} \pi(j) E(\omega_j(t), t) \leq \nu(t), \theta \in \Theta, t \in \boldsymbol{T}, \tag{8}$$
$$\sum_{j=1}^{n} \pi(j) = 1,\ \pi(j) \geq 0,\ j = 1, \cdots, n,$$

where $\Omega_{\theta t}$ is a set of invasion routes running in direction θ from watchmen' position at time t, defined by $\Omega_{\theta t} \equiv \{j \mid \phi(\omega_j(t), t) = \theta\}$.

We omit the derivation of Problem (P_I) and the proof of the validity of this system of equations because this theory can be obtained by modifying a little the discussion in the reference [13].

We can explain qualitative property of the optimal strategy as follows. The invader tends to choose the invasion route that runs through the places with poor visibility from the watchmen and the places far from the patrol route. This is what constraint (8) of (P_I) says. The watchmen distribute more attention to the invasion routes that the invader would choose more likely and keep the balance among all invasion routes in terms of the total amount of attention with the weight of the degree of detection. Constraint (7) of (P_P) claims that. From this property of the optimal strategy, we can say that the value of the game is a lower bound on the degree of detection which the relevant patrol route can afford to provide to all invasion routes.

By the solution of the distribution problem of attention, we can smooth the variance of the congeniality of the given patrol route to each invasion route by adjusting the amount of attention. On the other hand, the selection problem of patrol routes in Section IV gives us a way to smooth the variance of the congeniality of a patrol route by mixing several patrol routes or the mixed strategy. When we apply the formulation (P_P) to every patrol route, the patrol route with the largest optimal value W has the most adaptation or the least vulnerability to the invasion routes, and should be recommended as a desirable route.

VI. Optimal Invasion Route

Here we are going to find the worst invasion route with the minimum degree of detection on a network for several watchman's routes. Therefore, we set a network environment instead of the provision of invasion routes in Assumption A3 in Section II. We add some assumptions about the network to Assumption A1∼A5.

- B1. The invader decides his route on a network $G(\boldsymbol{V}, \boldsymbol{A})$ on a two-dimensional plane, where $\boldsymbol{V}$ is a set of nodes and $\boldsymbol{A}$ is a set of arcs. Node i has its position vector $\boldsymbol{v}_i$ on the plane and its visibility γ_i.
  We denote incident nodes to Node $i \in \boldsymbol{V}$ by $N(i)$, to which the invader can move from the node i.
- B2. The invader starts from node s and his destination node is e. The entrance s is assumed to be invisible, i.e., $\gamma_s = 0$.

We can develop a theory similar to Section III. The distance between node i and j is given by $d_{ij} = ||\boldsymbol{v}_i - \boldsymbol{v}_j||$ and it takes time points $n_{ij} = \lfloor d_{ij}/u \rfloor$ for the invader to move there. Therefore, the invader is moving during time points $z_i + 1, z_i + 2, \cdots, z_i + n_{ij}$ on the way to node j and arrives at node j at time $z_i + n_{ij} + 1$ if he starts from node i at time z_i. The invader's position at time $z_i + k$ is estimated by

$$\boldsymbol{v}_{ij}(k) = \boldsymbol{v}_i + \frac{k}{n_{ij} + 1}(\boldsymbol{v}_j - \boldsymbol{v}_i).$$

In this case, we gather the degree of detection from watchmen on all patrol routes to calculate the aggregate degree of detection of the invader's position $\boldsymbol{r}$ at time τ by

$$E(\boldsymbol{r}, \tau) \equiv \sum_{s=1}^{m} \frac{\alpha(\boldsymbol{r}) \delta_s(\boldsymbol{r}, \tau)}{||\boldsymbol{r} - p_s(\tau)||^2}. \tag{9}$$

From the expression, we have the degree of detection for the invader moving between nodes i and j at time $z_i + k$ by

$$D_k^{ij}(z_i) = E(\boldsymbol{v}_{ij}(k), z_i + k) \\ = \sum_{s=1}^{m} \frac{\alpha(\boldsymbol{v}_{ij}(k)) \delta_s(\boldsymbol{v}_{ij}(k), z_i + k)}{|\boldsymbol{v}_{ij}(k) - p_s(z_i + k)||^2}. \tag{10}$$

After the preliminary above, we can construct a dynamic programming formulation similar to Eq. (3) although each notation has the different meaning from (3).

$$g_j(t) = \min_{i\in N(j)} \min_{F_i\leq z\leq t-n_{ij}-1} \left[g_i(z) + \sum_{k=1}^{n_{ij}} D_k^{ij}(z) + \gamma_j \sum_{\tau=z+n_{ij}+1}^{t} E(\boldsymbol{v}_j,\tau)\right], \quad t = F_j, F_j+1,\cdots,T-H_j,\ j\in V \tag{11}$$

$$Initial\ value: \quad g_s(t) = 0,\ t = 1,\cdots,T-H_s$$

The optimized value $g_j(t)$ is the minimum degree of detection given by an optimal route and schedule before time t on condition that the invader starts from node j at the time t. F_j is the shortest time up to the node j from s without stopping at any node on the way. and H_j is the shortest time from the node j to e without stopping everywhere. We can compute F_j and H_j by the shortest path algorithm.

There is no need to explain the formulation in detail. We can see the validity of the formulation on analogy of the derivation of Eq. (3) with the exception of the doubly-layered optimization in respect of incident nodes $N(j)$ as well as time z. If we obtain optimal values i^* and z^* by the evaluation $g_j(t)$, i^* and z^* indicate the optimal node that the invader must stop by and the optimal leaving time from node i^*, respectively, just before arriving at node j until time t.

As seen in many shortest path algorithms using the DP formulation, we have to devise a computational algorithm to calculate the optimal invasion route in an effective way. We adopt the revision flag for a pair of node and time, (i,z) $(i\in V,\ z\in \boldsymbol{T})$. Let Γ be a set of revised pairs. We save the current minimum degree of detection among the feasible routes reaching the goal node e in S and use it as a lower bound such that we do not dare to revise the degree of nodes if the revised degree is larger than S. Our algorithm is as follows.

Algorithm for optimal invasion route

(S1) Initialize as follows. Set $g_s(t) = 0$ for all $t = 0,\ldots,T-H_s$ and give pairs of (s,t) revision flags by

$$\Gamma = \{(s,t)\ ,t = 1,\ldots,T-H_s\}.$$

We also set an initial value $S = \infty$.

(S2) If $\Gamma = \emptyset$, terminate the algorithm. The current $g_e(T)$ gives the minimum degree of detection by the optimal invasion route with an optimal schedule. Otherwise, execute the followings for all $(i,z)\in\Gamma$:

(i) For $j\in N(i)$ and $t = z+n_{ij}+1,\ldots,T-H_j$, calculate

$$x = g_i(z) + \sum_{k=1}^{n_{ij}} D_k^{ij}(z) + \gamma_j \sum_{\tau=z+n_{ij}+1}^{t} E(v_j,\tau).$$

If $x < g_j(t)$, set $g_j(t) = x$.
If $j = e$, put $S = x$. If $j \neq e$ and $x < S$, modify Γ by $\Gamma = \Gamma\cup\{(j,t)\}$.

(ii) Replace Γ by $\Gamma\backslash\{(i,z)\}$.

(S3) Go back to (S2).

VII. Numerical Examples

We cannot take examples enough to figure out all of our theory proposed in Section III through VI. That is why we focus on the optimal scheduling of invasion by a simple example and the optimal selection of patrol and invasion routes by an application of air defense problem in this section.

A. Optimal Invasion Scheduling

Figure 1 shows the floor design of an art gallery, which is divided by square cells for the sake of comprehensibility. The left wall is concave but it is glazed and transparent so as not to interrupt the vision of customers. Around the center, there are two obstacles, a black-colored square and a black-colored partition. In the gallery, we draw a big loop of patrol route which starts from a lower-left point at time $t = 1$, moves counterclockwise and comes back to the starting point at time $T = 35$. The watchmen move at the speed of a cell per time point. A small dot on the route indicates the position of watchmen at each time $t = 1,2,\cdots,T$. An invasion route, drawn by a broken line, runs from an upper-left point to a lower-left point, which is the destination of the invader. Seven waypoints are stationed on the route $(L = 7)$ and all are hidden $(\beta_j = 0,\ j = 1,\cdots,7)$. The invader desires to arrive at the goal of the seventh waypoint until $T = 35$. He runs at the speed of $u = 2$ (cells per time point), which doubles the watchmen' speed. Brightness is set to be $\alpha(\boldsymbol{r}) = 1$ for every place $\boldsymbol{r}$ in the gallery.

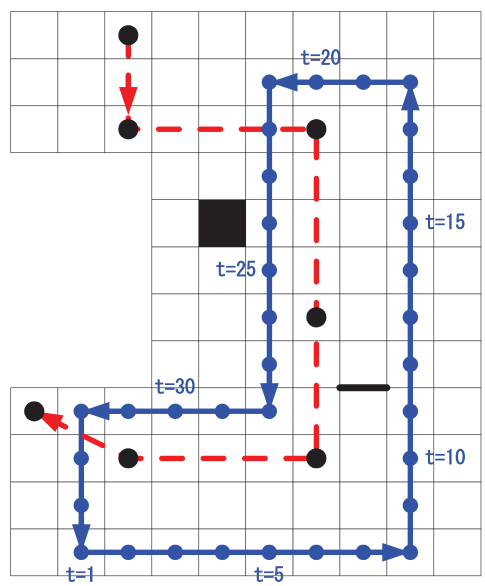

Fig. 1. A Patrol Route and an Invasion Route

An optimal invasion schedule computed by the proposed DP method is shown in Table 1. From the left, the number of waypoint (j), optimal arrival time (OAT) and departure time (z_j^*) (ODT) of the waypoint, the minimum degree of detection until j (MDD) and the $x-y$ coordinate of waypoint (COORD) are listed. Value $f_L(T)$ in the 4-th column and the last row is the minimum of the total degree of detection. The difference between the arrival and the departure time tells us that the invader's waiting time at the waypoint. The invader can arrive at his destination at $t = 24$ at the risk of the degree of detection 0.5801, which is the congeniality between the patrol route and the invasion one.

Table 1. Optimal Invasion Schedule

Waypoint (j)	OAT	ODT	MDD	COORD
1	1	3	0	(3,12)
2	5	5	0.0962	(3,10)
3	8	8	0.2118	(7,10)
4	11	17	0.2118	(7, 6)
5	19	19	0.4281	(7, 3)
6	22	22	0.5801	(3, 3)
7	24	–	0.5801	(1, 4)

From this table, we can easily see how cleverly the invader makes use of obstacles to keep himself out of sight of the watchmen when moving. Namely, no increase of the MDD during $j = 3 \sim 4$ and $6 \sim 7$ implies that the watchmen cannot watch any movement of the invader during these time periods. Generally speaking, the small increase of the MDD shows poor visibility of the watchmen for the invader. The increase of the MDD between sequent waypoints comes from the thoughtful management of motion by the invader to minimize the total degree of detection.

We illustrate the optimal invasion schedule in Fig. 2 by showing the correspondence between the invasion schedule and the patrol schedule. A line between a waypoint on the invasion route and a position of the watchmen shows the timing that the invader and the watchmen pass there. Two lines beaming from a waypoint indicate that the invader waits there while the watchmen continue walking. Let us check the details of the optimal invasion schedule.

(1) Departure from the starting point $j = 1$: The invader postpones the start until $t = 3$ such that the invader moves little far from the watchmen at the early time and he can manage to exploit obstacles for dead angle later.

(2) Movement between waypoints $j = 1$ and 2: The invader is in sight of the watchmen but the distance between them is too long to increase the degree of detection much. The degree there is $f_2(z_2^*) = 0.0962$.

(3) Movement between waypoints $j = 2$ and 3: The invader is sometimes exposed and sometimes in the dead angle made by the rectangular obstacle during the early half of time and made by the partition during the later time.

(4) Movement between waypoints $j = 3$ and 4: The watchmen walk very close to the invader. But the invader can make use of the partition to hide himself perfectly until reaching the 4-th waypoint. And then he stays there during $t = 11 \sim 17$ to let the watchmen pass through out of sight because the waypoint is hidden.

(5) Movement between waypoints $j = 4$ and 5: The invader moves in sight of the watchmen but there is a long distance between them. The invader keeps going without stopping at the 5-th waypoint while the distance is still long.

(6) Movement between waypoints $j = 5$ and 6: The invader keeps the long distance from the watchmen. He is exposed to the watchmen in the early time but is out of sight in the later time behind the rectangular obstacle.

(7) Movement between waypoints $j = 6$ and 7: The invader is perfectly in the dead angle made by the rectangular obstacle and arrives at the goal at $t = 24$.

From Fig. 2, we can see that the invader uses various ways to hide himself and lower the degree of detection.

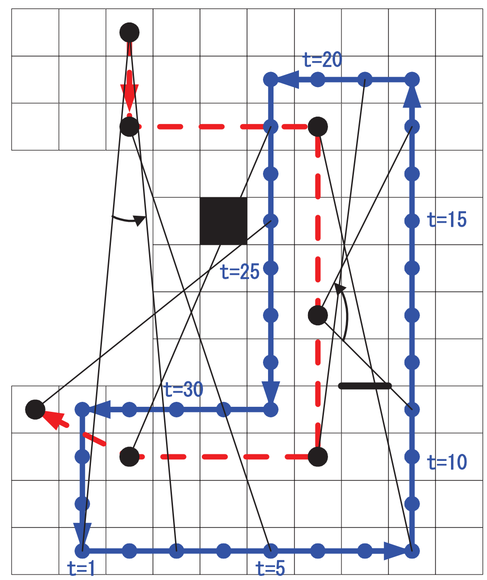

Fig. 2. Optimal Invasion Schedule

B. A Defense Problem in the Airspace around Tokyo

With a small modification but no essential change, we can apply the formulation and algorithms proposed for the security problem in the facility in the previous sections to air defense problems. We might modify the following points for the application. The watchman might use his eyesight as a sensor for the patrol for invaders. However, main sensor in the surveillance or patrol operation for the air defense aginst invasion aircrafts or missiles is radars equipped in airborne early warning aircrafts, such as E2C Hawkeye and AWACSs, or at radar sites in the land. Therefore, we have to change the definition of the degree of detection in Assumption A5 to correspond to the radar detection of invaders, namely, the radar signal power returned to receivers by taking account of radar gain, radar cross-section and the propagation of radar wave emitted from the radar in the air. The radar cross-section depends on the relative posture of the invader to the line of sight of the radar radiation. The visibility must be re-defined in a three-dimensional space instead of a two-dimensional space in the facility patrol problem. The algorithm of the visibility evaluates whether the radar wave can be reflected from invaders or obstacles such as mountains intervene the reflection from the geographical point of view.

The early warning aircrafts in the air defense correspond to the watchmen in the facility patrol. We can also regard the radar sites as the stationary watchmen in the air defense.

In the facility patrol problem, the invasion schedule is determined by the waiting time of the invader at the waypoint or the departure time z_j from waypoint q_j. In the air defense problem, the invader makes his invasion schedule by determining his flight height in each leg between each two waypoints assuming a relationship between the flight height and the flight speed, especially for unmanned air vehicles. In general, the air target flying high can speed up so that it can shorten the duration of exposure to the radar even though the high altitude makes it more detectable by the radars. The invader also has an option of flying lower that makes his speed lower but might provide him some blind legs from the radars under the cover of mountains or others. With the replacement of the degree of detection and the decision-making variables of the invader, the DP formulation (3) is still available to derive an optimal invasion schedule of air invaders. The replaced DP formulation gives us an optimal selection of height or velocity of the invader to minimize the degree of detection by radars.

Here we take an example of the air defense in the airspace around Tokyo in Japan. The Kanto Plain involves Tokyo as a part. It has the area of 17,000 square kilometers and faces the Pacific Ocean, which we can see in the lower-right of Fig. 3, southeast but is surrounded by high mountains or ranges in other directions. If a cruising missile, a small airplane or an unmanned-air vehicle (UAV) starts from the Japan sea, illustrated in the upper-left of Fig. 3, it once flies over the mountains running from northeast to southwest in the middle of the Japanese archipelago and crosses the plain to reach Tokyo. It possibly travels along the mountains located in the periphery of the plain and then traverses shorter ranges of the plain to get Tokyo. Here we take Yokosuka city near Tokyo as a destination of invaders, which is Point E in Fig. 3. The city has a home base of the Seventh Fleet of the US navy as well as the main harbor of the Japanese navy.

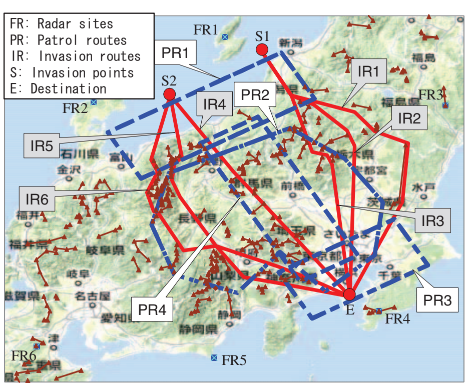

Fig. 3. Invasion Routes (IR1∼IR4) and E2C Patrol Routes (PR1∼PR4)

Other than the geographical space stated above, we set a time space $\boldsymbol{T} = \{1, \ldots, 30\}$ and six invasion routes illustrated in Fig. 3. Some of them, Route IR1, IR2 and IR3, start from point S1 in the Japan sea and cross the eastern areas of the Kanto Plain to reach point E (Yokosuka). Other two routes of IR5 and IR6 start from point S2 and cross the western areas of the plain to the destination point E. Route IR4 is a straight line running directly from S2 to E in the middle of the plain. On any route except for IR4, the invader can except hiding itself from the radar radiation by some mountains. We set waypoints at the points where the routes drastically change their angles of direction. On each route, the invader has just two options of its height: the height of 5000 meter and the speed of 217 kilometers per hour or 500m and 148km/h for the sake of simplicity.

To cover the airspace, Japanese air force has six radar sites on the top of mountains, FR1 through FR6. The effective detection range is assumed to be 500 km for all radars. The radars can detect targets only within the range. For the target flying 500m high, there are some places that all radars do not cover, in the plain. For the 5000m-height target, there is no uncovered place but there are some places near the high mountains that the network of the radar sites cannot provide enough covering to. To complement the weak covering of the radar sites, the air defense brings an E2C with four patrol routes of PR1 through PR4 in the airspace. PR1 or PR3 focuses on the side of the Japan sea or the seaside area around the destination, respectively. PR2 covers around the border of the Kanto Plain where many mountains lie. PR4 patrols around the mid area of the plain. The effective detection range of the E2C's radar is set to be 370 km. We derive an optimal invasion schedule to minimize the total degree of detection over all radars but we multiply the minimum degree by an adequate common coefficient to adjust the final number to be greater than 1 for the sake of presentation.

Table 2 shows us the minimum degree for every combination of 4 patrol routes and 6 invasion routes. It is also the payoff matrix for the air defense strategies and the invasion strategies. From the matrix, we obtain an optimal mixed strategy on the defender's side of taking PR3 and PR4 with their respective probabilities 0.725 and 0.275, and an optimal strategy on the invader's side with probabilities 0.309 and 0.691 of taking two routes IR4 and IR5, respectively.

Table 2. Minimum Degrees of Detection for Four Patrol Routes and Six Invasion Routes

	IR1	IR2	IR3	IR4	IR5	IR6
PR1	7.631	5.527	6.032	3.804	4.169	4.685
PR2	7.185	6.306	5.538	3.791	4.076	5.811
PR3	7.653	5.422	5.041	4.401	5.305	5.341
PR4	6.760	6.712	7.671	6.672	4.289	4.478

IR4 runs the middle of the Kanto Plain so that the target on the route is easily exposed to the radar radiation even though it flies low. The optimal schedule of IR4 recommends the invader to fly high on almost all legs against E2C on any patrol route to arrive at the destination as quickly as possible. Only on the second leg, the invader'd better take a low altitude to avoid the radar radiation from E2C on the patrol route PR1 when E2C flies near the Hida mountains. We show the optimal invasion schedule of IR5, which has 8 leg, in Table 3. A symbol 'H' recommends the heigh altitude and 'L' the low altitude. The invader should take a low altitude on Leg 2 and 7 for any patrol route when it flies near the Hida mountains and the Kiso mountains and the radar wave never reaches there from any radar site. From the defensive stand point, the patrol route PR3 provides larger degree of detection against any invasion route at the final stage of its flight. The route PR4 is most effective against the direct flight of the invader on IR4.

Table 3. An Optimal Invasion Scheduling of IR5

Legs	1	2	3	4	5	6	7	8
PR1	H	L	H	H	H	L	L	H
PR2	H	L	H	H	H	H	L	H
PR3	H	L	H	L	H	H	L	H
PR4	H	L	H	H	L	H	L	H

VIII. Conclusion

In this paper, we deal with patrol problems by watchmen in an art gallery or a facility, which has been studied especially by computational geometry. We combine an invasion scheduling with the patrol problem and consider four problems: an invasion scheduling problem and an invasion route problem on thief's side, a selection problem of patrol routes and a distribution problem of watching effort for the watchmen. The scheduling is the dynamic problem involving time and it is difficult to handle by geometrical approach. We take OR approach and search theory for solution. However the proposed methods are the first step to tackle practical security problems in the facility and we might have many obstacles to apply our methodology to practical problems.

One of the obstacles is that we need to separate what we require the rigidity for and what we are allowed to simplify, in order to apply our model to practical problems. In the concrete, we have to ask the following questions. How realistic is the optimal invasion schedule derived by the proposed DP formulation? How roughly can we set the axis of time and geographical space as the background of the problem? The so-called art gallery problem was originally modeled and extended by computational geometry. Even though the patrol problem is handled by computational geometry or OR methods, we need to make the derived solution be useful in the real world.

This paper originally aims to dedicate the proposed methodology to the development of automation by robots and other equipments in the field of security. Therefore, we need to take account of the limitation or constraints on the control or the sensor's capability of these automated systems in our modeling. To the end, we could utilize other modeling in computational geometry, e.g., robber route problem, watchman route problem, sweeper problem, zookeeper's route problem, safari route problem and aquarium keeper's route problem and so on.

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