Journal of the Operational Research Society (2012) 63, 1434–1446 0160-5682/12

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A smuggling game with asymmetrical information of players

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This paper deals with a smuggling game with multiple stages. Customs is allowed to patrol within the limited number of chances and obtain reward by the capture of a smuggler. The smuggler gets a reward depending on the amount of contraband he succeeds to ship in smuggling at each stage. The pay-off of the game is zero-sum. In almost all past studies, they adopt the alternative of smuggling or non-smuggling as the smuggler's strategy. From the point of view of information, some researchers assumed that both players could observe their opponent's behaviour at the past stage or a few assumed that both players had no information about their opponent. Other than these types of smuggling games with the symmetric information, we introduce the asymmetrical acquisition of information or the concept of perfect Bayesian equilibrium in the smuggling game for the first time.

Journal of the Operational Research Society (2012) **63,** 1434–1446. doi:10.1057/jors.2011.161

Published online 25 January 2012

Keywords: game theory; linear programming; military; decision analysis

1. Introduction

The Japanese archipelago is isolated from foreign countries by seas, and thus Customs or the Japanese coastguard can comparatively easily prevent illegal actions such as smuggling from flowing into Japan. However, they are always requested to efficiently use their budget, with which they patrol or inspect harbours, ships, airports and so on. This paper aims for the effective inspection strategy by Customs and the coastguard to deter the smuggling. The paper deals with an inspection game or a smuggling game with multiple stages, in which Customs and a smuggler participate. The inspection game has been applied to a variety of problems such as the smuggling problem of contraband, arms-control treaty violation or inspection by the International Atomic Energy Agency (IAEA) for nuclear facilities. Dresher (1962) formulated the compliance problem with the treaty of arms reduction as a multi-stage game and developed the field of the inspection game. Maschler (1966) generalized Dresher's problem. Dresher and Maschler consider the game where a violator wishes to violate the treaty in secret for his benefit and an inspector wants to prevent the illegal behaviour of the violator.

Their research was extended to two types of problems. One is the problem of the arms-reduction treaty, including the international inspection by the IAEA. We can count Canty *et al* (2001), Avenhaus and Canty (2005), Avenhaus and Kilgour (2004), Avenhaus *et al* (1996) and Hohzaki (2007) as contributors to this type of problem. Avenhaus *et al* (1996) survey past studies on compliance with regulations and treaties. Canty *et al* (2001) analyse a sampling inspection problem for nuclear materials by a sequential game model and propose an efficient inspection strategy to induce an inspectee to comply with the Treaty on the Non-Proliferation of Nuclear Weapons or related treaties. Avenhaus and Canty (2005) analyse the influence of two types of errors on effective inspection strategies by a sequential game model. Avenhaus and Kilgour (2004) discuss a non-zero-sum one-shot game with an inspector and two inspectee countries, where the inspector distributes his inspection resource to two countries in an effective manner and each inspectee decides legal or illegal action for his interest in an egoistic manner. Hohzaki (2007) also studies the distribution strategy of inspection resource to many inspectees and derives an optimal dispatching plan of inspection staffs to facilities in the inspectee countries.

The other branch from Dresher's research is the smuggling game with a smuggler and Customs. Thomas and Nisgav (1976) deal with a smuggling game, where one or two patrol boats are available for Customs, and propose a numerical algorithm to repeatedly solve a one-stage matrix game step by step. Baston and Bostock (1991) give a closed form of equilibrium for a game similar to the model of Thomas and Nisgave. Many researchers adopt the so-called perfect-capture assumption that Customs certainly captures the smuggler when both players meet.

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But Baston and Bostock model the problem into the imperfect-capture model and succeed in solving the game by introducing the capture probability depending on the number of patrol boats. But the smuggler is assumed to have at most one opportunity to ship contraband. Garnaev (1994) extends their work to a model with three patrol boats. Sakaguchi (1994) first introduced the assumption that the smuggler might take an action several times in the perfect-capture model. Ferguson and Melolidakis (1998) is an extension of the Sakaguchi model, assuming that the smuggler can get rid of the capture by means of side payment but he must pay penalty on his capture. Hohzaki *et al* (2006) and Hohzaki (2006) also extend Sakaguchi's model such that the capture of the smuggler terminates the game and the encounter of the smuggler and Customs stochastically results in capture, success of smuggling or nothing.

All past researches we survey above have treated a two-choice strategy of smuggling or non-smuggling as a smuggler's strategy. However, we can think of more natural smuggling strategy: how much contraband should be shipped at any one time. We have a few reports on the theme such as Hohzaki (2011) but the analytical study on the amount of contraband has just started. Other than the theme, information acquisition is a crucial issue to a general model of the game. The concept of complete information or incomplete information was proposed by Harsanyi (1967) and it has been applied to a huge variety of game models. In the smuggling game, players obtain information about their opponents in different ways. Customs is a public organization, but the smuggler group is a secret society in general. Therefore, the behaviour of Customs is comparatively clear and open to outside, but the smuggler keeps his action or his plan deeply in secret. From the situation, the information acquisition must be asymmetrical between players. In the past models, they never thought of the asymmetrical information. They assume that players come to know the behaviour or action their opponent took in the past or that the information about their opponents is perfectly hidden to competitors (Hohzaki and Maehara, 2010). In both models, the symmetry of information acquisition is kept between players. In this paper, we regard a smuggling game as a multi-stage game with incomplete information. The purpose of this paper is to introduce the asymmetry of information of players and evaluate the value of the information.

In the next section, we model our smuggling problem as a multi-stage game with incomplete information and formulate it by elucidating the pay-offs both players can recognize and showing the difference between them. In Section 3, we solve the optimization problem formulated in Section 2 and propose an algorithm to derive a perfect Bayesian equilibrium. In Section 4, we apply the proposed algorithm to a small size of problem to analyse an equilibrium point or optimal strategies of players. After that, we do a sensitivity analysis by a variety of parameter settings, by which we elucidate the characteristics of the optimal strategy.

2. Modelling and formulation

We consider the following multi-stage two-person zero-sum game played by Customs and a smuggler.

A1. Customs and a smuggler take an action per day and play the game during T days. We define the stage of the game by the residual days until the last day T.

A2. Customs has K chances of patrolling at most, but excess chances beyond the residual days are lost in the case of $K>T$. The smuggler initially has $M>0$ contraband and wants to smuggle them as much as possible.

A3. Customs chooses one of patrol (P) or non-patrol (NP), and the smuggler determines the amount of contraband to ship from x contraband remaining in his hand at the present stage.

A4. If the smuggler tries to smuggle y $(0 \leqslant y \leqslant x)$ contraband and Customs incidentally patrols, Customs can capture the smuggler with probability $p_1(y) \geqslant 0$ but the smuggler succeeds in smuggling with probability $p_2(y)$. Only one of the capture or the success of smuggling is possible and they are exclusive to each other, namely, $p_1(y)+p_2(y)=1$. The functions $p_1(y)$ and $p_2(y)$ are monotone non-decreasing and monotone non-increasing, respectively, and they satisfy the following boundary conditions.

$$p_1(0)=0,\ p_2(0)=1 \qquad (1)$$

A5. The success of smuggling gives the smuggler reward 1 per contraband and the capture of the smuggler brings $\alpha>0$ for Customs. The pay-off of the game is zero-sum, that is, the reward of Customs is the loss of the smuggler and vice versa. A discount rate $\beta(\leqslant 1)$ is set between the rewards of sequent two days. We define the pay-off by the reward of Customs.

A6. If the capture of the smuggler does not occur, the game transfers to the next day. Both players recognize the happening of the capture at the end of each stage. After the transition, the smuggler is informed of the previous strategy of Customs but Customs does not know which strategy the smuggler took on the previous day. Therefore, Customs does not have any information about the reward the smuggler gets in the process of the game, as well as the smuggler's strategy.

A7. The game ends if the capture of the smuggler occurs or the game reaches the last day T or the last stage 1. Customs behaves as a maximizer and the smuggler as a minimizer for the pay-off of the game.

A state at the beginning of the stage is represented by a triplet (n, k, x) of the number of stage n, the number of residual chances for patrolling k and the amount of residual contraband of the smuggler x. Only the smuggler recognizes the state but Customs does not. What Customs knows is n and k, and he anticipates x by his belief. Let us denote the belief by $\{q_n(x), x=0, 1, \ldots, M\}$, where $q_n(x)$ is the probability that the smuggler has x contraband at the current stage n and condition $\sum_{x=0}^{M} q_n(x)=1$ holds.

Let us confirm the strategies of players in the case of $k>0$. We denote a mixed strategy of Customs by $\pi=(\pi_1, \pi_2)$, where π_1 or π_2 is the probability of patrol (P) or no-patrol (NP), respectively. A mixed strategy of the smuggler with x contraband in hand is denoted by $\rho_x=(\rho_x(0), \rho_x(1), \ldots, \rho_x(x))$. $\rho_x(y)$ is the probability of taking a pure strategy $S(y)$, which is the smuggling of y contraband. The smuggler knows his present amount of contraband, x, and he is able to change his mixed strategy ρ_x adaptively to amount $x=0, \ldots, M$. Therefore, the whole set of the smuggler's strategies is $\rho=\{\rho_x, x=0, \ldots, M\}$. The feasible regions of their strategies are

$$\Pi \equiv \{(\pi_1, \pi_2) | \pi_1 + \pi_2 = 1, \pi_1, \pi_2 \geqslant 0\}$$

for the Customs' strategy π and

$$\Omega_x \equiv \left\{ \rho_x(y), y \in I_x | \sum_{y=0}^{x} \rho_x(y) = 1, \rho_x(y) \geqslant 0, y \in I_x \right\} \tag{2}$$

$$\Omega \equiv \{\{\rho_x, x \in I_M\} | \rho_x \in \Omega_x, x \in I_M\}$$

for the smuggler's strategy ρ, using notation I_x as a set of sequential integers $I_x=\{0, 1, \ldots, x\}$ for a positive integer x. The feasible region Ω of ρ has individual regions separated for x.

Let $w(n, k, x; q_n)$ be the expected pay-off of the smuggler, which is determined by sequentially rational and optimal strategies of both players after the stage n provided that Customs has belief q_n in the current state (n, k, x). We can say that the smuggler faces a decision making with the following pay-off matrix $W(n, k, x)$:

$$W(n,k,x) = \begin{array}{c} \\ P \\ NP \end{array} \begin{array}{cc} S(0) & S(y) \\ \left(\begin{array}{c} \beta w(n-1, k-1, x; \Gamma_P(q_n)) \\ \beta w(n-1, k, x; \Gamma_N(q_n)) \end{array} \right. & \left. \begin{array}{c} \alpha p_1(y) - y p_2(y) + (1-p_1(y)) \beta w(n-1, k-1, x-y; \Gamma_P(q_n)) \\ -y + \beta w(n-1, k, x-y; \Gamma_N(q_n)) \end{array} \right) \end{array} \tag{3}$$

Two rows correspond to the Customs' strategies, P and NP. There are originally $x+1$ columns corresponding to the smuggler's strategies, $S(0), \ldots, S(x)$, but we write only two columns for $S(0)$ (we sometimes use notation NS (No-smuggling) for this strategy) and $S(y)$ as their representative. We can explain the derivation of elements in the matrix as follows.

When Customs patrols (P) and the smuggler tries smuggling y contraband ($S(y)$), the expected pay-off $\alpha p_1(y) - y p_2(y)$ is born at the current stage and the game transfers to the next stage with probability $1-p_1(y)$. At the next stage, the allowed number of opportunities of future patrol decreases by one and the amount of contraband decreases by y. $\Gamma_P(q_n)$ is the belief that Customs revises from the current belief q_n by taking account of strategy P. For the second row of NP, Customs keeps the number of patrols but let the smuggler easily succeed in smuggling of strategy $S(y)$. The success brings Customs the loss of y and the game proceeds to the next stage in the state $(n-1, k, x-y)$. In this case, Customs revises his belief q_n to $\Gamma_N(q_n)$ by taking his strategy NP into account. Because elements in the first column are given by applying $y=0$ to elements in the second column for strategy $S(y)$, we take the elements in the second column as representative elements for a general discussion and denote them by

$$R_x(P, y) \equiv \alpha p_1(y) - y p_2(y) + (1-p_1(y)) \beta w(n-1, k-1, x-y; \Gamma_P(q_n)), \tag{4}$$

$$R_x(NP, y) \equiv -y + \beta w(n-1, k, x-y; \Gamma_N(q_n)). \tag{5}$$

Knowing n, k and x, the smuggler recognizes the pay-off matrix $W(n, k, x)$ and behaves as a minimizer to minimize the expected pay-off

$$\sum_{y=0}^{x} \rho_x(y)(\pi_1 R_x(P, y) + \pi_2 R_x(NP, y)) \tag{6}$$

adaptively to his competitor's strategy π.

We are going to discuss the pay-off of Customs who knows only n and k. Customs certainly knows an initial amount of contraband $x=M$, which the smuggler possesses at the initial stage $n=T$. At a general stage n, Customs anticipates the pay-off matrix of Equation (3) with probability $q_n(x)$. The anticipation implies that $M-x$ contraband have been already shipped. Although the present value of the pay-off concerning the shipment has already been lost, Customs has to manage a present strategy to minimize the pay-off that would be arisen at present and in the future from now. Customs expects the total pay-off from the present until the end of the game by

$$R(\pi, \rho; q_n) \equiv \sum_{x=0}^{M} q_n(x) \sum_{y=0}^{x} \rho_x(y) \times (\pi_1 R_x(P, y) + \pi_2 R_x(NP, y)) \tag{7}$$

knowing that the smuggler can change his strategy ρ corresponding to the amount of his contraband x.

3. Algorithm for the derivation of equilibrium point

The goal of this section is to develop a theory for the derivation of a perfect Bayesian equilibrium of the game with incomplete information, which we formulated in the previous section. Let us start with a discussion about the present value of the expected pay-off of Customs at Stage n, given by Equation (7). We say again that Customs is a maximizer who changes π adaptively to his adversary's strategy to maximize the pay-off. Noting that the expected pay-off of Equation (6) is the same as the expression after $\sum_y$ in Equation (7), the minimization of the expression (6) given x is equivalent to the minimization of the pay-off $R(\pi, \rho; q_n)$ with respect to variable ρ_x. As a result, we can make use of the general procedure for the solution of a two-person zero-sum matrix game to derive a perfect Bayesian equilibrium for our game. That is, we can derive an optimal strategy of Customs π^* by a maximin optimization of $R(\pi, \rho; q_n)$ and an optimal strategy of the smuggler $\rho^* = \{\rho_x^*, x = 0, \ldots, M\}$ by a minimax optimization of the pay-off.

Given the optimal strategy π^*, the optimized value $w(n, k, x; q_n)$ after stage n in the state (n, k, x) is calculated by

$$w(n,k,x;q_n) = \sum_{y=0}^{x} \rho_x^*(y)(\pi_1^* R_x(P,y) + \pi_2^* R_x(NP,y)) = \min_{y=0,\ldots,x} \{\pi_1^* R_x(P,y) + \pi_2^* R_x(NP,y)\}. \quad (8)$$

We are going to obtain a perfect Bayesian equilibrium, $\pi^*$and ρ^*, by the maximin or minimax optimization of the pay-off $R(\pi, \rho; q_n)$.

From feasible region (2), we transform $\min_{\rho \in \Omega} R(\pi, \rho; q_n)$ as follows.

$$\min_{\rho_x \in \Omega_x, x \in I_M} R(\pi, \rho; q_n) = \sum_{x=0}^{M} q_n(x) \min_{y \in I_x} \{\pi_1 R_x(P,y) + \pi_2 R_x(NP,y)\}$$

Because the summation with respect to x in the right-hand side is in effect only for $x \in Q_n^+ = \{x \in I_M | q_n(x) > 0\}$, the maximization problem of the expression above is formulated by the following linear programming problem, by which we can derive an optimal patrol plan π^*.

$$(P_P) \quad \max_{\pi_1, \pi_2, \{\mu_x, x \in Q_n^+\}} \sum_{x \in Q_n^+} q_n(x) \mu_x$$

$$\text{s.t.} \quad \pi_1 R_x(P,y) + \pi_2 R_x(NP,y) \geqslant \mu_x, y \in I_x, x \in Q_n^+,$$

$$\pi_1 + \pi_2 = 1, \pi_1, \pi_2 \geqslant 0$$

For comprehensibility, we show you another formulation by substituting $R_x(P,y)$ and $R_x(NP,y)$ with definitions (4) and (5):

$$(P_P) \quad \max_{\pi_1, \pi_2 \{\mu_x, x \in Q_n^+\}} \sum_{x \in Q_n^+} q_n(x) \mu_x$$

$$\text{s.t.} \quad \pi_1 \{\alpha p_1(y) - y p_2(y) + (1 - p_1(y)) \times w(n-1, k-1, x-y; \Gamma_P(q_n))\} + \pi_2 \{-y + w(n-1, k, x-y; \Gamma_N(q_n))\} \geqslant \mu_x, y = 0, \ldots, x, x \in Q_n^+, \quad (9)$$

$$\pi_1 + \pi_2 = 1, \quad \pi_1, \pi_2 \geq 0. \quad (10)$$

From Equation (8), optimal multiplier μ_x^* of (P_P) is nothing but $w(n, k, x; q_n)$ for $x \in Q_n^+$.

$$w(n,k,x;q_n) = \mu_x^*, x \in Q_n^+. \quad (11)$$

For $x \notin Q_n^+$, we have to calculate $w(n, k, x; q_n)$ from the minimization problem (8) and an optimal smuggler's strategy $\{\rho_x^*(y), y \in I_x\}$ by setting $\rho_x^*(y^*) = 1$ for an optimal y^* of the problem. The case of $x \notin Q_n^+$ implies that the state (n, k, x) is off the equilibrium path for belief q_n and any smuggler's strategy has no effect on the value of the game. Even in such a case, we have to show a rational strategy of the smuggler as the perfect Bayesian equilibrium.

Now let us discuss the minimax problem of the pay-off $R(\pi, \rho; q_n)$. By transforming the maximization of $R(\pi, \rho; q_n)$ with respect to $\pi \in \Pi$ into

$$\max_{\pi \in \Pi} R(\pi, \rho; q_n) = \max \left\{ \sum_{x=0}^{M} \sum_{y=0}^{x} \rho_x(y) q_n(x) R_x(P,y), \sum_{x=0}^{M} \sum_{y=0}^{x} \rho_x(y) q_n(x) R_x(NP,y) \right\}$$

the minimization problem of the expression above is formulated by

$$(P_S) \quad \min_{\{\rho_x(y), y \in I_x, x \in Q_n^+\}, \lambda} \lambda$$

$$\text{s.t.} \quad \sum_{x \in Q_n^+} \sum_{y=0}^{x} \rho_x(y) q_n(x) R_x(P,y) \leqslant \lambda,$$

$$\sum_{x \in Q_n^+} \sum_{y=0}^{x} \rho_x(y) q_n(x) R_x(NP,y) \leqslant \lambda,$$

$$\sum_{y=0}^{x} \rho_x(y) = 1, x \in Q_n^+,$$

$$\rho_x(y) \geqslant 0, \ y = 0, \ldots, x, x \in Q_n^+.$$

We can show another formulation using original definitions of $R_x(P, y)$ and $R_x(NP, y)$, as follows.

$$(P_S) \quad \min_{\{\rho_x(y), y \in I_x, x \in Q_n^+\}, \lambda} \lambda$$

$$\text{s.t.} \quad \sum_{x \in Q_n^+} \sum_{y=0}^{x} \rho_x(y) q_n(x) \{\alpha p_1(y) - y p_2(y) + (1 - p_1(y)) w(n-1, k-1, x-y; \Gamma_P(q_n))\} \leqslant \lambda,$$

$$\sum_{x \in Q_n^+} \sum_{y=0}^{x} \rho_x(y) q_n(x) \{-y + w(n-1, k, x-y; \Gamma_N(q_n)) \leqslant \lambda,$$

$$\sum_{y=0}^{x} \rho_x(y) = 1, x \in Q_n^+, \rho_x(y) \geq 0, y = 0, \ldots, x, x \in Q_n^+.$$

An optimal strategy of the smuggler $\{\rho_x^*(y), y = 0, \ldots, x\}$ is given by solving the problem (P_S) for $x \in Q_n^+$ and by the evaluation of the minimization of (8) for $x \notin Q_n^+$, in the same way as explained before.

There is a relationship of duality between Problem (P_P) and (P_S), as known for the solution of a general finite matrix game. In the concrete, there is a relation $\rho_x(y) = v_x(y)/q_n(x)$ between dual variable $v_x(y)$ corresponding to constraint (9) and variable ρ_x in Problem (P_S). A dual variable corresponding to condition (10) becomes variable λ in Problem (P_S). As the problems (P_P) or (P_S) possibly have multiple optimal solutions, we cannot say that the Bayesian equilibrium is uniquely determined. Nevertheless, we discuss the uniqueness in the Conclusion section later.

Now we are going to discuss how to revise the belief of Customs or calculate operators Γ_N and Γ_P on belief q_n. If Customs does not patrol, the capture of the smuggler never occurs and the game never ends. If the smuggler has s contraband in his hand at the beginning of stage n and keeps x at the next stage $n-1$, he must have smuggled $s-x$ contraband at stage n. The probability of these actions is $q_n(s)\rho_s^*(s-x)$. If Customs patrols and the smuggler keeps x contraband at the next stage $n-1$ without being arrested, the probability of these events is evaluated by $q_n(s)\rho_s^*(s-x)(1-p_1(s-x))$ conditioned by no-capture. Now we have the following formulas to compute operator $\Gamma_N(q_n)$ by no patrol and $\Gamma_P(q_n)$ by patrol.

$$\Gamma_N(q_n)(x) = \sum_{s=x}^{M} q_n(s) \rho_s^*(s-x), \tag{12}$$

$$\Gamma_P(q_n)(x) = \frac{\sum_{s=x}^{M} q_n(s) \rho_s^*(s-x)(1 - p_1(s-x))}{\sum_{z=0}^{M} \sum_{s=z}^{M} q_n(s) \rho_s^*(s-z)(1 - p_1(s-z))} = \frac{\sum_{s=x}^{M} q_n(s) \rho_s^*(s-x)(1 - p_1(s-x))}{\sum_{s=0}^{M} q_n(s) \sum_{u=0}^{s} \rho_s^*(u)(1 - p_1(u))} \tag{13}$$

At the beginning of the game or at stage $n = T$, Customs has an initial belief

$$q_T(M) = 1,\ q_T(x) = 0,\ M \neq x \in I_M. \tag{14}$$

In a special case of $k = 0$, Customs has only a strategy NP to choose and then we have $w(n, k, x; q_n) = -x$ for any $n > 0$, x and q_n, given by an immediately smuggling strategy.

Let us recall the procedure of computing the perfect Bayesian equilibrium discussed so far. We assume that we have evaluated every value with index $n-1$, such as $w(n-1, \cdot)$, at stage $n-1$. To solve Problem (P_P) or (P_S) for optimal strategy π^* or $\rho_x^*(y)$, we need the revised belief $\Gamma_P(q_n)$ or $\Gamma_N(q_n)$ involved in $w(n-1, \cdot)$. However, to have belief q_n revised by Equations (13) or (12), we need optimal strategy $\rho_x^*(y)$, which is given by solving the problem (P_S). Now we see that the computation for the solution forms a closed loop of the requirements and the equilibrium point is difficult to be derived. That is why we need an approximation algorithm to calculate the perfect Bayesian equilibrium.

To explicitly indicate that optimal strategies $\pi^*, \rho_x^*(y)$ and optimized value $w(n, k, x; q_n)$ are calculated using revised belief $\Gamma_P(q_n)$ and $\Gamma_N(q_n)$ embedded in $w(n-1, \ldots; \Gamma_P(q_n))$ and $w(n-1, \ldots; \Gamma_N(q_n))$, we write $\pi^*(\Gamma_P(q_n), \Gamma_N(q_n))$, $\rho_x^*(y{:}\Gamma_P(q_n), \Gamma_N(q_n))$ and $w(n, k, x; q_n{:}\Gamma_P(q_n), \Gamma_N(q_n))$. The next step for the approximation is to discretize belief $\{q_n(x), x = 0, \ldots, M\}$ such that $q_n(x) \in \Phi \equiv \{0, 1/m, 2/m, \ldots, m/m\}$. Because a discretized belief belongs to a product set $\Phi \times \Phi \times \cdots \Phi = \Phi^{M+1}$ and satisfies condition $\sum_{x=0}^{M} q_n(x) = 1$, the whole set of possible belief has $\binom{M+m}{m}$ elements in it. Now we are ready to describe an approximation algorithm to derive a perfect Bayesian equilibrium.

(S1) Initialize $w(n, k, x; q_n)$ to be 0 for any k, x and $q_n \in \Phi^{M+1}$ at stage $n = 0$ and set $n = 1$.

(S2) If $n = T$, set $k = K$ and make the belief be initialized by $q_T(M) = 1, q_T(x) = 0 (x \neq M)$. Go to (S3) and stop. Obtained $w(N, K, M{:}q_T)$ is the value of the game.
If $n \neq T$, execute (S3) for all $k \in \{0, 1, \ldots, K\}$ and $q_n \in \Phi^{M+1}$.

(S3) By doing (S4) for any q', $q'' \in \Phi^{M+1}$ at stage $n-1$, calculate $\pi^*(q', q'')$, $\rho_x^*(y{:}q', q'')$, $w(n, k, x; q_n{:}q', q'')$, $\Gamma_P(q_n)(x)$ and $\Gamma_N(q_n)(x)$. If $q' = \Gamma_P(q_n)$ and $q'' = \Gamma_N(q_n)$, save $\pi^*(q', q'')$ as an optimal strategy of Customs for information set of Customs $(n, k; q_n)$, $\rho_x^*(y{:}q', q'')$ as an optimal strategy of the smuggler $\rho_x^*(y)(y \in I_x, x \in I_M)$ for information set of the smuggler $(n, k, x; q_n)$ and $w(n, k, x; q_n{:}q', q'')$ as value $w(n, k, x; q_n)$. Otherwise, if the conditions above do not hold for any q', $q'' \in \Phi^{M+1}$, there is no equilibrium point on any path branching from the information set $(n, k; q_n)$.

(S4) Solve Problem (P_P) and (P_S) after substituting $w(n-1, \ldots; \Gamma_P(q_n))$ and $w(n-1, \ldots; \Gamma_N(q_n))$ with $w(n-1, \ldots; q')$ and $w(n-1, \ldots; q'')$, respectively, and obtain their optimal solutions $\pi^*(q', q'')$, $\rho_x^*(y{:}q', q'')$ and $w(n, k, x; q_n{:}q', q'')$. Revise belief q_n to $\Gamma_P(q_n)$ and $\Gamma_N(q_n)$ by substituting $\rho_x^*(y{:}q', q'')$ and q_n into formula (13) and (12) for $x \in I_M$. Then discretize the revised belief as follows. Depending on which one of the periods $[0, (1/2m)), [(1/2m), (3/2m)), \ldots, [((2k-1)/2m), ((2k+1)/2m)), \ldots, [((2m-1)/2m), 1]$ the calculated value belongs to, give the belief one of values 0, $(1/m), \ldots, (k/m), \ldots, 1$. The resultant values $\Gamma_P(q_n)$ and $\Gamma_N(q_n)$ must be in set Φ^{M+1}.

(S5) Set $n = n+1$. If $n \leqslant T$, go back to (S2). Otherwise, terminate the algorithm.

4. Numerical examples

First, we compare two cases. In Case 1, the probability of capture $p_1(y)$ is high for $y=1$ and increases approximately linearly to y larger than 1. In Case 2, the probability $p_1(y)$ is low for $y=1$, but increases in an exponential form for $y \geqslant 2$. We set $m=5$ for the discretization of belief in Step (S4) and $\beta=1$, meaning no discount for future's pay-off. We also set $\alpha=4$. In Table 1, we show the setting of $p_1(y)$ and the success probability of smuggling $p_2(y)$, where $p_1(y)+p_2(y)=1$. The last row of the table is for the tentative expected pay-off $\alpha p_1(y) - y p_2(y)$ on the coincidence of patrol and smuggling. Because the tentative pay-off is non-negative and increases by y in Case 1, the smuggler would not dare to smuggle myopically when the patrol is anticipated.

The parameter setting of Case 2 is given by $\alpha=2$ and Table 2. Even though Customs patrols, the smuggling of $y=1$ contraband gives the smuggler an incentive of action.

Table 1 Probability of capture, success probability of smuggling and tentative payoff (Case 1)

Y	0	1	2	3	4	5
$p_1(y)$	0	0.34	0.45	0.56	0.658	0.736
$P_2(y)$	1	0.66	0.55	0.44	0.342	0.264
$\alpha p_1(y) - y p_2(y)$	0	0.699	0.700	0.921	1.264	1.622

Table 2 Probability of capture, success probability of smuggling and tentative payoff (Case 2)

y	0	1	2	3	4	5
$p_1(y)$	0	0.1	0.45	0.7	0.8	0.85
$p_2(y)$	1	0.9	0.55	0.3	0.2	0.15
$\alpha p_1(y) - y p_2(y)$	0	−0.7	−0.2	0.5	0.8	0.95

In the next section, we discuss the characteristics of Bayesian equilibrium by comparing Cases 1 and 2 with specific parameters $(T, K, M) = (3, 2, 2)$. In Section 4.2, we change a triplet (T, K, M) of the number of stages, the number of opportunities for patrol and the initial amount of contraband of the smuggler to obtain the optimal strategy and the value of the game in Cases 1 and 2. Other than the examples, we take another game with complete information, where players get information about the strategies their opponent took at the previous stage and then recognize the present state (n, k, x), and quantitatively analyse the value of information by comparing the two kinds of games. Finally, we analyse the effect of discounting the pay-off on equilibrium by changing the discount rate β for Case 1 in Section 4.3.

4.1. Equilibrium in the case of $(T, K, M) = (3, 2, 2)$

We show an equilibrium of Case 1 on the game tree in Figure 1. Figure 2 is for Case 2. A black-coloured origin represents initial information set (N, K, M). The origin has two branches, which represent two pure strategies of Customs, $\{P, NP\}$, with two white nodes. On the white node, where the smuggler has x contraband in hand, the smuggler has to choose one of $x+1$ options or pure strategies without knowing the selection of Customs. These branching generate all combinations of strategies of two players in the same turn at a stage. In the figures, we abbreviate the smuggler's strategy $S(y)$ to S_y for simplicity. Customs makes his decision by anticipating the amount of contraband in the smuggler's hand based on the history of his strategies and no occurrence of the capture of the smuggler in the past. On the other hand, the smuggler chooses his strategy unknowing of his opponent's choice. The player cannot distinguish an individual node from the others in the same information set denoted by an oval. As a symbol of information set, we also use a line connecting nodes at the last stage $n=1$ for simplicity. In the oval or beside the line, we write recognizable state of the information set for each player: (n, k) for Customs and (n, k, x) for the smuggler.

At Stage $n=3$, we draw a triangle and a square, which symbolize the capture of the smuggler and the fork between capture and non-capture, respectively, and possibly occur on the path with the combination of patrol and smuggling. However, a branch with the capture is substituted with a broken line at the later stages, for simplicity. The game at the fork or the square steps forward to the triangle with probability $p_1(y)$ and goes to another branch with probability $p_2(y) = 1 - p_1(y)$. This branch comes from chance move by natural stochastic law. Because optimal strategies of players are evident at the last stage $n=1$, we omit branches other than the equilibrium path. For example, strategy P of Customs is always optimal in the case of $n=k$. In the same situation

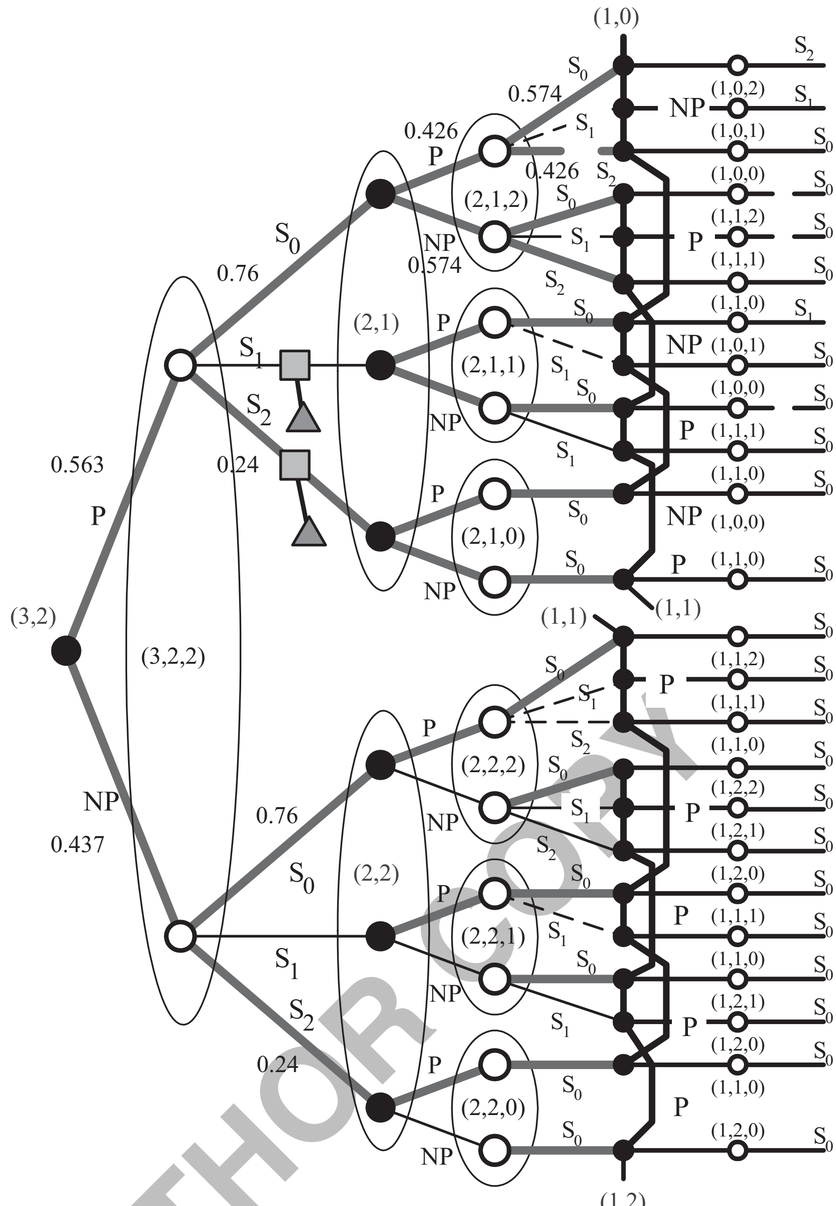

Figure 1 Game tree and equilibrium (Case 1).

of $n = k$, the smuggler must choose S_0 in Case 1 and S_1 in Case 2 against his opponent's strategy P, as seen from Tables 1 and 2.

To illustrate the equilibrium path, we make a branch or a pure strategy with positive selection probability bold. Only a bold branch from an information set means that the optimal selection probability of the pure strategy is one. When a mixed strategy is optimal, we add the optimal probability of taking the branch beside it. In the same information set, the same number of branches goes out of every node and an optimal mixed strategy for the branches from a node has to be applied to the other nodes, as we know about the game tree. In Figure 1, two branches $\{P, NP\}$ from every node in the information set $(2, 1)$ have their probabilities $\{0.426, 0.574\}$. In a similar manner, we can see that optimal mixed strategies at Stage 3 are probabilities $\{0.563, 0.437\}$ for Customs' strategy $\{P, NP\}$ and $\{0.76, 0, 0.24\}$ for the smuggler's strategy $\{S_0, S_1, S_2\}$ in Case 1, and $\{0.667, 0.333\}$ for $\{P, NP\}$ and $\{0.423, 0.577, 0\}$ for $\{S_0, S_1, S_2\}$ in Case 2. Below we describe the characteristics of optimal strategy in Cases 1 and 2.

Case 1

(1) The smuggler attitude is passive because of high capture probability. Especially, no-smuggling strategy is myopically optimal for the smuggler when Customs patrols. As seen from Figure 1, Customs always takes

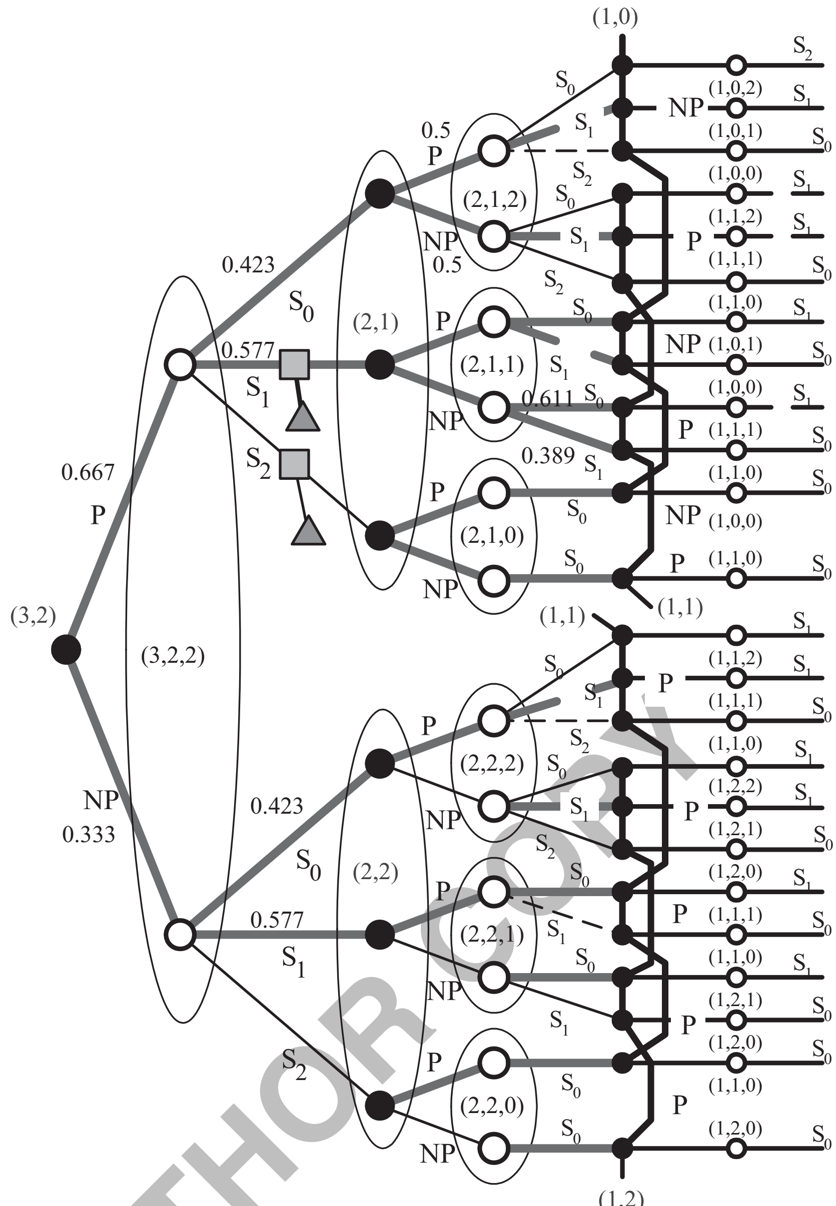

Figure 2 Game tree and equilibrium (Case 2).

strategy $Pr(P)=1$ and the smuggler never tries to smuggle on the equilibrium path after information set $(2,2)$ of $n=k$. Customs could take NP in $(3,2)$ or $(2,1)$. On the basis of the estimation, the smuggler mixes the strategies S_2 and S_0 at $(3,2,2)$ and $(2,1,2)$ because strategy S_2 brings him a little larger reward than S_1 against NP but approximately the same reward $\alpha p_1(y)-yp_2(y)$ against P. The selection probability of S_0 is larger than that of S_2. Concerning the ratio of the number of allowed patrols to the number of residual stages, k/n, state $(3,2,2)$ is larger than $(2,1,2)$, and then the optimal probability of no-smuggling in the former state is larger than that in the latter one.

(2) Customs estimates the passive behaviour of the smuggler by the reason explained above and then makes patrol probability $Pr(P)$ in $(3,2)$ and $(2,1)$ lower in Case 1 than in Case 2, where the active behaviour of the smuggler is estimated, as we see later. On the basis of the optimal selection of S_0 and S_2 by the smuggler at initial information set $(3,2,2)$, Customs has the belief that $q_2(2)=4/5$, $q_2(1)=0$, $q_2(0)=1/5$ at $(2,1)$, that is, the guess that the smuggler would have 2 contraband or nothing remaining at the present. Because any strategy of patrol has no effect on no-contraband state of the smuggler $(2,1,0)$ at Stage 2, $Pr(P)=0.426$ is the Customs' behaviour optimally corresponding to the

2-contraband state $(2, 1, 2)$ without worrying about the state $(2, 1, 0)$. It would be helpful that we show the result by the complete-information game (CIG) with the same setting of parameter as our incomplete-information game (ICIG), where Customs recognizes state (n, k, x) of stage n, the residual number of allowed patrols k and the amount of residual contraband of the smuggler x. According to the result of the CIG, the probability of patrol in the state $(n, k, x) = (2, 1, 2)$, $(2, 1, 1)$ and $(2, 1, 0)$ is 0.426, 0.371 and 0, respectively. Optimal patrol strategy in the state $(2, 1)$ of the ICIG is the same as that in the state $(2, 1, 2)$ of the CIG. Therefore, optimal smuggling strategy in initial state $(3, 2, 2)$, $(Pr(S_0), Pr(S_1), Pr(S_2)) = (0.76, 0, 0.24)$, is the same for both the ICIG and the CIG. Thus, Customs, who selects strategy P at initial stage, has the perfect anticipation about the smuggler's optimal strategy since then and makes an optimal plan of patrolling in $(3, 2)$ of the ICIG as same as in $(3, 2, 2)$ of the CIG, that is, $(Pr(P), Pr(NP)) = (0.563, 0.437)$. Customs who selects strategy NP at the initial stage keeps strategy P at all stages since then, which dominates any other patrol strategy, and therefore he does not need any information about the smuggler's behaviour. In result, patrol strategy in the initial state $(3, 2)$, $Pr(P) = 0.563$, in the ICIG is the same as in the CIG and the value of the ICIG, $w(3, 2, 3; q_3) = -0.479$, entirely equal the value of the CIG.

Case 2

(1) In this case, the smuggler gains some reward by smuggling strategy S_1 and S_2 even when Customs patrols and then he would have an active behaviour. That is why the probability of patrolling in information set $(3, 2)$ and $(2, 1)$ in Case 2 is larger than in Case 1. By optimal smuggling strategy of taking S_0 or S_1 in $(3, 2)$, Customs has his belief $q_2(2) = 2/5$, $q_2(1) = 3/5$, $q_2(0) = 0$ in $(2, 1)$. The belief leads him to the estimation that the game steps forward to states $(2, 1, 2)$ or $(2, 1, 1)$ with positive probability.

(2) In the state $(2, 1, 2)$, the smuggler had better smuggle one contraband (S_1) at sequent stages $n = 2$ and 1 such that he can expect some reward against the patrol by Customs. After state $(2, 1, 1)$, he optimally smuggles once by sequential strategies (S_0, S_1) or (S_1, S_0) during two stages. Taking account of the possibility, Customs takes strategy $Pr(P) = 0.5$, which is the same as optimal patrol strategy in the state $(2, 1, 1)$ of the CIG. In the state $(2, 1, 1)$, the same pay-off $w(2, 1, 1; q_2) = -0.85$ is brought for both the ICIG and the CIG. In $(2, 1, 0)$, the pay-off is zero for both games, of course.

Let us confirm again that if Customs selects NP in state $(3, 2)$, he should always patrol (P) afterwards and that he never needs any information about his opponent. Optimal pay-offs in states $(2, 2, 2)$, $(2, 2, 1)$ and $(2, 2, 0)$, which are reached by the strategy NP from $(3, 2)$, are -1.33, -0.7 and 0, respectively, for both the CIG and the ICIG.

At Stage $n = 2$, there is only the difference between the ICIG and the CIG in the information set $(2, 1, 2)$. The belief of $q_2(2) = 2/5$ and $q_2(1) = 3/5$ makes Customs estimate the transition of the game from the initial state $(3, 2, 2)$ to $(2, 1, 2)$ with probability 2/5. The patrol strategy $Pr(P) = 0.5$ is optimal in $(2, 1, 1)$ but not in $(2, 1, 2)$. Therefore, the expected pay-off in this state is $w(2, 1, 2; q_n) = -1.65$ for the ICIG, which is a little smaller than the pay-off for the CIG -1.629.

Even in the ICIG, Customs knows the initial state $(3, 2, 2)$ at the first stage $n = 3$ and then recognizes his pay-off matrix, which is put in the upper position below, by deliberating on the rational strategies coming in future, as explained above.

$$\begin{array}{c} \\ P \\ NP \end{array} \begin{array}{c} \begin{array}{ccc} S_0 & S_1 & S_2 \end{array} \\ \begin{pmatrix} -1.65 & -1.465 & -0.2 \\ -1.33 & -1.7 & -2 \end{pmatrix} \end{array}$$

$$\begin{array}{c} \\ P \\ NP \end{array} \begin{array}{c} \begin{array}{ccc} S_0 & S_1 & S_2 \end{array} \\ \begin{pmatrix} -1.629 & -1.465 & -0.2 \\ -1.33 & -1.7 & -2 \end{pmatrix} \end{array}$$

The matrix is generated in such a way that $w(2, 1, 2; q_2) = -1.65$ is filled up in $(1, 1)$-entry for the combination of strategy P and S_0, for example. By solving the matrix game, Customs concludes that he would have the value of the game $w(3, 2, 2; q_3) = -1.543$ by an optimal mixed strategy of patrol $Pr(P) = 0.667$. On the other hand, the CIG has another pay-off matrix with -1.629 as $(1, 1)$-entry, mentioned above, which is the only different entry from the ICIG. The pay-off matrix for the CIG is put below the matrix of the ICIG. From the matrix, Customs obtains optimal strategy $Pr(P) = 0.714$ and the value of the game -1.537, which is a little larger than the ICIG.

There are some information sets such as $(n, k) = (2, 2)$, where Customs evidently does not need any information even in the ICIG. In Case 1, Customs uses those rational estimation about the smuggler's strategy and he is able to choose the best strategy without any information about his opponent's behaviour as if he knew, as we analyse before. On the other hand, a small shortage of rationality caused by no information in the process makes a difference between the ICIG and the CIG, and Customs is driven to take a little worse strategy in Case 2.

4.2. Equilibrium in other cases of (T, K, M)

Here we compare the CIG and the ICIG to evaluate the value of information. We solve the game for every

Table 3a Value of the game in Case 1 for the CIG (upper) and the ICIG (lower)

T	K	M = 1	2	3	4
2	1	−0.371	−0.851	−1.3	−1.727
		−0.371	−0.851	−1.3	−1.727
3	1	−0.541	−1.194	−1.814	−2.412
		−0.541	−1.194	−1.814	−2.412
	2	−0.179	−0.479	−0.747	−0.988
		−0.179	−0.479	−0.747	−0.988
4	1	−0.639	−1.379	−2.089	−2.78
		−0.639	−1.379	−2.089	−2.78
	2	−0.323	−0.797	−1.229	−1.629
		−0.323	−0.797	−1.229	−1.629
	3	−0.095	−0.301	−0.48	−0.632
		−0.095	−0.301	−0.48	−0.632

Table 4a Value of the game in Case 2 for the CIG (upper) and the ICIG (lower)

T	K	M = 1	2	3	4
2	1	−0.85	−1.621	−1.991	−2.307
		−0.85	−1.629	−1.991	−2.307
3	1	−0.9	−1.761	−2.557	−3.086
		−0.9	−1.767	−2.57	−3.09
	2	−0.8	−1.537	−2.166	−2.398
		−0.8	−1.543	−2.166	−2.398
4	1	−0.925	−1.823	−2.685	−3.488
		−0.925	−1.825	−2.7	−3.5
	2	−0.85	−1.652	−2.392	−3.017
		−0.85	−1.655	−2.416	−3.057
	3	−0.775	−1.487	−2.123	−2.649
		−0.775	−1.49	−2.147	−2.697

Table 3b Optimal probability of patrol at the initial stage in Case 1 for the CIG (upper) and the ICIG (lower)

T	K	M = 1	2	3	4
2	1	0.371	0.426	0.433	0.432
		0.371	0.426	0.433	0.432
3	1	0.27	0.299	0.302	0.302
		0.27	0.299	0.302	0.302
	2	0.483	0.563	0.575	0.572
		0.483	0.563	0.575	0.572
4	1	0.213	0.23	0.232	0.232
		0.213	0.23	0.232	0.232
	2	0.398	0.445	0.452	0.45
		0.398	0.445	0.452	0.45
	3	0.532	0.629	0.643	0.64
		0.532	0.629	0.643	0.64

Table 4b Optimal probability of patrol at the initial stage in Case 2 for the CIG (upper) and the ICIG (lower)

T	K	M = 1	2	3	4
2	1	0.5	0.714	0.364	0.357
		0.5	0.714	0.364	0.357
3	1	0.333	0.356	0.561	0.233
		0.333	0.333	0.574	0.243
	2	0.667	0.693	1	0.417
		0.667	0.667	1	0.421
4	1	0.25	0.258	0.29	0.44
		0.250	0.25	0.25	0.381
	2	0.5	0.512	0.577	0.9
		0.5	0.5	0.54	0.79
	3	0.75	0.76	0.839	1
		0.750	0.751	0.817	0.889

combination (T, K, M) of $T=2, \ldots, 4, K<T$ and $M=1, \ldots, 4$, and list the value of the game with the parameter setting of Case 1 for two kinds of games (the CIG and the ICIG) in Table 3a. Upper figures are for the CIG and lower ones for the ICIG. All figures coincide. Table 3b shows optimal patrol probability $Pr(P)$ at the initial stage in Case 1, which is the same for both the CIG and the ICIG. We take the example of $(T, K, M)=(3, 2, 2)$ in Case 1 and analyse why there is no difference between optimal strategies for the two games. As seen from Tables 3a and 3b, Customs can perfectly anticipate the optimal strategy of the smuggler even in the ICIG as if he acted in the CIG and loses nothing in all cases of (T, K, M). Now we conclude that the information Customs loses has no value in many cases.

From Table 3a, we can see the monotone decreasingness for the number of stages T and the amount of contraband M, and the monotone increasingness for the number of allowed patrols K as the characteristics of the value of the ICIG. Table 3b shows us the properties of optimal strategy that the patrol probability $Pr(P)$ decreases for larger T and increases for larger K. These properties are self-evident. The probability increases in a monotonic manner for $M=1$, 2 and 3 but decreases for larger M. Even if the smuggler can afford to smuggle more, he becomes less active in fear of the encounter with the patrol. The tendency on the smuggling strategy depresses the increase of $Pr(P)$.

In Case 2, Tables 4a and 4b show the value of the game and the optimal probability of patrol, respectively, for the CIG and the ICIG.

Customs with no information tends to hold more chances of patrol untouched at the earlier stages and keep them for use at the later stages in the ICIG, and the probability of patrol $Pr(P)$ at the initial stage is smaller than in the CIG in many cases. We cannot easily explain the sensitivity of $Pr(P)$ in Case 2, where the smuggler becomes more active than in Case 1. Let us take the following example. For the case $(3, 2, M)$ in the CIG, $Pr(P)$ changes to 0.667, 0.693, 1 and 0.417 for $M = 1, 2, 3$ and 4, respectively. We can explain the sensitivity as follows. For $M = 1, 2$, Customs increases $Pr(P)$ by his anticipation that the smuggler would take more active behaviour. In the case of $M = 3$, the smuggler can repeat the efficient smuggling of one contraband, S_1, through all stages and then Customs takes strategy $Pr(P) = 1$. By the same reason, $Pr(P)$ is comparatively high in the cases of $(T, M) = (2, 2), (4, 4)$. Customs focuses on the hitting of smuggling by patrolling for $M = 1 \sim 3$. For more M, Customs is interested mainly in the amount of contraband the smuggler tries to smuggle on the day of patrol, from the standpoint of the expected pay-off.

Customs wants to hit the smuggling with more contraband than S_1 by his patrol and decreases $Pr(P)$ a little. The probability $Pr(P)$ does not change monotonically for M more than 3.

4.3. Effect of discounting future's pay-off

Here, let us compare two versions of Case 1 with no-discount of $\beta = 1$ and with discount $\beta = 0.5$. Other parameters except β are fixed in the two versions. We already have Tables 3a and 3b as the results of the no-discount version. We compute equilibriums for the discount version of the game and compare two versions in terms of the value of the game in Table 5a. Tables 5b and 5c show the comparison in terms of optimal patrol strategy and optimal smuggling strategy, respectively.

Table 5a Value of the game for the no-discount version (upper) and the discount version (lower)

T	K	M = 1	2	3	4
2	1	−0.371	−0.851	−1.300	−1.727
		−0.227	−0.541	−0.830	−1.101
3	1	−0.541	−1.194	−1.814	−2.412
		−0.278	−0.638	−0.975	−1.295
	2	−0.179	−0.479	−0.747	−0.988
		−0.063	−0.182	−0.287	−0.379
4	1	−0.639	−1.379	−2.089	−2.780
		−0.290	−0.658	−1.003	−1.333
	2	−0.323	−0.797	−1.229	−1.629
		−0.089	−0.240	−0.374	−0.495
	3	−0.095	−0.301	−0.480	−0.632
		−0.018	−0.065	−0.106	−0.139

Table 5b Optimal patrol probability at the initial stage for the no-discount version (upper) and the discount version (lower)

T	K	M = 1	2	3	4
2	1	0.371	0.426	0.433	0.432
		0.455	0.541	0.553	0.551
3	1	0.270	0.299	0.302	0.302
		0.425	0.504	0.516	0.514
	2	0.483	0.563	0.575	0.572
		0.552	0.673	0.692	0.688
4	1	0.213	0.230	0.232	0.232
		0.418	0.497	0.509	0.507
	2	0.398	0.445	0.452	0.450
		0.536	0.652	0.670	0.666
	3	0.532	0.629	0.643	0.640
		0.578	0.717	0.738	0.733

Table 5c Optimal smuggling probability for $\{S_0, \ldots, S_M\}$ at the initial stage for the no-discount version (upper) and the discount version (lower)

T	K	M = 1	2	3	4
2	1	{0.629, 0.371}	{0.574, 0, 0.426}	{0.567, 0, 0, 0.433}	{0.568, 0, 0, 0, 0.432}
		{0.773, 0.227}	{0.73, 0, 0.270}	{0.723, 0, 0, 0.277}	{0.725, 0, 0, 0, 0.275}
3	1	{0.73, 0.27}	{0.701, 0, 0.299}	{0.698, 0, 0, 0.302}	{0.698, 0, 0, 0, 0.302}
		{0.815, 0.185}	{0.787, 0, 0.213}	{0.783, 0, 0, 0.217}	{0.784, 0, 0, 0, 0.216}
	2	{0.821, 0.179}	{0.76, 0, 0.24}	{0.751, 0, 0, 0.249}	{0.753, 0, 0, 0, 0.247}
		{0.937, 0.063}	{0.909, 0, 0.091}	{0.904, 0, 0, 0.096}	{0.905, 0, 0, 0, 0.095}
4	1	{0.787, 0.213}	{0.77, 0, 0.23}	{0.768, 0, 0, 0.232}	{0.768, 0, 0, 0, 0.232}
		{0.825, 0.175}	{0.799, 0, 0.201}	{0.795, 0, 0, 0.205}	{0.796, 0, 0, 0, 0.204}
	2	{0.824, 0.176}	{0.791, 0, 0.209}	{0.786, 0, 0, 0.214}	{0.787, 0, 0, 0, 0.213}
		{0.94, 0.060}	{0.922, 0, 0.078}	{0.919, 0, 0, 0.081}	{0.92, 0, 0, 0, 0.080}
	3	{0.905, 0.095}	{0.849, 0, 0.151}	{0.84, 0, 0, 0.16}	{0.842, 0, 0, 0, 0.158}
		{0.982, 0.018}	{0.967, 0, 0.033}	{0.965, 0, 0, 0.035}	{0.965, 0, 0, 0, 0.035}

The discount of the pay-off gives Customs some advantage to become active for patrol and the smuggler's activity is depressed at the early stage. In the discount version, active Customs lets the smuggler hesitate to smuggle at the early stage, and even though the smuggler is allowed to become active and gets some profit at the late stages the reward would be discounted. The value of the game increases in the discount model, although the value is still negative. The increasing rate of the value is more distinguishing for the game with the larger number of stages or larger T.

5. Conclusion

This paper deals with a smuggling game with multiple stages. Customs is allowed to patrol within the limited number of chances and obtain reward by the capture of a smuggler. The smuggler decides the amount of contraband to try to smuggle at each stage and gets reward depending on the amount of contraband he succeeds to smuggle. The pay-off of the game is assumed to be zero-sum. In almost all past studies on the smuggling game, they focus on when the smuggler should take the action of smuggling within the limited number of chances. From the practical point of view, the smuggler is interested in his benefit by smuggling as much contraband as possible. By this reason, we adopt the strategy on the amount of contraband for the smuggler. Furthermore, we model our problem as a multi-stage game with incomplete information, where Customs cannot obtain information about the past behaviour of the smuggler but the smuggler gets information about Customs. Information acquisition is the key to make the game practical but many researchers have not taken the assumption on information acquisition or the asymmetry of information in their models so far. We embed the asymmetrical information in the game to evaluate the value of the information and analyse the effect of the information on optimal strategy.

In this paper, we clarify the difference between the pay-offs two players recognize and formulate the game with incomplete information using the belief of Customs on the amount of contraband. We propose a theory and a numerical algorithm to derive the perfect Bayesian equilibrium of the game. Generally speaking, Customs stands on more disadvantageous position for lack of information and the value of the game is estimated to become smaller than the game without the information asymmetry. In some cases, however, we show that Customs exactly anticipates the rational behaviour of the smuggler and responds to it in such a rational manner that he does not lose anything. Through these examples and the sensitivity analysis, we elucidate some properties of optimal strategy. Many smuggling games studied so far have multiple stages as our model does, but it was not easy or was never tried to solve the games by the concept of perfect Bayesian equilibrium. The methodology proposed in this paper could be one of the effective approaches to evaluate the value of information involved in the smuggling game.

The proposed method is an approximation numerical algorithm. As seen in Sections 4.1 and 4.2, the algorithm seems to perform well for the problem with the comparatively small number of stages $T=3$ or 4. In order to obtain the value of the game at the initial state (T, K, M), we use not the optimal value of the objective of Problem (P_P), which discretized belief $q_n(x)$ directly affects, but given by constraint (9) in the formulation. This might make the approximation algorithm work.

Finally, let us mention the uniqueness of the Bayesian equilibrium in our game. We could not prove the uniqueness of the perfect Bayesian equilibrium. We use the linear programming problems (P_P) and (P_S) to derive the equilibrium. There is no wonder that the problems could have multiple optimal solutions. However, in Sections 4.1 and 4.2, we showed that optimal solutions of our model with incomplete information are very similar to the solutions of the complete-information model, for which we proved in Hohzaki (2011) that optimal solutions are basically unique with the exception of special cases. Thus, we reasonably think that the equilibrium of this model is also uniquely determined in many cases.

From another point of view, we could discuss the uniqueness in this way. We could imagine that there are the following relations among elements in the pay-off matrix of Equation (3), although we could not prove them practically.

$$\beta w(n-1,k-1,x;\Gamma_P(q_n)) \leqslant \alpha p_1(y) - y p_2(y) + (1-p_1(y)) \times \beta w(n-1,k-1,x-y;\Gamma_P(q_n)), \quad (16)$$

$$\beta w(n-1,k,x;\Gamma_N(q_n)) \geqslant -y + \beta w(n-1,k,x-y;\Gamma_N(q_n)), \quad (17)$$

$$\beta w(n-1,k-1,x;\Gamma_P(q_n)) \leqslant \beta w(n-1,k,x;\Gamma_N(q_n)), \quad (18)$$

$$\alpha p_1(y) - y p_2(y) + (1-p_1(y)) \times \beta w(n-1,k-1,x-y;\Gamma_P(q_n)) > -y + \beta w(n-1,k,x-y;\Gamma_N(q_n)). \quad (19)$$

In inequality (16), the left-hand side and the right-hand side represent the rewards that Customs expects by wasting a chance of patrol against the smuggler's no-smuggling strategy and by the coincidence of patrolling and smuggling, respectively. Hence, the right-hand side would be larger than the left-hand side. Similarly, other inequalities also seem to be valid. If the relations (16)-(19) hold, they restrict the existence of equilibrium point and lessen the number of equilibriums, even though the game has multiple equilibriums.

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Received March 2011;
accepted December 2011 after one revision