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"On the Problem of the Minimum Wave making Resistance of Ships"

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Abstract

The author discusses on the the minimum problem of the wave resistance of three types of the singularity distribution.

He finds no minimum solution in general, but in the cases with some restrictions. He points out also the difficulty of the numerical computation, and asserts that there may be quasi-waveless solutions.

The principal idea of this paper is based on the fact that a singularity distribution gives a value of wave resistance, but inversely a value of wave resistance does not correspond to one distribution but to infinitely many ones. This is said exactly in the former paper and approximately in the present.

Introduction. The problem to minimize the wave making resistance of ships has made a great progress in recent works, but also thrown back many questions to the theory[8),10)].

The author intends to pick up and find out their difficulties as far as possible.

First of all, we must classificate problems and questions.

The problems are distinguished by the types of ships and the types of minimum conditions. Firstly, we consider three cases for the types of ships, or mathematically, of their singularity distribution.

I a) Mitchell-Havelock distributions for the model of displacement ships[12)],
  b) pressure distributions for surface ships,
  c) submerged ships.

Secondly, our minimum proplem assumes naturally the given velocity and length or Froude number, and the given displacement or total sum of the singularity distribution of the given type.

Under these circumstances, problems are considered for which

II a) A) there is not another restriction,
  B) the moment of the distribution is given,
  C) the second moment is given.

These conditions are concerned with integrated quantities of the distribution, but by the familiar notation to naval architects we may ask the next problem,

II b) under the same conditions as I a) A), the block coefficient must be taken as a given value[8].

Since the displacement is given, assuming the given draft, this condition determines the breadth. This is not of integrated quantity, so that this problem may differ from the ones described above.

Lastly, we ask the following questions for these problems.

1) Have the minimum problem a solution?

This is a question partly solved by S. Karp and others[10], so that we may ask as "In which cases the problem has a solution?".

2) If we have the solution, is it uniquely determined?

G. Weinblum and others have described the instability of their numerical solution, that is, they obtained fairly different minimum ship forms respectively[8]. From what cause such phenomenon appears?

3) In these connection, we must remind the theory of the waveless ship by T. Inui[9]. If a ship would have no wave resistance, then has the minimum problem their meaning?

What is the relation between the minimum wave resistance ship form and the waveless ship form?

These are concerns of the author, and he discusses them under the various types of distributions cited above.

Chapter 1. Submerged ship

The wave resistance of a submerged ship is somewhat simpler than the usual floating ship, and discussed in detail numerically and theoretically in the literatures[4],[11].

In all of those works, they have discussed with doublet distributions on the given segment of the longitudinal axis.

Hence, we suffice too with such simplified treatment.

1.1 Influence function. Consider the water flow of unit velocity, and take the origin of the coordinates at the center of the doublet distribution occupying the segment $|x|\leqq 1$, submerged under the water surface with immersion f, x-axis horizontally and upper steam direction, z-axis vertically upwards.

The wave resistance of the distribution is given by the formula[11],

$$R=\frac{\rho g^4}{\pi}\int_0^{\frac{\pi}{2}}|F(g\sec^2\theta,\theta)|^2\sec^5\theta\,d\theta\,, \tag{1.1.1}$$

$$F(\kappa,\theta)=\int_{-1}^{1}H(x)\,e^{-\kappa f-i\kappa x\cos\theta}\,dx\,, \tag{1.1.2}$$

where ρ is the water density and g the gravity constant of our unit system, and so Froude number, $Fr.$, based on the ship length equals to $1/\sqrt{2g}$.

$H(x)$ equals approximately to the sectional area as usually known.

Now, if we add a small quantity $\Delta H(x)$ to $H(x)$ in the vicinity of x, neglecting higher order term, the wave resistance will increase ΔR.

Taking the variation of (1.1.1), we have

$$\Delta R = 2\rho g G(x) \Delta H(x) \Delta x \,, \tag{1.1.3}$$

and

$$G(x) = Re. \frac{g^3}{\pi} \int_0^{\frac{\pi}{2}} F(g \sec^2 \theta, \theta) e^{-gf \sec^2 \theta + igx \sec \theta} \sec^5 \theta \, d\theta$$
$$= \frac{g^3}{\pi} \int_{-1}^{1} H(\xi) P_{-5}(g \overline{x-\xi}, 2gf) \, d\xi \,, \tag{1.1.4}$$

where the function P_{-5} is seen in **Appendix A.**

Hence, the function $G(x)$ tells us the wave resistance variation for small variation of the distribution. We will call $G(x)$ hereafter the influence function according to E. Hogner[2)].

We may write now (1.1.1) with the aid of this function as

$$R = \rho g \int_{-1}^{1} G(x) H(x) \, dx \,. \tag{1.1.5}$$

Now let us consider the minimum problem II) a A).

Then the wave resistance must be stationary for any variation of the distribution. Accordingly, we can assert the influence function must be constant over the length of distribution, that is,

$$G(x) = C \,. \tag{1.1.6}$$

Write the displacement volume or total sum of the distribution as

$$\nabla = \int_{-1}^{1} H(x) \, dx \,. \tag{1.1.7}$$

If (1.1.6) would hold good, then the wave resistance might be from (1.1.5)

$$R = \rho g \nabla C \,, \tag{1.1.8}$$

so that C must be positive.

Thus, the problem reduces to solve the integral equation (1.1.6) with (1.1.4)

Now, the kernel P_{-5} in (1.1.4) is regular in x, therefore $G(x)$ is to be regular in x too, assumed the integrability of the right hand expression.

If $G(x)$ would be constant in $|x| \leq 1$, and it might be constant at infinity too. But $P_{-5}(gx, 2gf)$ vanishes at infinity, so that we may conclude this constant to be zero. Therefore $H(x)$ should be zero identically.

Namely, we have no solution of the problem except the one vanishing identically.

However, the integral equation (1.1.6) with (1.1.4) might have a solution numerically in almost every case, in fact, G. Weinblum solved those problems[4]. What is the relation between the fact that there are those numerical solutions and the fact that the minimum solution is to be identically zero?

1.2 Quasi-waveless solution. If there is no minimum value of the wave resistance, it seems that the least value might be zero.

In fact, we will show the wave resistance to make small at our disposal.

Consider the next distribution

$$H(x)=\sum_{n=0}^{\infty} a_{2n}\frac{\cos 2n\theta}{\sin\theta}, \qquad x=-\cos\theta, \tag{1.2.1}$$

Since the kernel of (1.1.4) is expanded as (A. 10), integrating term by term, we have

$$\begin{aligned} G(x)&=g^3\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{m}\frac{(-1)^m g^{2m} U_{m+2}(2gf)\, x^{2r} a_{2r}}{2^{2m-2r}(2r)!(n+m-r)!(m-n-r)!} \\ &=g^3\sum_{r=0}^{\infty}(-1)^r\frac{(gx)^{2r}}{(2r)!}\sum_{n=0}^{\infty}(-1)^n a_{2n} C_{2n,\,2r}, \end{aligned} \tag{1.2.2}$$

where

$$C_{2n,\,2r}=(g/2)^{2n}\sum_{s=0}^{\infty}(-1)^s(g/2)^{2s}\frac{U_{r+n+s+2}(2gf)}{s!(2n+s)!}. \tag{1.2.3}$$

Now, taking a sufficiently large integer N, and put

$$\sum_{n=0}^{\infty}(-1)^n a_n C_{2n,\,2r}=0, \qquad \text{for } r=0,\cdots,(N-1). \tag{1.2.4}$$

Then we have from (1.2.2)

$$G(x)=g^3 O\left[\frac{(gx)^{2N}}{(2N)!}\right]. \tag{1.2.5}$$

Therefore, since the series expansion (1.2.2) is shown convergent, this value will be sufficiently small by selecting appropriately large N.

Take M unknown a_n's ($M>N$), and we have a solution except $(M-N+1)$ undetermined coefficients, for we have $(N+1)$ equations (1.2.4) with (1.1.7).

If the influence function would be small, and the wave resistance might be small by (1.1.5).

Thus, we may have many solutions by appropriate selections of M and N, and reduce the wave resistance smaller at our disposal.

In these circumstances, we call these solutions quasi-waveless for a convenience.

However, we can not expect that the solutions obtained converge to a definite distribution, because the minimum solution should be zero identically.

1.3 The case of small immersion. The numerical computations of the above method will be very laborious when the immersion is small. There is another method to obtain the distributions with the smaller wave resistance.

Define the function

$$\Gamma(x)=\frac{1}{\pi g}\int_{-1}^{1} H(\xi)P_{-1}(g\,\overline{x-\xi}, 2gf)\,d\xi\,, \tag{1.3.1}$$

$$(d/dx)^4\Gamma(x)=G(x)\,. \tag{1.3.2}$$

If we put with arbitrary constants a_n as

$$\Gamma(x)=a_0+a_1x+a_2x^2+a_3x^3\,, \tag{1.3.3}$$

then we have by (1.3.2)

$$G(x)=0\,, \tag{1.3.4}$$

and so by (1.1.5)

$$R=0\,. \tag{1.3.5}$$

The integral equation (1.3.1) with (1.3.3) has a solution numerically in almost every cases, but may be shown to have only identically vanishing solution by the same way as in § **1.1** theoretically.

Meanwhile, if the immersion tends to zero, the integral equation has a solution not identically zero by **Appendix B.**

Accordingly, when the immersion becomes small enough, we may expect a solution like the one in its limit.

And now we call these solutions quasi-waveless too.

Chapter 2. Mitchell-Havelock type distribution[12)]

We consider the doublet distribution over the rectangle on the $x-z$ plane, and similates the displacement ship.

2.1 General discussion. The wave resistance is the same form as (1.1.1) with

$$F(\kappa,\theta)=\int_{-t}^{0}\int_{-1}^{1} H(x,z)\,e^{-\kappa z-i\kappa x\cos\theta}\,dx\,dz\,, \tag{2.1.1}$$

where t is the ratio of the draft to the half length, and $H(x,z)$ equals approximately to the breadth of the ship.

Define the influence function as (1.1.4),

$$G(x,z)=\frac{g^3}{\pi}\int_{-t}^{0}\int_{-1}^{1} H(\xi,\zeta)P_{-5}(g\,\overline{x-\xi}, -g\,\overline{z+\zeta})\,d\xi\,d\zeta\,, \tag{2.1.2}$$

then the wave resistance is written as

$$R=\rho g\int_{-t}^{0}\int_{-1}^{1}H(x,z)\,G(x,z)\,dx\,dz\,. \tag{2.1.3}$$

The minimum value is attained, when

$$G(x,z)=C:\ \text{Constant}, \tag{2.1.4}$$

under the condition

$$\nabla=\int_{-t}^{0}\int_{-1}^{1}H(x,z)\,dx\,dz\,. \tag{2.1.5}$$

Then

$$R=\rho g\nabla C\,. \tag{2.1.6}$$

At first, we have the next differential equation from (A. 2),

$$\left(\frac{\partial}{\partial z}+\frac{1}{g}\frac{\partial}{\partial x^2}\right)G(x,z)=0\,,\qquad \text{for } z<0. \tag{2.1.7}$$

Moreover, introduce the auxiliary function vanishing at $x=\pm\infty$ by the next equation[12]

$$H(x,z)=\left(\frac{\partial}{\partial z}-\frac{1}{g}\frac{\partial^2}{\partial x^2}\right)\sigma(x,z)\,, \tag{2.1.8}$$

and we may write

$$G(x,z)=\frac{g^3}{\pi}\int_{-\infty}^{\infty}\sigma(\xi,0)P_{-5}(g\,\overline{x-\xi},\,-gz)\,d\xi\,. \tag{2.1.9}$$

Since the above formula is regular in x with the appropriate class of the function σ, we will deduce the same conclusion as of the preceding chaper.

In fact, if we pick up the distributions uniform in z and represented by Fourier transforms of functions which have the next quality

$$F(\kappa,\theta)=0\,,\qquad \text{for } \kappa>g, \tag{2.1.10}$$

we may have waveless distributions extending to infinity but confined practically in $|x|\leq 1$ by suitable combinations of such functions[12].

2.2 The case of infinite draft. Thus, it seems that the minimum problem has no solution in general, but this fact does not hold always in more restricted cases.

Consider that the distribution is uniform in z and extends to infinitely great depth, and this is the case discussed by S. Karp and others mathematically and solved numerically[10].

Here, we consider it analytically in more detail.

Integrating (2.1.2) and (2.1.3), we have

$$c_w=\frac{R}{\frac{\rho}{2}(\bar{B}/2)^2}=(8g^2/\bar{B})\int_{-1}^{1}H(x)\Gamma(x)\,dx\,, \tag{2.2.1}$$

where

$$\Gamma(x)=\frac{1}{\pi\bar{B}}\int_{-1}^{1}H(\xi)P_{-1}(g\,\overline{x-\xi},0)\,d\xi\,, \tag{2.2.2}$$

Assume the next expansion by Mathieu functions, hereafter we follow the notations of the text by McLachlan[5],

$$H(-\cos\theta)=\frac{\bar{B}}{\sin\theta}\sum_{n=0}^{\infty}a_n\,ce_n(\theta,q)\,,\qquad q=g^2/4\,. \tag{2.2.3}$$

Then, the mean breadth defined as

$$\bar{B}=\frac{1}{2}\int_{-1}^{1}H(x)\,dx\,, \tag{2.2.4}$$

proposes a condition between the coefficients in (2.2.3), that is,

$$\sum_{n=0}^{\infty}a_{2n}A_0^{(2n)}=2/\pi\,. \tag{2.2.5}$$

By the way, define the next quantity

$$\delta=\bar{B}/H(0)=1\Big/\left[\sum_{n=0}^{\infty}a_{2n}\,ce_{2n}(\pi/2,q)\right], \tag{2.2.6}$$

this equals approximately to the water plane area coefficient.

Moreover, we define the first and second moment as follows,

$$\int_{-1}^{1}H(x)\,x\,dx=-4\bar{B}\alpha\,,\qquad \alpha=(\pi/8)\sum_{n=0}^{\infty}a_{2n+1}A_1^{(2n+1)}\,, \tag{2.2.7}$$

$$\int_{-1}^{1}H(x)\,x^2\,dx=8\bar{B}m^2\,,\qquad m^2=1/8+(\pi/32)\sum_{n=0}^{\infty}a_{2n}A_2^{(2n)}\,. \tag{2.2.8}$$

Now, put (2.2.3) into (2.2.2) and use (B. 13), we have

$$\Gamma(-\cos\theta)=\sum_{n=0}^{\infty}\mu_n a_n\,ce_n(\theta,q)\,. \tag{2.2.9}$$

Owing to the orthogonality of Mathieu functions, the wave resistance is written from (2.2.1) with this equation as

$$c_w=4\pi g^2\sum_{n=0}^{\infty}\mu_n a_n^2\,. \tag{2.2.10}$$

Hence, the minimum problem reduces to a simple calculation.

A) Obtain the minimum of (2.2.10) under the condition (2.2.5).

As explaned in the preceding chapter or obtained by Lagrange's method, we have it, when

$$\Gamma(-\cos\theta)=\lambda: \quad \text{Constant,} \tag{2.2.11}$$

then

$$c_w = 16g^2\lambda\,. \tag{2.2.12}$$

Since

$$\sum_{n=0}^{\infty} A_0^{(2n)} ce_{2n}(\theta, q)=1/2\,,$$

comparing with (2.2.9), we have

$$a_{2n+1}=0\,, \qquad a_{2n}=2\lambda A_0^{(2n)}/\mu_{2n}\equiv a_{2n}^*\,, \tag{2.2.13}$$

Putting this into (2.2.5), we have the constant, that is,

$$\lambda=1/(\pi C_{0,0})\,, \qquad C_{0,0}=\sum_{n=0}^{\infty}[A_0^{(2n)}]^2/\mu_{2n}\,, \tag{2.2.14}$$

putting this value into (2.2.12),

$$c_w=16g^2/(\pi C_{0,0})\equiv c_{w_0}\,. \tag{2.2.15}$$

Lastly, the coefficient by (2.2.6) becomes

$$\delta=\pi C_{0,0}/2D_0\equiv\delta_0\,, \qquad D_0=\sum_{n=0}^{\infty} A_0^{(2n)} ce_{2n}(\pi/2, q)/\mu_{2n}\,. \tag{2.2.16}$$

B) Solve the same problem with the other condition (2.2.7).
It is to be, by Lagrange's method,

$$\Gamma(-\cos\theta)=\lambda_1+\lambda_2\cos\theta\,. \tag{2.2.17}$$

Then,

$$c_w=16g^2\lambda_1+32g^2\alpha\lambda_2\,, \tag{2.2.18}$$

The constant λ_1 is given by (2.2.14) as easily seen, and since

$$\cos\theta=\sum_{n=0}^{\infty} A_1^{(2n+1)} ce_{2n+1}(\theta, q)\,,$$

comparing with (2.2.9), we have

$$a_{2n+1}=\lambda_2 A_1^{(2n+1)}/\mu_{2n+1}\,, \tag{2.2.19}$$

and that from (2.2.7)

$$\lambda_2=8\alpha/(\pi C_{1,1})\,, \qquad C_{1,1}=\sum_{n=0}^{\infty}[A_1^{(2n+1)}]^2/\mu_{2n+1}\,. \tag{2.2.20}$$

When we write as

$$c_{w_1}=16g^2/(\pi C_{1,1})\,, \tag{2.2.21}$$

we have from (2.2.18) with (2.2.15) finally

$$c_w=c_{w_0}+16\alpha^2 c_{w_1}\,. \tag{2.2.22}$$

C) Consider A problem with the other condition (2.2.8), then we have in the same way,

$$\Gamma(-\cos\theta)=\lambda_1+\lambda_2\cos 2\theta\,, \tag{2.2.23}$$

and

$$c_w=16g^2(\lambda_1-\lambda_2)+128g^2\lambda_2 m^2\,. \tag{2.2.24}$$

Since

$$\cos 2\theta=\sum_{n=0}^{\infty}A_2^{(2n)}ce_{2n}(\theta,q)\,,$$

we have the coefficients in like manner as in A), that is,

$$a_{2n+1}=0\,,\qquad a_{2n}=[2\lambda_1 A_0^{(2n)}+\lambda_2 A_2^{(2n)}]/\mu_{2n}\,. \tag{2.2.25}$$

Using these formulae, we have

$$\lambda_1=1/(\pi C_{0,0})-2\gamma C_{0,2}/(\pi\Delta)\,,\qquad \lambda_2=2C_{0,0}/(\pi\Delta)\,, \tag{2.2.26}$$

where

$$\gamma=16(m^2-m_0^2)\,, \tag{2.2.27}$$

m_0 is the value of (2.2.8) taken for the solution A, namely

$$m_0^2=1/8+C_{0,2}/(16C_{0,0})\,, \tag{2.2.28}$$

and

$$\left.\begin{aligned}&\Delta=C_{0,0}C_{2,2}-C_{0,2}^2\,,\\ &C_{0,2}=\sum_{n=0}^{\infty}A_0^{(2n)}A_2^{(2n)}/\mu_{2n}\,,\qquad C_{2,2}=\sum_{n=0}^{\infty}[A_2^{(2n)}]^2/\mu_{2n}\,,\end{aligned}\right\} \tag{2.2.29}$$

Then, we have from (2.2.25) with (2.2.15)

$$c_w=c_{w_0}+\gamma^2 c_{w_2}\,, \tag{2.2.30}$$

where

$$c_{w_2}=16g^2C_{0,0}/(\pi\Delta)\,. \tag{2.2.31}$$

Here, we rewrite (2.2.25) as

$$\left.\begin{aligned}&a_{2n}=a_{2n}^*+\gamma b_{2n}^*\,,\\ &b_{2n}^*=\frac{2}{\pi\Delta\mu_{2n}}[C_{0,0}A_2^{(2n)}-C_{0,2}A_0^{(2n)}]\,.\end{aligned}\right\} \tag{2.2.32}$$

Table

$4q$ / g / $Fr.$		1 / 1 / 0.7071	4 / 2 / 0.500	10 / $\sqrt{10}$ / 0.3976	16 / 4 / 0.3536	24 / $\sqrt{24}$ / 0.3195	36 / 6 / 0.2887	50 / $\sqrt{50}$ / 0.2659	64 / 8 / 0.2500	80 / $\sqrt{80}$ / 0.2364	100 / 10 / 0.2236
μ_0		0.644094	0.998125(1)	0.814806(2)	0.132126(2)	0.193168(3)	0.189457(4)	0.202547(5)	0.294995(6)	0.419816(7)	0.478425(8)
μ_1		1.096150	0.732746	0.194796	0.470898	0.851693(2)	0.992818(3)	0.122314(3)	0.199029(4)	0.313925(5)	0.397238(6)
μ_2		0.547239	0.686380	0.662995	0.397181	0.138960	0.234523(1)	0.342655(2)	0.624663(3)	0.109448(3)	0.154365(4)
μ_3		0.344356	0.385909	0.500315	0.582640	0.510346	0.227603	0.547913(1)	0.120592(1)	0.237545(2)	0.373999(3)
μ_4		0.254270	0.268533	0.306237	0.360255	0.448653	0.495383	0.320119	0.128702	0.344321(1)	0.634796(2)
μ_5		0.202113	0.208842	0.224475	0.244169	0.279845	0.358126	0.442317	0.402119	0.230739	0.712211(1)
μ_6		0.167868	0.171617	0.179844	0.189269	0.204374	0.235323	0.290597	0.362817	0.406359	0.305693
μ_7		—	0.145917	0.150846	0.156245	0.164335	0.179028	0.202103	0.235452	0.290416	0.363293
μ_8		—	0.127026	0.130230	0.133654	0.138613	0.147071	0.159004	0.174202	0.197839	0.241970
c_{w_0}		6.5518	4.4226	1.1384	0.34299	0.85324(1)	0.14156(1)	0.23083(2)	0.46094(3)	0.87171(4)	0.13191(4)
c_{w_2}		2.789	12.73	23.60	20.36	12.01	4.566	1.441	0.4719	0.1398	0.0330
δ_0		1.606	1.114	0.7960	0.6827	0.6039	0.5378	0.4910	0.4591	0.4323	0.4074
m_0		0.7109	0.6358	0.5371	0.4837	0.4401	0.3997	0.3694	0.3480	0.3298	0.3123
$16c_{w_2}/g^3$		44.60	25.46	11.94	5.090	1.636	0.3349	0.6522(1)	0.1475(1)	0.3126(2)	0.0528(3)
c_p		0.5917	0.6031	0.6078	0.5813	0.5409	0.4931	0.4566	0.4305	0.4085	0.3869
c_w	$\delta=0.5$	20.26	53.46	46.35	12.26	1.639	0.6213(1)	0.2905(2)	0.3574(2)	0.2083(2)	0.7188(3)
	$\delta=0.6$	14.43	28.29	14.90	2.038	0.8688(1)	0.1046	0.6283(1)	0.2610(1)	0.8602(2)	0.2134(2)
	$\delta=0.7$	11.24	15.80	3.563	0.3974	0.7623	0.4658	0.1657	0.5553(1)	0.1603(1)	0.3609(2)
	$\delta=0.8$	9.395	9.438	1.142	2.261	2.224	0.9176	0.2758	0.8488(1)	0.2312(1)	0.4970(2)
γ	$\delta=0.5$	-2.2170	-1.963	-1.384	-0.7648	-0.3596	-0.1025	0.0204	0.0812	0.1196	0.1462
	$\delta=0.6$	-1.6803	-1.369	-0.7638	-0.2885	-0.0114	0.1407	0.2049	0.2331	0.2471	0.2534
	$\delta=0.7$	-1.2971	-0.9456	-0.3206	0.0517	0.2374	0.3145	0.3367	0.3416	0.3381	0.3300
	$\delta=0.8$	-1.0096	-0.6277	0.0118	0.3069	0.4240	0.4448	0.4356	0.4230	0.4064	0.3874
a_0^*		0.89565	0.93186	1.05195	1.13513	1.20958	1.28477	1.34651	1.39366	1.43700	1.48113
a_2^*		0.09256	0.04368	0.00792	0.00286	0.00141	0.00088	0.00065	0.00052	0.00043	0.00035
a_4^*		0.00105	0.00268	0.00151	0.00061	0.00017	0.00003	0.00001	0.00000	0.00000	0.00000
a_6^*		0.00001	0.00003	0.00005	0.00039	0.00002	0.00001	0.00000	0.00000	0.00000	0.00000
b_0^*		-0.05566	-0.19362	-0.32323	-0.38490	-0.45604	-0.55957	-0.67128	-0.77320	-0.88073	-1.00420
b_2^*		0.63372	0.59764	0.50740	0.47106	0.50503	0.63798	0.80724	0.96082	1.11919	1.29827
b_4^*		0.02856	0.12674	0.23008	0.18878	0.09742	0.03211	0.01299	0.00828	0.00669	0.00591
b_6^*		0.00021	0.00366	0.01698	0.04590	0.02089	0.01157	0.00455	0.00157	0.00047	0.00014
b_8^*		0.00000	0.00004	0.00006	0.00101	0.00135	0.00116	0.00058	0.00037	0.00016	—

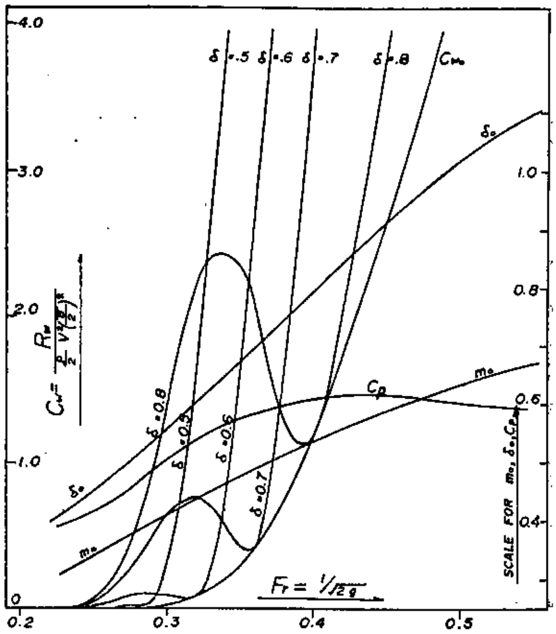
The numbers (n) in parenthesis indicate that the results must be multiplied by 10^{-n}, for example 0.3441(2) means 0.003441.

We show some values calculated by the tables of Mathieu functions in **Table** and **Figure**, in which the values of c_w refer to the quantity δ, more familiar than γ, with the aid of the next formula,

$$1/\delta=1/\delta_0+2\gamma(C_{0,0}D_2-C_{0,2}D_0)/(\pi\Delta), \quad (2.2.33)$$

where δ_0 is given by (2.2.16), and

$$D_2=\sum_{n=0}^{\infty}A_2^{(2n)}ce_{2n}(\pi/2,q)/\mu_{2n}. \quad (2.2.34)$$



Figure

2.2.1 Approximate relations. Let us consider the approximate relations by making use of the results given in **Appendix C.**

At first, suppose g is sufficiently small, then

A) $$\left.\begin{aligned}&\lambda\doteqdot\frac{2}{\pi}\log(4/\gamma g),\\&a_0^*\doteqdot2\sqrt{2}/\pi,\quad a_{2n}^*\doteqdot0,\qquad\text{for } n\geq1,\end{aligned}\right\} \quad (2.2.1.1)$$

$$H(-\cos\theta)\doteqdot2/(\pi\sin\theta), \quad (2.2.1.2)$$

and

$$c_{w_0}\doteqdot\frac{32g^2}{\pi}\log(4/\gamma g),\qquad \log\gamma=C\text{: Euler's Const.} \quad (2.2.1.3)$$

with

$$\delta_0\doteqdot\pi/2. \quad (2.2.1.4)$$

This is the case considered by S. Karp and others[10].

B) $$\lambda_2\doteqdot4\alpha/\pi,\quad a_1\doteqdot8\alpha/\pi,\quad a_{2n+1}=0,\qquad\text{for } n\geq1, \quad (2.2.1.5)$$

and

$$c_{w_1}\doteqdot128g^2/\pi. \quad (2.2.1.6)$$

In the same way, we may obtain the approximate values for the problem C. However, since we neglect higher order terms in (2.2.1.1) to (2.2.1.4), we cannot expect their accuracy up to the degree written down.

In other words, as seen in (2.2.1.6) compared with (2.2.1.3), the longitudinal variation of the displacement distribution does not affect appreciably to the wave resistance in very high speed.

Secondly, suppose g is very large, then we can observe in the Table that the wave resistance correspond to Mathieu function of lower order is exceedingly smaller than the one of higher order when the order is not so large.

Hence, we have an almost minimum value whenever we take for the distribution a Mathieu function of lower order as far as possible.

Asymptotic characters of various quantities are as follows;

A) $$a_0^* \doteqdot (\pi g)^{\frac{1}{2}}/\pi\,, \quad a_{2n}^* \doteqdot 0\,, \qquad \text{for } n \geq 1, \tag{2.2.1.7}$$

$$c_{w_0} \doteqdot 64g^2 \exp.(-2g)\,, \tag{2.2.1.8}$$

$$\delta_0 \doteqdot \sqrt{(\pi/2g)}\,, \tag{2.2.1.9}$$

B) $$c_{w_1} \doteqdot 64g^4 \exp.(-2g)\,, \tag{2.2.1.10}$$

C) $$c_{w_2} \doteqdot 16g^6 \exp.(-2g)\,, \tag{2.2.1.11}$$

$$m_0^2 \doteqdot 1/(4g)\,, \tag{2.2.1.12}$$

and

$$\delta \doteqdot \sqrt{(\pi/2g)}\Big/\left(1-\frac{\gamma}{4}g\right). \tag{2.2.1.13}$$

Owing to their exponential term, these values of the wave resistance are very small compared with the existing ones in very low speed in spite of infinite draft.

2.2.2 The problem II b). Consider the problem to minimize the wave resistance with given $\bar{B}$ or δ. This is the problem considered by G. Weinblum[8].

Introduce Lagrange's constants and proceed in usual manner, we have

$$a_{2n} = [2\lambda_1 A_0^{(2n)} + \lambda_2 ce_{2n}(\pi/2, q)]/\mu_{2n}\,. \tag{2.2.2.1}$$

Put this into (2.2.9), the influence function becomes to

$$\Gamma(-\cos\theta) = \lambda_1 + \lambda_2 \sum_{n=0}^{\infty} ce_{2n}(\pi/2, q)\, ce_{2n}(\theta, q)\,. \tag{2.2.2.2}$$

The last series is rewritten by changing the order of summation and using the orthogonality relations as follows,

$$\begin{aligned}\sum_{n=0}^{\infty} ce_{2n}(\pi/2, q)\, ce_{2n}(\theta, q) &= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{n=0}^{\infty}(-1)^r A_{2r}^{(2n)} A_{2s}^{(2n)} \cos 2s\theta \\ &= \frac{1}{2} + \sum_{r=0}^{\infty}(-1)^r \cos 2r\theta \\ &= \frac{1}{2}\lim_{N\to\infty}\left[\frac{\sin(2N+1)(\theta-\pi/2)}{\sin(\theta-\pi/2)}\right].\end{aligned} \tag{2.2.2.3}$$

This series does not converge to any smooth function, however N becomes large. The same difficulty will appear in the case of which $H(x)$ vanishes at end points.

Accordingly, we may expect that the numerical solution in these cases will show the instability as observed by G. Weinblum[8].

Meanwhile, the wave resistance converges to a finite value, namely, putting (2.2.2.3) into (2.2.1), we have by making use of Dirichlet integral,

$$c_w = 8g^2(2\lambda_1 + \pi\lambda_2/2\delta),$$

and then determining the constants from (2.2.5) and (2.2.6), finally

$$c_w = c_{w_0} + 4\pi g^2 C_{0,0}\left(\frac{1}{\delta} - \frac{1}{\delta_0}\right)^2 \Big/ (C_{0,0}E - D_0^2), \tag{2.2.2.4}$$

where

$$E = \sum_{n=0}^{\infty} [ce_{2n}(\pi/2, q)]^2/\mu_{2n}. \tag{2.2.2.5}$$

2.3 The case of finite draft. The preceding analysis is very clear and easy to compute numerically, but unfortunately usual ships have very shallow draft and it seems hardly to apply such results.

Therefore, consider the case of the distribution draftwise uniform and of their draft t for a moment.

We have in the same manner as in the preceding the wave resistance coefficient as

$$c_w = \frac{8g}{\bar{B}} \int_{-1}^{1} H(x)\Gamma(x)\,dx, \tag{2.3.1}$$

where

$$\Gamma(x) = \frac{g}{\pi \bar{B}} \int_{-1}^{1} H(\xi) K_{-1}(g\overline{x-\xi}, gt)\,d\xi, \tag{2.3.2}$$

with

$$K_{-1}(u, \tau) = P_{-1}(u, 0) - 2P_{-1}(u, \tau) + P_{-1}(u, 2\tau). \tag{2.3.3}$$

Owing to the singularity of this kernel, we may expect that there will be a minimum solution, but here we will consider no accurate numerical value but only some approximate characters.

Firstly, when the speed is very high and g is small, we cannot treat as in the preceding, because we have no simple approximate expression of (2.3.3) for the usual magnitude of the draft. But, when the draft is sufficiently small, we have another simple results, and this problem will be left in § 2.4.

Secondly, when g is sufficiently large and the distribution is expanded by Mathieu functions as (2.2.3), we have, considered only even functions,

$$c_w = 4\pi g^2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{2n} a_{2m} M_{2n,2m}, \tag{2.3.4}$$

where

$$M_{2n,2m}=2\left[\frac{A_0^{(2n)}A_0^{(2m)}}{ce_{2n}(\pi/2,q)\,ce_{2m}(\pi/2,q)}\right]\int_0^\infty(1-e^{-gt\cosh^2 u})^2 Ce_{2n}(u,q)\,Ce_{2m}(u,q)\,du\,. \qquad (2.3.5)$$

If we put (C. 8) into (2.3.5), we obtain

$$M_{2n,2m}\doteqdot\frac{(-1)^{n+m}}{g}ce_{2n}(0,q)\,ce_{2m}(0,q)L_{2n-2m}(gt)\,, \qquad (2.3.6)$$

where

$$L_{2\nu}(gt)=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}(1-e^{-gt\sec^2\theta})^2\cos 2\nu\theta\,d\theta\,. \qquad (2.3.7)$$

The last function is duduced from the error function as easily seen, and when gt is sufficiently larger than one,

$$L_{2\nu}(gt)\underset{gt\gg1}{\doteqdot}\begin{cases}0\,, & \text{for } \nu\neq0,\\ 1\,, & \text{for } \nu=0,\end{cases} \qquad (2.3.8)$$

Accordingly, the conclusions for the case of infinite draft hold good.

When gt is very small but g is very large, this functions is proportional to $\sqrt{gt}$ and of the same order for ν, which is not so large.

Hence, the fundamental character, the lower the order of Mathieu function the smaller the wave resistance, would not change.

2.3.1 Two classes of the wave resistance. The use of Mathieu function is very convenient theoretically as explaned, but not for the purpose of numerical computation.

In this circumstance, we remind naturally the simple Fourier expansion.

Assume the next expansion

$$H(x)=\frac{\bar{B}}{\sin\theta}\sum_{n=0}^{\infty}a_n\cos n\theta\,,\qquad x=-\cos\theta\,. \qquad (2.3.1.1)$$

Then, we will easily find the next formulae in like manner as (2.3.4),

$$c_w=8g^2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}a_n a_m R_{n,m}\,, \qquad (2.3.1.2)$$

where

$$R_{n,m}=\text{Re. }i^{n-m}\int_0^\infty J_n(g\cosh u)J_m(g\cosh u)(1-e^{-gt\cosh^2 u})^2du\,. \qquad (2.3.1.3)$$

Consider that both g and gt are very large, and using the assymptotic expansions of Bessel functions, we have

$$\left.\begin{aligned}&R_{n,m}=0\,, && \text{for } (n-m)\text{: odd integer},\\ &R_{n,m}\doteqdot(1/2g)\left[1+(-1)^n\frac{2}{\pi}P_0(2g,0)+\cdots\right], && \text{for } (n-m)\text{: even.}\end{aligned}\right\} \qquad (2.3.1.4)$$

Namely, all of the function $R_{n,m}$ are nearly equal each other for even or odd suffixes respectively, which are not so large.

Thence, for example we have, taken N even functions,

$$c_w \doteqdot 8g^2 R_{0,0} \sum_{n=0}^{N} \sum_{m=0}^{N} a_{2n} a_{2m}\,, \qquad \text{for } 2N \ll g\,, \tag{2.3.1.5}$$

This vanishes, for example, when

$$\sum_{n=0}^{N} a_{2n} = 0\,, \quad \text{or} \quad H(\pm 1) = 0\,, \tag{2.3.1.6}$$

namely, c_w becomes to zero with respect to the order g, but in general not to the lower order, and at most it will be

$$c_w = 0(1)\,. \tag{2.3.1.8}$$

Continuing this procedure, we will have the distributions which have sharper ends and are finer and that their wave resistance of the next order,

$$c_w = 0(g^{-n})\,, \qquad n:\ \text{some positive integer,} \tag{2.3.1.8}$$

These belong to a different class with the exponental ones obtained in § 2.2, and will be higher than those in increasing g.

And we are easily seen that the distributions represented by polynomials belong also to this class[12].

2.4 The case of small draft. When the draft is sufficiently small, the minimum problem has another interesting feature.

Assume $4\tau/u^2$ is very small in (2.3.3), and we have an approximation by (A. 9), that is,

$$K_{-1}(u,\tau) \doteqdot \tau^2 P_{-5}(u,0)\,. \tag{2.4.1}$$

Thence, we may rewrite (2.3.1) and (2.3.2) as follows,

$$R = \rho g \int_{-1}^{1} H(x)\,\Gamma(x)\,dx\,, \tag{2.4.2}$$

where

$$\Gamma(x) = (g^3/\pi) \int_{-1}^{1} H(\xi)\,P_{-5}(g\,\overline{x-\xi}, 0)\,d\xi\,, \tag{2.4.3}$$

and $H(x)$ equals approximately to the sectional area, or if we consider this problem as the limit of the pressure distribution, it equals to the water head of the lift per unit length. We write its total sum as

$$\int_{-1}^{1} H(x)\,dx = \nabla\,. \tag{2.4.4}$$

Consider (2.4.3) as the integral equation for the given $\Gamma(x)$, and $H(x)$ may be determined except the function (B. 11) from the result of Appendix, and that this undetermined function contributes nothing to the wave resistance by (2.4.2), because it does not to $\Gamma(x)$.

The waveless distribution can not have a finite displacement[12], but these can do.

However, the integral of (2.4.3) has not a definite value, if $H(x)$ and its derivatives do not vanish at end point owing to the singularity of P_{-5}.

These conditions are counted four, and the arbitrary constants in (B. 11) four too, so that we may have a unique solution.

For the purpose of calculation, rewrite (2.4.2) to (2.4.4), integrating partially,

$$R=\rho\int_{-1}^{1} H''(x)\Gamma^*(x)\,dx\,, \tag{2.4.5}$$

$$\Gamma^*(x)=\frac{1}{\pi}\int_{-1}^{1} H''(\xi)P_{-1}(g\overline{x-\xi},0)\,d\xi\,, \tag{2.4.6}$$

$$\nabla=\frac{1}{2}\int_{-1}^{1} H''(x)x^2\,dx\,, \tag{2.4.7}$$

with the conditions

$$H(\pm1)=H'(\pm1)=0\,,$$

or

$$\left.\int_{-1}^{1} H''(x)\,dx=\int_{-1}^{1} H''(x)\,x\,dx=0\,,\right\} \tag{2.4.8}$$

Assume the expansion

$$H''(-\cos\theta)=\nabla\sum_{n=0}^{\infty} d_{2n}\,ce_{2n}(\theta,q)\,,\qquad q=g^2/4\,, \tag{2.4.9}$$

then, (2.4.7) and (2.4.8) are written

$$\left.\begin{aligned}&\int_{-1}^{1} H''(x)\,dx=\pi\nabla\sum_{n=0}^{\infty} d_{2n}A_0^{(2n)}=0\,,\\&\int_{-1}^{1} H''(x)x^2\,dx=(\pi\nabla/4)\sum_{n=0}^{\infty} d_{2n}A_2^{(2n)}=2\nabla\,.\end{aligned}\right\} \tag{2.4.10}$$

Consider the minimum problem under these conditions, and it is solved in the similar way as the problem C in § 2.2, then we have

$$\Gamma^*(-\cos\theta)=\frac{\lambda_1}{2}+\lambda_2\cos 2\theta\,, \tag{2.4.11}$$

$$d_{2n}=(\lambda_1 A_0^{(2n)}+\lambda_2 A_2^{(2n)})/\nabla\mu_{2n}\,, \tag{2.4.12}$$

$$\lambda_1=-8\nabla C_{0,2}/(\pi\Delta)\,,\qquad \lambda_2=8\nabla C_{0,0}/(\pi\Delta)\,, \tag{2.4.13}$$

and

$$R=4\rho\nabla\lambda_2\,, \quad \text{or} \quad R/[\rho g\nabla(\nabla/8)]=16c_{w_2}/g^3\,, \tag{2.4.14}$$

and that, comparing with (2.2.32), we see

$$d_{2n}=4b^*_{2n}\,. \tag{2.4.15}$$

Finally, we define the coefficient, which equals approximately to the prismatic coefficient of the ship form,

$$c_p=\nabla/2H(0)\,, \qquad H(0)=\nabla\sum_{u=0}^{\infty}d_{2n}\int_0^{\frac{\pi}{2}}ce_{2n}(\theta,q)\cos\theta\,d\theta\,. \tag{2.4.16}$$

These quantities have been studied in detail in § **2.2.1**, and values of (2.4.14) and (2.4.16) are shown in the Table.

Consider here only the case when g is very small, then we have

$$\Gamma^*(-\cos\theta)\doteqdot(4\nabla/\pi)\cos 2\theta\,, \tag{2.4.17}$$

$$\left.\begin{aligned} H''(-\cos\theta)&\doteqdot(8\nabla/\pi)\frac{\cos 2\theta}{\sin\theta}\,,\\ H(-\cos\theta)&\doteqdot(2\nabla/\pi)\left(\sin\theta-\frac{1}{3}\sin 3\theta\right), \end{aligned}\right\} \tag{2.4.18}$$

$$R/[\rho g\nabla(\nabla/8)]\doteqdot 128/(\pi g)\,, \tag{2.4.19}$$

and

$$c_p\doteqdot 3\pi/16\,. \tag{2.4.20}$$

The value of (2.4.19) might be very high compared with the one of (2.2.1.3). Moreover, it will be seen much higher than that considered in § **2.3**, when g is very large, but it should be remembered that the value of (2.4.14) decreases exponentially as g increases, and so it will be smaller than the class like (2.3.1.8) in the range of very low speed.

Chapter 3. Pressure distribution

The velocity potential of the pressure distribution is represented by the one of the doublet in x-direction on the water surface[12], and therefore the wave resistance has the same form as the one of the preceding chapter.

Thence, the discussion like in § **2.1** might hold generally. We do not repeat here such discussion, and proceed to simpler cases.

3.1 The distribution with large aspect ratio. Consider the case in which the wave length generated is sufficiently longer than the longitudinal length of the distribution, so that we may put as

$$\int p(x,y)\,e^{-i\kappa x\cos\theta}\,dx=\rho gH(y)\,, \tag{3.1.1}$$

where $p(x, y)$ means the pressure per unit area.

Then, we have in the same way as in the preceding chapter,

$$R=\rho g\int_{-1}^{1} H(y) G(y) dy\,, \tag{3.1.2}$$

$$G(y)=\frac{g^3}{\pi}\int_{-1}^{1} H(\eta) P_{-5}(0, g\overline{y-\eta}, 0) d\eta\,, \tag{3.1.3}$$

where we take the half breadth for unit length.

The minimum value of (3.1.2) will be attained, when

$$G(y)=C:\ \text{Constant}, \tag{3.1.4}$$

under the conditions

$$\nabla=\int_{-1}^{1} H(y) dy\,, \tag{3.1.5}$$

and

$$H(\pm 1)=0\,. \tag{3.1.6}$$

The last condition is the one by which the integral (3.1.3) has a proper meaning.

The integral equation (3.1.4) is solved in **Appendix B**, and has the next solution by (B. 21), that is,

$$H(-\cos\theta)=\frac{2}{g^3\sin\theta}\sum_{n=0}^{\infty}(-1)^n(2CA_0^{(2n)}+ac_{2n})ce_{2n}(\theta, -q)/\lambda_{2n}\,, \tag{3.1.7}$$

where a is an arbitrary constant and $q=(g/4)^2$.

Two constants C and a are determined by (3.1.5) and (3.1.6), that is,

$$\left.\begin{aligned} \nabla&=\frac{\pi}{g^2}\sum_{n=0}^{\infty}(-1)^n[4C(A_0^{(2n)})^2+ac_{2n}A_0^{(2n)}]/\lambda_{2n}\,,\\ 0&=\sum_{n=0}^{\infty}(-1)^n(2CA_0^{(2n)}+ac_{2n})ce_{2n}(0, -q)/\lambda_{2n}\,. \end{aligned}\right\} \tag{3.1.8}$$

Then, the wave resistance is given as

$$R=\rho g\nabla C\,. \tag{3.1.9}$$

If the condition (3.1.6) would be removed, we have a quasi-waveless solution in the same meaning as in § **2.4.**

These are in neat forms mathematically, but not so practical, because the speed range to be considered for such distributions is generally very high. Hence, consider that g is sufficiently small, and we may have an approximate solution by making use of results in **Appendix C**, but it is too cumbersome to calculation.

It is easier to calculate by making use of the next approximation

$$K_0(gy/2) \underset{gy\to 0}{\doteqdot} -\log[\gamma gy/4]\,, \qquad \log\gamma\text{: Euler's constant,}$$

Then, it is a simple calculation to find

$$H(-\cos\theta)\doteqdot(2\nabla/\pi)\sin\theta\,, \tag{3.1.10}$$

$$C\doteqdot g\nabla/2\pi\,, \tag{3.1.11}$$

and

$$R\doteqdot(4g/\pi)\rho g\nabla(\nabla/8)\,. \tag{3.1.12}$$

The solution (3.1.10) was obtained by H. Maruo[8], and, as he said, the wave resistance is very small compared with the ones in the preceding chapter.

3.2 Symmetrical distribution over a circular disc. The last problem we consider is the case of the distribution symmetrical about the origin over a circular disc with unit radius. The wave resistance is given as

$$R=\frac{\rho g^4}{\pi}\int_0^{\frac{\pi}{2}}|F(g\sec^2\theta,\theta)|^2\sec^5\theta\,d\theta\,, \tag{3.2.1}$$

where

$$F(\kappa,\theta)=\frac{1}{\rho g}\iint p(x,y)\,e^{-i\kappa(x\cos\theta+y\sin\theta)}\,dx\,dy\,.$$

Putting $p(x,y)=\rho gH(r)$, $r=\sqrt{x^2+y^2}$, and integrating on the circle, we have[12]

$$F(\kappa,\theta)\equiv F(\kappa)=2\pi\int_0^1 H(r)\,J_0(\kappa r)\,r\,dr\,, \tag{3.2.2}$$

Now, if we expand as

$$H\left(\sin\frac{\theta}{2}\right)=\frac{2\pi}{\kappa}\sum_{n=0}^{\infty}a_n P_n(\cos\theta)\,, \tag{3.2.3}$$

where P_n means Legendre function, and put this into (3.2.2), we have[12]

$$F(\kappa)=\frac{2\pi}{\kappa}\sum_{n=0}^{\infty}a_n J_{2n+1}(\kappa)\,. \tag{3.2.4}$$

The displacement is calculated as

$$\nabla=2\pi\int_0^1 H(r)\,r\,dr=\pi a_0/2\,, \tag{3.2.5}$$

and the wave resistance

$$R=\rho\sum_{u=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{n+m}a_n a_m R_{n,m}\,, \tag{3.2.6}$$

where

$$R_{n,m}=(-1)^{n+m}\frac{g^2}{\pi}\int_0^{\frac{\pi}{2}} J_{2n+1}(g\sec^2\theta)J_{2m+1}(g\sec^2\theta)\sec\theta\,d\theta\,. \tag{3.2.7}$$

For a while, we consider as g is very large.

Then, we have an asymptotic expansion for $R_{n,m}$, when its suffixes are not so large, that is,

$$R_{n,m}\doteqdot\left[1-\frac{\sqrt{\pi}}{2\sqrt{2g}}\sin\left(2g+\frac{\pi}{4}\right)+\cdots\right]. \tag{3.2.8}$$

The situation is similar to the case § 2.3.1, so that we may have similar conclusions, that is, the lower the pressure near the periphery becomes, the smaller the wave resistance reduces.

Nextly, we introduce the influence function.

$$R=2\pi\rho g\int_0^1 H(r)G(r)\,r\,dr\,, \tag{3.2.9}$$

$$G(r)=2g\int_0^1 H(r')K^*(r,r')\,r'dr'\,, \tag{3.2.10}$$

where

$$K^*(r,r')=g^2\int_0^{\frac{\pi}{2}} J_0(gr\sec^2\theta)J_0(gr'\sec^2\theta)\sec^5\theta\,d\theta\,. \tag{3.2.11}$$

This integral does not converge, therefore introducing the next function

$$K(r,r')=\int_0^{\frac{\pi}{2}} J_1(gr\sec^2\theta)J_1(gr'\sec^2\theta)\sec\theta\,d\theta\,, \tag{3.2.12}$$

we define it by differentiation as follows,

$$K^*(r,r')=\left(\frac{1}{r}\frac{d}{dr}r\right)\left(\frac{1}{r'}\frac{d}{dr'}r'\right)K(r,r')\,. \tag{3.2.13}$$

Since

$$J_1(\kappa r)J_1(\kappa r')=\frac{1}{\pi}\int_0^{\pi} J_0(\kappa R)\cos\varphi\,d\varphi=\frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi} e^{i\kappa R\cos u}\cos\varphi\,d\varphi\,du\,,$$

where

$$R=\sqrt{r^2+r'^2-2rr'\cos\varphi}\,,$$

and

$$\int_0^{\frac{\pi}{2}} e^{iz\sec^2\theta}\sec\theta\,d\theta=e^{\frac{iz}{2}}\int_0^{\infty} e^{\frac{iz}{2}\cosh 2v}\,dv=\frac{\pi i}{4}e^{\frac{iz}{2}}H_0^{(1)}(z/2)$$

putting these into (3.2.12), we have

$$K(r, r')=\frac{i}{4\pi}\int_0^\pi\int_0^\pi e^{\frac{ig}{2}R\cos u}H_0^{(1)}\left(\frac{gR\cos u}{2}\right)\cos\varphi\, d\varphi\, du\,. \tag{3.2.14}$$

The minimum will be attained when

$$G(r)=C:\ \text{Constant}, \tag{3.2.15}$$

under the condition (3.2.5), and given as (3.1.9).

The integral representation of (3.2.14) is not always convenient, but usefull in the limit when g is very small.

Then, we have the next approximation

$$\frac{i}{4\pi}e^{\frac{ig}{2}R\cos u}H_0^{(1)}\left(\frac{g}{2}R\cos u\right)\doteqdot\frac{1}{2\pi^2}\log\left(\frac{\gamma}{4}gR\cos u\right), \qquad \log\gamma:\ \text{Euler's Const.},$$

so that $K(r, r')$ may be integrated as

$$\begin{aligned} K(r, r')&=-\frac{1}{2\pi^2}\int_0^\pi\int_0^\pi\log\left(\frac{\gamma}{4}gR\cos u\right)\cos\varphi\, d\varphi\, du \\ &=\begin{cases} r'/(4r)\,, & \text{for } r>r', \\ r/(4r')\,, & \text{for } r<r', \end{cases} \end{aligned} \tag{3.2.16}$$

by the well known integral.

Put the above kernel into (3.2.13) and (3.2.10), and consider (3.2.15), then it is integrated as

$$\frac{1}{2}Cr=\frac{g}{r}\int_0^r H(r')\,r'dr', \tag{3.2.17}$$

The solution of this integral equation is easily found, that is,

$$H(r) = C/g\,. \tag{3.2.18}$$

The constant is determined by (3.2.5), namely

$$C \doteqdot g\nabla/\pi\,, \tag{3.2.19}$$

Putting this value into (3.1.9), we have finally

$$R\doteqdot(4g/\pi)\rho g\nabla(\nabla/8)\,. \tag{3.2.20}$$

This is coincident with (3.1.12), so that we may have the same resistance in both cases when the breadth of each distribution is the same. In fact, it is easily found that the distribution (3.2.18) integrated in x equals to the one of (3.1.10).

Conclusion. We have studied various minimum problems of the wave making resistance as far as possible, and had conclusions as follows.

1. In general, the minimum problem of the wave resistance has no solution, so that we may always obtain the distribution with smaller resistance than any one theoretically.
2. Since their magnitude will be of various order, we will be necessary to distinguish them, that is, the singularity distributions which represent some ships should be classified by the magnitude of their wave resistance.
3. In some cases, we have the minimum solution, and these solutions present one of such classes.
4. It will be expected that the numerical solution may be unstable in various cases. This is a natural consequence of the conclusion 1., that is, the difference between any two solutions for one problem might be of the lower order in its numerical value of the wave resistance.

These conclusions will be considered one of the extensions of those in the former paper in which we saw many examples of waveless distributions[12].

The author thanks to Prof. Maruo for his kind discussions and suggestions especially for the problem of § 2.4.

References

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Appendix A. Auxiliarly functions. We define the functions P_n by the next integrals,

$$\left.\begin{aligned} P_{2n}(x, y, t) &= (-1)^n \int_0^{\frac{\pi}{2}} e^{-t\sec^2\theta} \sin(x\sec\theta)\cos(y\sec^2\theta\sin\theta)\cos^{2n}\theta\, d\theta\,, \\ P_{2n+1}(x, y, t) &= (-1)^{n+1} \int_0^{\frac{\pi}{2}} e^{-t\sec^2\theta} \cos(x\sec\theta)\cos(y\sec^2\theta\sin\theta)\cos^{2n+1}\theta\, d\theta\,, \end{aligned}\right\} \quad \text{(A. 1)}$$

where $t>0$ and n, m: integer.

When the confusion does not occur, we describe as

$$P_n(x, 0, 0) \equiv P_n(x)\,, \qquad P_n(x, 0, t) \equiv P_n(x, t)\,.$$

$P_n(x)$ is introduced and discussed by T. H. Havelock[1], and $P_n(x, t)$ by the author[11]. Firstly, we have the relations by differentiation

$$\left.\begin{aligned} &\frac{\partial}{\partial x}P_n = P_{n-1}\,, \quad \frac{\partial}{\partial t}P_n = P_{n-2}\,, \quad \frac{\partial^2}{\partial x^2}P_n = \frac{\partial}{\partial t}P_n\,, \\ &\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial t^2}\right)P_n = 0\,, \\ &\left(\frac{\partial^4}{\partial x^4}+\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)P_n = 0\,, \end{aligned}\right\} \qquad \text{(A. 2)}$$

Nextly we consider only P_{-1}, and then other functions will be derived by the above relations.

We may write from (A. 1) changing the variable

$$P_{-1}(x, y, t) = \frac{1}{2}e^{-\frac{t}{2}}\,\mathrm{Re.}\int_{-\infty}^{\infty} \exp.\,[-\rho \cosh (2u - i\gamma) + ix \cosh u]\,du\,,$$

where $2\rho = \sqrt{t^2+y^2}$, $\tan\gamma = y/t$ and $x, y, t > 0$.

Since

$$\exp.\,[-z \cosh \varphi] = I_0(z) + 2\sum_{u=1}^{\infty}(-1)^n I_n(z) \cosh n\varphi\,,$$

and

$$I_n(z) = \frac{e^z}{\sqrt{2\pi z}}\int_0^{\infty} e^{-\frac{v^2}{8z}} J_{2n}(v)\,dv\,,$$

putting these into the above representation, we have

$$P_{-1}(x, y, t) = -\frac{1}{4}\sqrt{\frac{\pi}{2\rho}}\,e^{-\frac{t}{2}+\rho}\int_{-\infty}^{\infty} e^{-\frac{v^2}{8\rho}}\,Y_0(R)\,dv\,, \qquad \text{(A. 3)}$$

where

$$R = \sqrt{x^2+v^2-2xv \cos\frac{\gamma}{2}}\,.$$

Expanding $Y_0(R)$ by the addition theorem, and integrating term by them, we have two expansions,

$$P_{-1}(x, y, t) = -\frac{\pi}{2}e^{-\frac{t}{2}}\left[I_0(\rho)Y_0(x) + 2\sum_{u=1}^{\infty} I_n(\rho)Y_{2n}(x)\cos n\gamma\right], \qquad \text{(A. 4)}$$

$$P_{-1}(x, y, t) = \frac{1}{2}e^{-\frac{t}{2}}\left[K_0(\rho)J_0(x) + 2\sum_{n=1}^{\infty} K_n(\rho)J_{2n}(x)\cos n\gamma\right]. \qquad \text{(A. 5)}$$

It is easily seen that the former does not converge, but gives asymptotic expansion, when $x^2/(4\sqrt{y^2+t^2})$ is very large.

Thus, if ρ vanishes in any fixed x, we have

$$P_{-1}(x, 0, 0)=-\frac{\pi}{2}Y_0(x)\,, \tag{A. 6}$$

Other limits are

$$P_{-1}(0, 0, t)=\frac{1}{2}e^{-\frac{t}{2}}K_0(t/2)\,, \tag{A. 7}$$

$$P_{-1}(0, y, 0)=\frac{1}{2}K_0(y/2)\,, \tag{A. 8}$$

When y vanishes, we have another expansions[11] from (A. 1), as easily seen,

$$P_n(x, t)=\sum_{m=0}^{\infty}\frac{t^m}{m!}P_{n-2m}(x, 0)\,, \tag{A. 9}$$ [2]

$$\left.\begin{aligned} P_{2n}(x, t)&=\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+1}}{(2m+1)!}U_{m-n}(t)\,,\\ P_{2n+1}(x, t)&=\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{(2m)!}U_{m-n-1}(t)\,,\end{aligned}\right\} \tag{A. 10}$$

where

$$U_m(t)=(-1)^m P_{-2m-1}(0, t)\,, \tag{A. 11}$$

The expansion (A. 9) is also asymptotic.

Appendix B. Integral equations. Consider the integral equation

$$f(\theta)=\frac{1}{\pi}\int_0^{\pi}P_{-1}(g\,\overline{\cos\theta-\cos\vartheta})\,\varphi(\vartheta)\,d\vartheta\,, \tag{I}$$

This is solved by J. Dörr[7], here we give another method to obtain the same result. Let us evaluate the next integral

$$I_n(\theta)=\frac{1}{\pi}\int_0^{\pi}P_{-1}(g\,\overline{\cos\theta-\cos\vartheta})\,ce_n(\vartheta, q)\,d\vartheta\,, \qquad q=g^2/4\,, \tag{B. 1}$$

where ce_n means Mathieu function, and all notations are followed to the text by McLachlan[5].

Since

$$P_{-1}(x)=\int_0^{\infty}\cos(x\cosh u)\,du\,,$$

by (A. 1), putting into (B. 1) and using next formulae[5]

$$Ce_{2n}(z, q)=\frac{ce_{2n}(\pi/2, q)}{\pi A_0^{(2n)}}\int_0^{\pi}\cos(g\cosh z\cos u)\,ce_{2n}(u, q)\,du\,,$$

$$Fey_{2n}(z,q)=-\frac{2ce_{2n}(\pi/2,q)}{\pi A_0^{(2n)}}\int_0^\infty \cos(g\cosh z\cosh u)\,Ce_{2n}(u,q)\,du\,,$$

we have

$$I_{2n}(iz)=-\frac{\pi}{2}\left[\frac{A_0^{(2n)}}{ce_{2n}(\pi/2,q)}\right]^2 Fey_{2n}(z,q)\,, \tag{B. 2}$$

When z is imaginary,

$$\text{Re.}\,[Fey_{2n}(-i\theta,q)]=\frac{Fey_{2n}(0,q)}{ce_{2n}(0,q)}ce_{2n}(\theta,q)\,,$$

therefore we have

$$I_{2n}(\theta)=\mu_{2n}\,ce_{2n}(\theta,q)\,, \tag{B. 3}$$

$$\mu_{2n}=-\frac{\pi}{2}\left[\frac{A_0^{(2n)}}{ce_{2n}(\pi/2,q)}\right]^2\frac{Fey_{2n}(0,q)}{ce_{2n}(0,q)}\,. \tag{B. 4}$$ [7]

In the same way, we have

$$\left.\begin{aligned} I_{2n+1}(\theta)&=\mu_{2n+1}\,ce_{2n+1}(\theta,q)\,,\\ \mu_{2n+1}&=-\frac{\pi}{2}\left[\frac{kA_1^{(2n+1)}}{ce'_{2n+1}(\pi/2,q)}\right]^2\frac{Fey_{2n+1}(0,q)}{ce_{2n+1}(0,q)}\,,\end{aligned}\right\} \tag{B. 5}$$ [7]

In another way, since

$$\mu_n=\frac{2}{\pi}\int_0^\pi I_n(\theta,q)\,ce_n(\theta,q)\,d\theta\,,$$

integrating in θ and ϑ, we have

$$\left.\begin{aligned} \mu_{2n}&=2\left[\frac{A_0^{(2n)}}{ce_{2n}(\pi/2,q)}\right]^2\int_0^\infty Ce_{2n}^2(u,q)\,du\,,\\ \mu_{2n+1}&=2\left[\frac{kA_1^{(2n+1)}}{ce'_{2n+1}(\pi/2,q)}\right]^2\int_0^\infty Ce_{2n+1}^2(u,q)\,du\,.\end{aligned}\right\} \tag{B. 6}$$

The similar integral equation

$$F(\theta)=\frac{1}{\pi}\int_0^\pi P_{-5}(g\,\overline{\cos\theta-\cos\theta})\,\psi(\vartheta)\,d\vartheta\,, \tag{II}$$

are easily solved by the same method as used by J. Dörr[7].

Consider the integral

$$I_n^{(4)}(\theta)=\frac{1}{\pi}\int_0^\pi P_{-5}(g\,\overline{\cos\theta-\cos\vartheta})\,ce_n(\vartheta)\,d\vartheta\,, \tag{B. 7}$$

then we have by differentiation of (B. 1)

$$I_n^{(4)}(\theta)=\frac{1}{g^4}\left(\frac{d}{dx}\right)^4 I_n(\theta)\,,\qquad x=-\cos\theta\,. \tag{B. 8}$$

Hence, we have formally by (B. 3)

$$I_n^{(4)}(\theta)=\frac{\mu_n}{g^4}\left(\frac{d}{dx}\right)^4 ce_n(\theta, q). \tag{B. 9}$$

However, the solution of (II) does not determined up to the function which becomes to zero by differentiation as (B. 8), that is,

$$a_0'+a_1'x+a_2'x^2+a_3'x^3=a_0+a_1\cos\theta+a_2\cos 2\theta+a_3\cos 3\theta, \tag{B. 10}$$

where a's are arbitrary constants.

Since[5]

$$1=2\sum_{n=0}^{\infty}A_0^{(2n)}ce_{2n}(\theta, q),$$

$$\cos 2r\theta=\sum_{n=0}^{\infty}A_{2r}^{(2n)}ce_{2n}(\theta, q), \qquad \text{for } r\geq 1,$$

$$\cos(2r+1)\theta=\sum_{u=0}^{\infty}A_{2r+1}^{(2n+1)}ce_{2n+1}(\theta, q), \qquad \text{for } r\geq 0,$$

the undetermined function will be

$$\sum_{u=0}^{\infty}\frac{1}{\mu_{2n}}(2a_0A_0^{(2n)}+a_2A_2^{(2n)})ce_{2n}(\theta, q)+\sum_{u=0}^{\infty}\frac{1}{\mu_{2n+1}}(a_1A_1^{(2n+1)}+a_3A_3^{(2n+1)})ce_{2n+1}(\theta, q), \tag{B. 11}$$

Nextly, consider the integral equation

$$f(\theta)=\frac{1}{2\pi}\int_0^{\pi}K_0\left(\frac{g}{2}\,\overline{\cos\theta-\cos\vartheta}\right)\varphi(\vartheta)\,d\vartheta, \tag{III}$$

In like manner as the former, define the integral

$$J_{2n}(\theta)=\frac{1}{2\pi}\int_0^{\pi}K_0\left(\frac{g}{2}\,\overline{\cos\theta-\cos\vartheta}\right)ce_{2n}(\vartheta, -q)\,d\vartheta, \tag{B. 12}$$

where $q=(g/4)^2=k^2$.

Since

$$\frac{1}{\pi}\int_0^{\pi}\cos(2k\cos\theta\sinh u)\,ce_{2n}(\theta, -q)\,d\theta=\frac{(-1)^nA_0^{(2n)}}{ce_{2n}(0, q)}Ce_{2n}(u, q),$$

$$\frac{1}{\pi}\int_0^{\infty}\cos(2k\cosh z\sinh u)\,Ce_{2n}(u, q)\,du=\frac{(-1)^nA_0^{(2n)}}{ce_{2n}(0, q)}Fek_{2n}(z, -q),$$

and

$$\text{Re.}\,[Fek(-i\theta, -q)]=\frac{Fek_{2n}(0, -q)}{ce_{2n}(0, -q)}ce_{2n}(\theta, -q),$$

we have

$$J_{2n}(\theta)=\lambda_{2n}\, ce_{2n}(\theta,\, -q)\,, \tag{B. 13}$$

with

$$\lambda_{2n}=\frac{\pi}{2}\left[\frac{A_0^{(2n)}}{ce_{2n}(0,\, q)}\right]^2 \frac{Fek_{2n}(0,\, -q)}{ce_{2n}(0,\, -q)}\,, \tag{B. 14}$$

or

$$\lambda_{2n}=\left[\frac{A_0^{(2n)}}{ce_{2n}(0,\, q)}\right]^2 \int_0^\infty Ce_{2n}^2(u,\, q)\, du=\frac{\mu_{2n}}{2}\left[\frac{ce_{2n}(\pi/2,\, q)}{ce_{2n}(0,\, q)}\right]^2\,, \tag{B. 15}$$

Lastly, consider the equation

$$F(\theta)=\frac{1}{\pi}\int_0^\pi P_{-5}(0,\, g\overline{y-\eta},\, 0)\,\psi(\vartheta)\, d\vartheta\,, \tag{IV}$$

and define the integral

$$J_{2n}^*(\theta)=\frac{1}{\pi}\int_0^\pi P_{-5}(0,\, g\overline{y-\eta},\, 0)\, ce_{2n}(\vartheta,\, -q)\, d\vartheta\,, \tag{B. 16}$$

where $y=-\cos\theta,\ \eta=-\cos\vartheta$.

Since

$$\begin{aligned} P_{-5}(0,\, gy,\, 0)&=\int_0^\infty \cos\left(\frac{gy}{2}\sinh 2u\right)\cosh^4 u\, du \\ &=\frac{3}{16}K_0(gy/2)-\frac{1}{16}K_2(gy/2)=\frac{1}{4}\left(1-\frac{2}{g^2}\frac{d^2}{dy^2}\right)K_0(gy/2)\,. \end{aligned} \tag{B. 17}$$ [3]

Hence, we have

$$\left.\begin{aligned} F(\theta)&=\frac{1}{2}\left(1-\frac{2}{g^2}\frac{d^2}{dy^2}\right)f(\theta)\,, \\ J_{2n}^*(\theta)&=\frac{1}{2}\left(1-\frac{2}{g^2}\frac{d^2}{dy^2}\right)J_{2n}(\theta)\,. \end{aligned}\right\} \tag{B. 18}$$

These are differential equations, so that their solution consists of homogeneous solutions and a special one.

For example, put in (IV)

$$F(\theta)=C:\ \text{Constant}, \tag{B. 19}$$

then we have the solution of (B. 18) as follows,

$$f(\theta)=2C+a\cosh\,(gy/\sqrt{2})\,, \tag{B. 20}$$

where a is an arbitrary constant.

Since

$$1=2\sum_{n=0}^{\infty}(-1)^n A_0^{(2n)} ce_{2n}(\theta, -q),$$

and

$$\cosh(g\cos\theta/\sqrt{2})=2\sum_{n=0}^{\infty}(-1)^n c_{2n} ce_{2n}(\theta, -q),$$

where

$$c_{2n}=\frac{2}{\pi}\int_0^{\pi}\cosh\left(\frac{g}{\sqrt{2}}\cos\theta\right)ce_{2n}(\theta, -q)\,d\theta=\frac{A_0^{(2n)}}{ce_{2n}(0, q)}Ce_{2n}(\sinh^{-1}\sqrt{2}, q),$$

we have the solution of (IV) as

$$\psi(\theta)=2\sum_{u=0}^{\infty}(-1)^n(2CA_0^{(2n)}+ac_{2n})ce_{2n}(\theta, -q)/\lambda_{2n},$$

Appendix C. Approximate relations of Mathieu functions. Almost all formulae in the following are found in the texts[5),6)], but some important results not found. That is the reason why we rewrite them.

Firstly, if $q(=k^2)$ is sufficiently small, we have[5)]

$$\left.\begin{aligned}&ce_0(\theta, q)\doteqdot 1/\sqrt{2},\\ &ce_m(\theta, q)\doteqdot\cos m\theta, \quad \text{for } m\geq 1,\end{aligned}\right\} \quad (\text{C. }1)$$

$$\left.\begin{aligned}&Ce_m(u, q)\doteqdot p'_m J_m(ke^u),\\ &Fey_m(u, q)\doteqdot p'_m Y_m(ke^u),\end{aligned}\right\} \quad (\text{C. }2)$$

where

$$p'_0\doteqdot 1/[\sqrt{2}J_0(k)], \quad p'_m\doteqdot 1/J_m(k), \quad \text{for } m>0, \quad (\text{C. }3)$$

and

$$\left.\begin{aligned}&ce_0(0, q)\doteqdot ce_0(\pi/2, q)\doteqdot A_0^{(0)}=1/\sqrt{2},\\ &ce_{2n}(0, q)\doteqdot ce_{2n}(\pi/2, q)\doteqdot 1, \quad A_0^{(2n)}\doteqdot J_{2n}(k), \quad \text{for } n\geq 1.\end{aligned}\right\} \quad (\text{C. }4)$$

Then, (B. 4) reduces to

$$\left.\begin{aligned}&\mu_0\doteqdot -\log(\gamma k/2), \quad \log\gamma=C:\ \text{Egler's constant},\\ &\mu_m\doteqdot -(\pi/2)J_m(k)Y_m(k)\doteqdot 1/2m, \quad \text{for } m\geq 1.\end{aligned}\right\} \quad (\text{C. }5)$$

Even if q is negative, these relations are valid, and so we have from (B. 15)

$$\lambda_m\doteqdot\mu_m/2. \quad (\text{C. }6)$$

Secondly, if q is sufficiently large, but m or n not so large, we have

$$ce_m(\theta, q)\doteqdot\frac{C_m}{\cos^{m+1}\theta}\left[e^{g\sin\theta}\left(\cos\left(\frac{\theta}{2}+\frac{\pi}{4}\right)\right)^{2m+1}+e^{-g\sin\theta}\left(\sin\left(\frac{\theta}{2}+\frac{\pi}{4}\right)\right)^{2m+1}\right], \quad (\text{C. }7)$$

$$Ce_m(z, q) \doteqdot \frac{C_m}{2^{m-\frac{1}{2}}\sqrt{\cosh z}} \cos\left[g \sinh z-(2m+1)\tan^{-1}\left(\tanh\frac{z}{2}\right)\right], \tag{C. 8}$$

where

$$\left.\begin{aligned} C_{2n}&=(-1)^n 2^{2n-\frac{1}{2}}\frac{ce_{2n}(0, q)\,ce_{2n}(\pi/2, q)}{\sqrt{\pi k}A_0^{(2n)}}, \\ C_{2n+1}&=(-1)^{n+1}2^{2n+\frac{1}{2}}\frac{ce_{2n+1}(0, q)\,ce'_{2n+1}(\pi/2, q)}{k\sqrt{\pi k}A_1^{(2n+1)}}. \end{aligned}\right\} \tag{C. 9}$$

Put θ and z equal to zero in (C. 7) and (C. 8), then we have

$$C_m \doteqdot 2^{m-\frac{1}{2}}\,ce_m(0, q), \tag{C. 10}$$

and also from (C. 9)

$$\left.\begin{aligned} ce_{2n}(\pi/2, q)&\doteqdot(-1)^n\sqrt{\pi k}\,A_0^{(2n)}, \\ ce'_{2n+1}(\pi/2, q)&\doteqdot(-1)^{n+1}k\sqrt{\pi k}\,A_1^{(2n+1)}. \end{aligned}\right\} \tag{C. 11}$$

Nextly, integrating (C. 7), we have

$$\left.\begin{aligned} A_0^{(2n)}/ce_{2n}(0, q)&\doteqdot e^{2k}\Gamma\left(n+\frac{1}{2}\right)\Big/[\pi 2^{n+1}(2k)^{n+\frac{1}{2}}], \\ A_1^{(2n+1)}/ce_{2n+1}(0, q)&\doteqdot e^{2k}\Gamma\left(n+\frac{3}{2}\right)\Big/[\pi 2^{n}(2k)^{n+\frac{3}{2}}]. \end{aligned}\right\} \tag{C. 12}$$ [6]

Moreover, we have by the recurrence formula[5]

$$A_2^{(2n)} \doteqdot 2A_0^{(2n)}[-1+(8n+2)/\sqrt{q}+\cdots]. \tag{C. 13}$$

In the neighbourhood of $\theta=\pi/2$, we have the other approximation, that is

$$\left.\begin{aligned} ce_m(\theta, q)&\doteqdot\frac{(\pi k/2)^{\frac{1}{4}}}{\sqrt{m!}}D_m(z), \qquad z=2\sqrt{k}\cos\theta, \\ \text{where}\qquad D_m(z)&=(-1)^m e^{\frac{z^2}{4}}(d/dz)^m e^{-\frac{z^2}{2}}. \end{aligned}\right\} \tag{C. 14}$$ [6]

Hence, we have at $\theta=\pi/2$,

$$\left.\begin{aligned} ce_{2n}(\pi/2, q)&\doteqdot(-1)^n(\pi k/2)^{\frac{1}{4}}\sqrt{(2n)!}/(2^n n!), \\ ce'_{2n+1}(\pi/2, q)&\doteqdot(-1)^{n+1}\sqrt{k}\,(\pi k/2)^{\frac{1}{4}}\sqrt{(2n+1)!}/(2^{n-1}n!). \end{aligned}\right\} \tag{C. 15}$$

Putting these into (C. 11), we have

$$\left.\begin{aligned} A_0^{(2n)}&\doteqdot\sqrt{(2n)!}/[(2^n n!)(2\pi k)^{\frac{1}{4}}], \\ A_1^{(2n+1)}&\doteqdot\sqrt{(2n+1)!}/[(2^{n-1}n!)(2\pi k)^{\frac{1}{4}}\sqrt{k}], \end{aligned}\right\} \tag{C. 16}$$

then, from (C. 12)

$$\left.\begin{aligned} ce_{2n}(0,q) &\doteqdot \frac{2^{3n+1}}{\sqrt{(2n)!}} k^n (2\pi k)^{\frac{1}{4}} e^{-2k}, \\ ce_{2n+1}(0,q) &\doteqdot \frac{2^{3n+3}}{\sqrt{(2n+1)!}} k^{n+\frac{1}{2}} (2\pi k)^{\frac{1}{4}} e^{-2k}, \end{aligned}\right\} \tag{C. 17}$$

Lastly, put (C. 8) into (B. 6) and integrate, and we have

$$\mu_m \doteqdot \frac{1}{2k} [ce_m(0,q)]^2,$$

and by (C. 17) this equals approximately to

$$\mu_{2n} \doteqdot \frac{2^{4n+2}\sqrt{\pi}\,(2k)^{2n-\frac{1}{2}}}{(2n)!} e^{-4k}, \qquad \mu_{2n+1} \doteqdot \frac{2^{4n+5}\sqrt{\pi}\,(2k)^{2n+\frac{1}{2}}}{(2n+1)!} e^{-4k}, \tag{C. 18}$$

and also

$$\lambda_{2n} \doteqdot (2n)!\sqrt{(\pi/2k)}/[2^{2n+2}(n!)^2], \tag{C. 19}$$