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On the Fundamental Function in the Theory of the Wave-Making Resistance of Ships

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Abstract

The author introduces the functions which play important roles in the theory of the wave-making resistance of ships, converts their double integral to a simple one, obtains various integral representations and analyzes their properties especially in the relations of the well-known simple functions.

Then finally, he gives the list of the available tables of his or similar functions.

1. Introduction; Definition and Differentiation

It is difficult but necessary to compute the fundamental function in the theory of the wave-making resistance of ships for the development of this theory.

The author has tried to analyze this function and finds its double integral to be convertible to a simple one, making use of the velocity potential of T. H. Havelock[16], but the preceding works[2)3)5)6] were limited to the case of two variables x and t (see the definition below).

The functions are defined as follows[6]:

$$O_n^{(1)}(x, y, t)=\lim_{\mu\to+0}\frac{(-i)^n}{4\pi}\int_{-\pi}^{\pi}\int_0^{\infty}\frac{\exp.[-kt+ik(x\cos u+y\sin u)]}{k\cos^2 u-1+\mu i\cos u}\cos^{n+2}u\,dkdu\,, \tag{1.1}$$

$$O_n^{(2)}(x, y, t)=\lim_{\mu\to+0}\frac{(-i)^n}{4\pi}\int_{-\pi}^{\pi}\int_0^{\infty}\frac{\exp.[-kt+ik(x\cos u+y\sin u)]}{k\cos^2 u-1-\mu i\cos u}\cos^{n+2}u\,dkdu\,, \tag{1.2}$$

$$P_n(x, y, t)=\frac{1}{2}[O_n^{(1)}(x, y, t)-O_n^{(2)}(x, y, t)]\,, \tag{1.3}$$

$$Q_n(x, y, t)=\frac{1}{2}[O_n^{(1)}(x, y, t)+O_n^{(2)}(x, y, t)]\,, \tag{1.4}$$

where x, y and t are assumed real positive and n integer greater than -2.

All functions are real and they have usually oscillatory parts but $O_n^{(1)}$ is monotonic for the positive x.

From the definition (1.3), we can easily find that

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$$\left.\begin{aligned} P_{2n}(x, y, t)&=(-1)^n\int_0^{\pi/2} e^{-t\sec^2 u} \sin(x\sec u)\cos(y\sec^2 u\sin u)\cos^{2n} u\,du \\ P_{2n+1}(x, y, t)&=(-1)^{n+1}\int_0^{\pi/2} e^{-t\sec^2 u} \cos(x\sec u)\cos(y\sec^2 u\sin u)\cos^{2n+1} u\,du \end{aligned}\right\} \tag{1.5}$$

These are the dircect generalizations of Havelock's P_n function[12)13)].

For the negative x, we can easily find also that

$$\left.\begin{aligned} O_n^{(1)}(-x, y, t)&=(-)^n O_n^{(2)}(x, y, t)\,, \\ P_n(-x, y, t)&=(-1)^{n+1}P_n(x, y, t)\,, \\ Q_n(-x, y, t)&=(-1)^n Q_n(x, y, t)\,. \end{aligned}\right\} \tag{1.6}$$

and then

$$O_n^{(1)}(-x, y, t)=(-1)^n[O_n^{(1)}(x, y, t)-2P_n(x, y, t)]\,. \tag{1.7}$$

Accordingly, we don't consider the function $O_n^{(2)}$ in the following, because it is the same one as $O_n^{(1)}$ when the sign of x is reversed.

For the negative y, we have also

$$\left.\begin{aligned} O_n^{(1)}(x, -y, t)&=O_n^{(1)}(x, y, t)\,, \\ P_n(x, -y, t)&=P_n(x, y, t)\,, \\ Q_n(x, -y, t)&=Q_n(x, y, t)\,. \end{aligned}\right\} \tag{1.8}$$

Now, the velocity potential of the unit source in the uniform flow of the unit velocity flowing from the positive x direction down to the negative at the point (x', y', z') under the water surface where the z-axis is taken positive vertically upwards, is written by T. H. Havelock[16)] as follows.

$$\begin{aligned} S(x, y, z; x', y', z')=&\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}-\frac{1}{\sqrt{(x-x')^2-(y-y')^2+(z+z')^2}} \\ &+\frac{g}{\pi}\int_{-\pi}^{\pi}\int_0^{\infty}\frac{\exp.[k(z+z')+ik(x-x')\cos\theta+ik(y-y')\sin\theta]}{k\cos^2\theta-g+\mu i\cos\theta}dk\,d\theta\,, \end{aligned}$$

where g means the gravity constant in this unit system.

This is written by making use of the definition (1.1) as

$$\begin{aligned} S(x, y, z; x', y', z')=&\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}-\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}} \\ &+4gO_{-2}^{(1)}[g(x-x'),\ g(y-y'),\ -g(z+z')]\,, \end{aligned} \tag{1.9}$$

Hence, the most fundamental functions are $O_{-2}^{(1)}$ and its derivatives.

In this respect, the definition (1.1) is not very convenient but we follow it merely because the author has used it to the present.

Differentiating the definition formula partially, we have

$$\left.\begin{aligned}&\frac{\partial}{\partial x}\left\{\begin{matrix}O_n^{(1)}\\ Q_n\end{matrix}\right\}=\left\{\begin{matrix}O_{n-1}^{(1)}\\ Q_{n-1}\end{matrix}\right\}+q_{n-1}\ , \quad \frac{\partial}{\partial x}P_n=P_{n-1}\ ,\\ &\frac{\partial}{\partial t}\left\{\begin{matrix}O_n^{(1)}\\ Q_n\end{matrix}\right\}=\left\{\begin{matrix}O_{n-2}^{(1)}\\ Q_{n-2}\end{matrix}\right\}+q_{u-2}\ , \quad \frac{\partial}{\partial t}P_n=P_{n-2}\ ,\end{aligned}\right\} \tag{1.10}$$

where

$$\left.\begin{aligned}q_{2n}(x,\ y,\ t)&=\frac{(-1)^n}{4\pi}\int_{-\pi}^{\pi}\int_0^\infty e^{-kt}\cos(kx\cos u)\cos(ky\sin u)\cos^{2n+2}u\,dkdu\ ,\\ q_{2n+1}(x,\ y,\ t)&=\frac{(-1)^n}{4\pi}\int_{-\pi}^{\pi}\int_0^\infty e^{-kt}\sin(kx\cos u)\cos(ky\sin u)\cos^{2n+3}u\,dkdu\ ,\end{aligned}\right\} \tag{1.11}$$

and we learn from this definition that

$$\left.\begin{aligned}&\frac{\partial}{\partial x}q_n=\frac{\partial}{\partial t}q_{n+1}\ ,\\ &q_n+t\frac{\partial}{\partial t}q_n+x\frac{\partial}{\partial x}q_n+y\frac{\partial}{\partial y}q_n=0\ ,\end{aligned}\right\} \tag{1.12}$$

and especially by integration

$$\left.\begin{aligned}q_{-3}(x,\ y,\ t)&=\frac{tx}{2r\rho^2}\ , \qquad \rho^2=t^2+y^2\ ,\ r^2=x^2+y^2+t^2\ ,\\ q_{-2}(x,\ y,\ t)&=-\frac{1}{2r}\ ,\\ q_{-1}(x,\ y,\ t)&=-\frac{1}{2r(r+t)}\ ,\\ q_0(x,\ y,\ t)&=\frac{\rho^2+rt}{2r(r+t)^2}\ .\end{aligned}\right\} \tag{1.13}$$

Here we are to notice that the restriction for the order n may be rejected by the introduction of the relations (1.10) and (1.12).

Lastly, we can verify the next differential relations.

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial t^2}\right)\left\{\begin{matrix}O_n^{(1)}\\ P_n\\ Q_n\end{matrix}\right\}=0\ , \tag{1.14}$$

$$\left.\begin{aligned}&\left(\frac{\partial^2}{\partial x^2}-\frac{\partial}{\partial t}\right)\left\{\begin{matrix}O_n^{1)}\\ P\\ Q_n\end{matrix}\right\}=\frac{\partial}{\partial x}q_n\ ,\\ &\left(\frac{\partial^2}{\partial x^2}-\frac{\partial}{\partial t}\right)P_n=0\ .\end{aligned}\right\} \tag{1.15}$$

And then, combining two equations, we have

$$\left.\begin{aligned}&\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial t^2}+\frac{\partial}{\partial t}\right)\begin{Bmatrix}O_n^{(1)}\\Q_n\end{Bmatrix}=-\frac{\partial}{\partial x}q_n\,,\\&\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial t^2}+\frac{\partial}{\partial t}\right)P_n=0\,,\end{aligned}\right\}\tag{1.16}$$

or

$$\left.\begin{aligned}&\left(\frac{\partial^4}{\partial x^4}+\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\begin{Bmatrix}O_n^{(1)}\\Q_n\end{Bmatrix}=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial}{\partial t}\right)\frac{\partial}{\partial x}q_n\,,\\&\left(\frac{\partial^4}{\partial x^4}+\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)P_n=0\,.\end{aligned}\right\}\tag{1.17}$$

2. Integral Representation I.

Let us consider deforming the integral (1.1) in the simplest case $n=-1$.

Considering the integration with respect to u in (1.1), we can write also it as

$$O_{-1}^{(1)}(x,y,t)=\frac{i}{4\pi}\int_{\pi/2}^{\pi/2}\cos u\,du\int_0^\infty e^{-kt+iky\sin u}\left[\frac{e^{ikx\cos u}}{k\cos^2 u-1+\mu i\cos u}-\frac{e^{-ikx\cos u}}{k\cos^2 u-1+\mu i\cos u}\right]dk\,.$$

If we change the variable k to m as $m=k\cos u$ and then u to v as $\sec u=\cosh v$, this becomes

$$O_{-1}^{(1)}(x,y,t)=\frac{1}{4\pi}\int_{-\infty}^\infty dv\int_0^\infty e^{-m\rho\cosh(v-i\alpha)}\left[\frac{e^{imx}}{m+\mu i-\cosh v}-\frac{e^{-imx}}{m-\mu i-\cosh v}\right]dm\,,$$

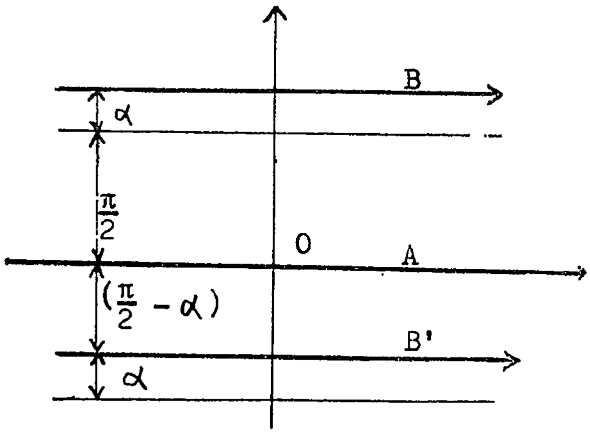

Fig. 1. v-plane

where $\rho=\sqrt{t^2+y^2}$ and $\alpha=\tan^{-1}(y/t)$.

Now let us consider the integration with respect to v in the complex v-plane (see Fig. 1) and deform the path of integration A to the line B for the above first term and to B' for the second term.

Then, this integral on B and B' goes to

$$\frac{1}{4\pi i}\int_{-\infty}^\infty du\int_{-\infty}^\infty\frac{e^{im(\rho\sinh u-x)}}{m+i\sinh(u+i\alpha)-\mu i}\,dm\,,$$

and then integrate this with respect to m in the complex m-plane.

This is a well known integral and vanishes except $0<u<\beta$, and equals

$$-\frac{1}{2}\int_0^\beta e^{-(x-\rho\sinh u)\sinh(u+i\alpha)}du,\ \text{where}\ \beta=\sinh^{-1}(x/\rho)\,.$$

Other parts of the integral are residues at the poles lying between the line A and B and between A and B'.

Calculating these residues and changing the variable slightly, we have finally

$$O_{-1}^{(1)}(x, y, t)=\frac{1}{2}\int_{L_1+L_2} \exp.[\{\rho \sinh (u-i\alpha)-x\} \sinh u]\, du\,, \tag{2.1}$$

where L_1 and L_2 are the paths of integration shown as in Fig. 2.

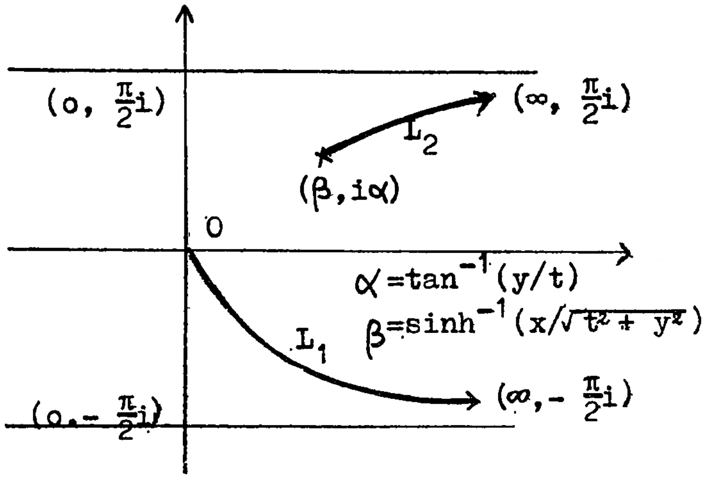

Fig. 2. u-plane

In the same way, we have generally

$$O_{-n}^{(1)}(x, y, t)=\frac{(-1)^{n-1}}{2}\int_{L_1+L_2} \exp.[\{\rho \sinh (u-i\alpha)-x\} \sinh u] \sinh^{n-1} u\, du, \text{ for } n\geqq 1\,, \tag{2.2}$$

The equivalent formula for the case $n=2$ and $y=0$ was obtained by R. Guilloton[8] who introduced it from Mitchell's integral, so that we may see the identity of his one with Havelock's[11].

For the case $n\leqq 0$, the integrand diverges near the origin, so that an artificial technique must be used.

Let us consider the case $n=0$ for example; we can write its integral as:

$$O_0^{(1)}(x, y, t)=\frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{dv}{\cosh v}\int_0^{\infty} e^{-mt\cosh v}\cosh(my\sinh v)\Big[\frac{e^{imx}}{m+\mu i-\cosh v} + \frac{e^{-imx}}{m-\mu i-\cosh v}\Big]dm\,,$$

but

$$\frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{\sinh v}{\cosh v}dv\int_0^{\infty} e^{-mt\cosh v}\cos(my\sinh v)[\text{the same as the above}]dm=0\,,$$

as easily from the symmetric character.

Hence, adding the latter integral to the former and integrating in the same way as the above, we have:

$$O_0^{(1)}(x, y, t)=-\frac{1}{2}\int_{L_1+L_2} \exp.[\rho \sinh(u-i\alpha)-x\sinh u]\frac{(1-\cosh u)}{\sinh u}du\,, \tag{2.3}$$

and moreover

$$O_1^{(1)}(x, y, t)=\frac{1}{2}\int_{L_1+L_2} \exp.[\text{the same as the above}]\frac{(1-\cosh u)}{\sinh^2 u}du\,, \tag{2.4}$$

$$O_2^{(1)}(x, y, t)=-\frac{1}{2}\int_{L_1+L_2} \exp.[\text{the same as the above}]\frac{(1-\cosh u+\frac{1}{2}\cosh u\sinh^2 u)}{\sinh^3 u}du\,. \tag{2.5}$$

All these formulas contain imaginary parts but only their real parts are to be taken.

Now, since we have already the integral representation of P_n as (1.5), subtracting it from the above formula, we have Q_n by the definition (1.3) and (1.4) as follows:

$$\left.\begin{aligned} Q_{-2n}(x, y, t)&=(-1)^{n-1}\int_0^{\pi/2} e^{-t\sin^2 u}\cosh(y\sin u\cos u)\cos(x\sin u)\sin^{2n-1}u\,du \\ &\quad+\frac{1}{2}\int_0^{\beta+i\alpha}\exp[\rho\sinh(u-i\alpha)-x\sinh u]\sinh^{2n-1}u\,du, \quad \text{for } n\geqq 1 \\ Q_{-2n-1}(x, y, t)&=(-1)^{n}\int_0^{\pi/2} e^{-t\sin^2 u}\cosh(y\sin u\cos u)\sin(x\sin u)\sin^{2n}u\,du \\ &\quad-\frac{1}{2}\int_0^{\beta+i\alpha}\exp.[\text{the same as the above}]\sinh^{2n}u\,du, \quad \text{for } n\geqq 0\,, \end{aligned}\right\} \tag{2.6}$$

and

$$\begin{aligned} Q_0(x, y, t)&=-\int_0^{\pi/2} e^{-t\sin^2 u}\cosh(y\sin u\cos u)\cos(x\sin u)\frac{(1-\cos u)}{\sin u}du \\ &\quad+\frac{1}{2}\int_0^{\beta+i\alpha}\exp.[\{\rho\sinh(u-i\alpha)-x\}\sinh u]\frac{(1-\cosh u)}{\sinh u}du \\ &\quad+\int_0^{\infty} e^{-t\cosh^2 u}\cos(x\cosh u)\cos(y\cosh u\sinh u)\tanh u\,du, \cdots \end{aligned} \tag{2.7}$$

The last one is equivalent to the next formula which is obtained by the integration of (2.6) making use of the relation (1.10).

$$\begin{aligned} Q_0(x, y, t)&=Q_0(0, y, t)-\frac{1}{2}\log\left(\frac{r+t}{\rho+t}\right) \\ &\quad+\int_0^{\pi/2} e^{-t\sin^2 u}\cosh(y\sin u\cos u)[1-\cos(x\sin u)]\frac{du}{\sin u} \\ &\quad-\frac{1}{2}\int_0^{\beta-i\alpha}[1-\exp.[\{\rho\sinh(u-i\alpha)-x\}\sinh u]]\frac{du}{\sinh u}\,, \end{aligned} \tag{2.8}$$

3. Integral Representation II.

The preceding analysis enables us to compute their values numerically, but it is felt that their integrands converge very slowly in the range for small t and y, especially in the formula (1.5) for the P_n function.

For such a range, the following analysis will be convenient for numerical works.

For convenience' sake, let us consider the function P_{-1} and other functions will be deduced by the differentiation and integration.

At first, we have from (1.5) after the substitution of the variable in its integral,

$$P_{-1}(x, y, t)=\frac{1}{2}\exp.\left(-\frac{1}{2}t+\frac{1}{2}\rho\right)\int_{-\infty}^{\infty}\exp.\left[ix\cosh u-\rho\cosh^2\left(u-i\frac{\alpha}{2}\right)du\right., \qquad (3.1)$$

where its real part is to be taken.

Since we have the integral

$$e^{-\rho\cosh^2\left(u-i\frac{\alpha}{2}\right)}=\frac{1}{2\sqrt{\pi\rho}}\int_{-\infty}^{\infty}\exp.\left[-\frac{v^2}{4\rho}-iv\cosh\left(u-i\frac{\alpha}{2}\right)\right]dv\,,$$

we can rewrite it as

$$\frac{e^{-\frac{t}{2}+\frac{\rho}{2}}}{4\sqrt{\pi\rho}}\int_{-\infty}^{\infty}\exp\left(-\frac{v^2}{4\rho}\right)dv\int_{-\infty}^{\infty}\exp.\left[ix\cosh u-iv\cosh\left(u-i\frac{\alpha}{2}\right)\right]du\,,$$

If we introduce here the new variables as shown in Fig. 3, that is,

$$R\cos\phi=x-v\cos\frac{\alpha}{2}\,,\quad R\sin\phi=v\sin\frac{\alpha}{2}\,,$$

$$\text{or}\quad R^2=x^2+v^2-2xv\cos\frac{\alpha}{2}\,,$$

then we may write

$$x\cosh u-v\cosh\left(u-i\frac{\alpha}{2}\right)=R\cosh(u+i\phi)\,.$$

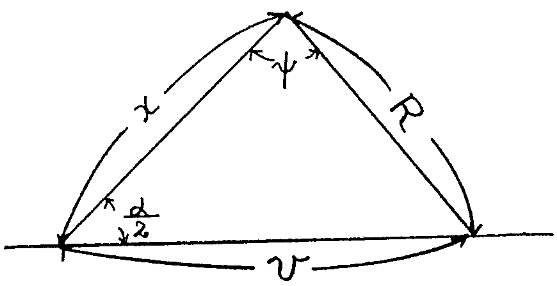

Fig. 3.

Thence, shifting the path of u-integration in parallel by ϕ and carrying out the integration which gives the Bessel function of the second species, we have finally

$$P_{-1}(x, y, t)=-\frac{\sqrt{\pi}}{4\sqrt{\rho}}e^{-\frac{t}{2}+\frac{\rho}{2}}\int_{-\infty}^{\infty}\exp.\left(-\frac{v^2}{4\rho}\right)Y_0(R)dv\,, \qquad (3.2)$$

This is suitable for numerical computations when ρ is small.

When y vanishes, this becomes

$$P_{-1}(x, 0, t)=-\frac{\sqrt{\pi}}{4\sqrt{t}}\int_{-\infty}^{\infty}\exp.(-v^2/4t)Y_0(|x-v|)dv\,, \qquad (3.3)$$

which was given by E. T. Goodwin[10], and we see that the right hand side satisfies the differential equation (1.15), that is, of the heat conduction.

Moreover, taking care of this point, we have generally

$$P_n(x, y, t)=\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty}\exp.(-v^2/4t)P_n(x-v, y, 0)dv, \tag{3.4}$$

From the stand point that these function can be represented in the form of the solution of the partial differential eqation, we can apply this method to the equations (1.14) and (1.16).

In the former case, Laplace's equation, there are many formulas but we will not consider them here.

In the latter case, we can change it to the next equation

$$\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}+\frac{1}{4}\right)[P_n(x, y, t)e^{\frac{t}{2}}]=0, \tag{3.5}$$

This is analogous to the equation of the diffraction of the wave, but its wave number is imaginary, so that it will be treated in the same way[1].

4. Expansion of P_n Function

Let us consider the function P_{-1} once more in the form (3.1), which equals also

$$P_{-1}(x, y, t)=\frac{1}{2}e^{-\frac{t}{2}}\int_{+\infty}^{\infty}e^{-\frac{1}{2}\rho\cosh(2u-i\alpha)}\cos(x\cosh u)du, \tag{4.1}$$

Remembering the expansion [23][28]

$$\cos(x\cosh u)=\sum_{n=0}^{\infty}(-1)^n\varepsilon_n J_{2n}(x)\cosh(2nu),$$

where ε_n means 1 for $n=0$, 2 for $n\geqq 1$, and the definition

$$K_n\left(\frac{1}{2}\rho\right)=\int_0^{\infty}e^{-\frac{1}{2}\rho\cosh u}du,$$

we can integrate (4.1) term by term as follows,

$$P_{n-1}(x, y, t)=\frac{1}{2}e^{-t/2}\sum_{n=0}^{\infty}(-1)^n\varepsilon_n K_n\left(\frac{1}{2}\rho\right)J_{2n}(x)\cos n\alpha, \tag{4.2}$$

This series is convergent but its convergence is very slow for large value of the argument $(x^2/4\rho)$, and so for such range the next asymptotic expansion may be suitable.

Namely, we proceed as the above expanding $\exp.\left[-\frac{1}{2}\rho\cosh(2u-i\alpha)\right]$ in the Bessel function with the imaginary argument, and can obtain the expansion,

$$P_{-1}(x, y, t)=-\frac{\pi}{2}e^{-t/2}\sum_{n=0}^{\infty}\varepsilon_n I_n\left(\frac{1}{2}\rho\right)Y_{2n}(x)\cos n\alpha. \tag{4.3}$$

In general, we can not have such an elegant expression as this, but we can expand it in the Taylor series by the definition (1.10) as follows,

$$P_n(x, y, t)=\sum_{m=0}^{\infty}\frac{x^m}{m!}P_{n-m}(0, y, t)\,, \tag{4.4}$$

$$P_n(x, y, t)=\sum_{m-0}^{\infty}\frac{t^m}{m!}P_{n-2m}(x, y, 0)\,, \tag{4.5}$$

The former is convergent but the latter is divergent and an asymptotic expansion, and the functions in the right hand sides are expressible by known functions in the next three cases.

i) $P_n(x, 0, 0)$ is the same one as defined by T. H. Havelock[12], that is,

$$\left.\begin{aligned} P_n(x, 0, 0)&=-\frac{\pi}{2}\int_{\infty}^{x}\cdots\int_{\infty}^{x}Y_0(x)dx^{n+1}\,, \text{ for } n\geqq 0\,, \\ P_{-n}(x, 0, 0)&=-\frac{\pi}{2}\left(\frac{d}{dx}\right)^{n-1}Y_0(x)\,, \qquad \text{for } n\geqq 1\,,^{17)} \end{aligned}\right\} \tag{4.6}$$

ii) $P_n(0, y, t)$ is zero for even n because it is odd in x by its definition, namely,

$$P_{2n}(0, y, t)=0\,, \tag{4.7}$$

When n is odd, the simplest case is given by (4.2) putting in it x to zero, that is,

$$P_{-1}(0, y, t)=\frac{1}{2}e^{-t/2}K_0\left(\frac{1}{2}\rho\right)\,, \tag{4.8}$$

and moreover by the definition

$$P_{-2n-1}(0, y, t)=\frac{1}{2}\left(\frac{d}{dt}\right)^n\left[e^{-t/2}K_0\left(\frac{1}{2}\rho\right)\right]\,, \text{ for } n\geqq 0\,, \tag{4.9}$$

$$P_{2n-1}(0, y, t)=\frac{1}{2}\int_{\infty}^{t}\cdots\int_{\infty}^{t}e^{-t/2}K_0\left(\frac{1}{2}\rho\right)dt^n\,, \text{ for } n\geqq 0\,, \tag{4.10}$$

iii) When y vanishes too, the above relations become simpler and we can obtain the next recurrence formula by partial integration of the definition formula.

$$\left(n+\frac{1}{2}\right)P_{2n+1}(0, 0, t)=(t-n)P_{2n-1}(0, 0, t)+tP_{2n-3}(0, 0, t)\,, \tag{4.11}$$

In another way, we have from (4.8) and (4.9), especially,

$$\left.\begin{aligned} P_{-1}(0, 0, t)&=\frac{1}{2}e^{-t/2}K_0\left(\frac{1}{2}t\right)\,, \\ P_{-3}(0, 0, t)&=-\frac{1}{4}e^{-t/2}\left[K_0\left(\frac{1}{2}t\right)+K_1\left(\frac{1}{2}t\right)\right]\,. \end{aligned}\right\} \tag{4.12}$$

Then we can get all functions by arithmetic.

Finally, it is easy to see that

$$P_{2n+1}(0, 0, 0)=(-1)^{n+1}\frac{\Gamma(\frac{3}{2})\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}, \tag{4.13}$$

5. $O_n^{(1)}(x, 0, 0)$ and $O_n^{(1)}(0, y, t)$

The functions $O_n^{(1)}(x, y, t)$ or $Q_n(x, y, t)$ can not have such simple series expansions as in the preceding case, because they have a part expressed by indefinite integral as we see in § 2.

Then let us consider here their degenerate cases only.

Firstly, getting to zero y and t in (2.1), we have

$$O_{-1}^{(1)}(x, 0, 0)=\frac{1}{2}\int_0^\infty e^{-x\sinh u}du=\frac{\pi}{4}[\boldsymbol{H}_0(x)-Y_0(x)], \tag{5.1}$$

where $\boldsymbol{H}_0$ means Struve's function[23][28].

By the integration and differentiation, we have directly

$$O_0^{(1)}(x, 0, 0)=-\frac{\pi}{4}\int_x^\infty\left[\boldsymbol{H}_0(x)-Y_0(x)-\frac{2}{\pi x}\right]dx, \tag{5.2}$$

$$O_{-2}^{(1)}(x, 0, 0)=\frac{1}{2x}+\frac{1}{2}-\frac{\pi}{4}[\boldsymbol{H}_1(x)-Y_1(x)], \tag{5.3}$$

and the recurrence formula

$$\begin{aligned}&nO_n^{(1)}(x, 0, 0)+(n-1)O_{n-2}^{(1)}(x, 0, 0)\\&=x[O_{n-1}^{(1)}(x, 0, 0)+O_{n-3}^{(1)}(x, 0, 0)+q_{n-1}(x, 0, 0)+q_{n-3}(x, 0, 0)],\end{aligned} \tag{5.4}$$

where q_n is easily found to be

$$\left.\begin{aligned}&q_{2n}(x, 0, 0)=0,\ \ q_{2n-1}(x, 0, 0)=\frac{(-1)^{n-1}}{2x\sqrt{\pi}}\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)},\\&q_{-2}(x, 0, 0)=-\frac{1}{2x}\ \text{and}\ q_{-n}(x, 0, 0)=\infty\ \text{for}\ n\geqq 3.\end{aligned}\right\} \tag{5.5}$$

These functions correspond to Havelock's $\boldsymbol{Q}_n$ function which are defined as[15][29]

$$\left.\begin{aligned}&\boldsymbol{Q}_0(x)=\frac{\pi}{2}\int_0^x[\boldsymbol{H}_0(x)-Y_0(x)]dx,\\&\boldsymbol{Q}_0(x)=\int_0^x\boldsymbol{Q}_{n-1}(x)dx.\end{aligned}\right\} \tag{5.6}$$

After fairly long calculation in the formulas (2.3) to (2.5), we can obtain the following relations.

$$\left.\begin{aligned} Q_0(x)&=\log(2\gamma x)+2O_0^{(1)}(x,0,0)\,,\\ Q_1(x)&=x\log(2\gamma x)+1-x+2O_1^{(1)}(x,0,0)\,,\\ Q_2(x)&=\frac{(x^2-1)}{2}\log(2\gamma x)+\frac{1}{4}+x-\frac{3}{4}x^2+2O_2^{(1)}(x,0,0)\,,\end{aligned}\right\} \tag{5.7}$$

Here γ means Euler's constant 1.78108$\cdots$.

One of the conveniences of using our $O_n^{(1)}$ function is that each of it has a simple asymptotic nature, and tends to zero when x tends to infinity.

Secondly, when x vanishes, we have by its symmetrical nature

$$\left.\begin{aligned} O^{(1)}_{2n}(0,y,t)&=Q_{2n}(0,y,t)\,,\\ O^{(1)}_{2n+1}(0,y,t)&=P_{2n+1}(0,x,t)\,.\end{aligned}\right\} \tag{5.8}$$

Since the latter case is already discussed, we will consider the former only.

By the integral (2.2), we may write it as

$$Q_{-2}(0,y,t)=O^{(1)}_{-2}(0,y,t)=-\frac{1}{2}e^{-t/2}\int_{L_1+L_2}\exp.\left[\frac{1}{2}\rho\cosh(2u-i\alpha)\right]\sinh u\,du\,,$$

Here L_1 and L_2 are the paths shown in Fig. 2 but here $\beta=0$.

Now, let us deform the paths L_1 and L_2 as shown in Fig. 4.

Then, the integrations along the lines parallel to the real axis cancel out each other and after some calculations we have

$$\begin{aligned} &Q_{-2}(0,y,t)\\ &=\cos\left(\frac{1}{2}\alpha\right)e^{-t/2}\int_0^{\frac{\pi-\alpha}{2}}e^{-\frac{1}{2}\rho\cos 2v}\cos v\,dv\\ &=\frac{(\rho+t)}{2\rho}E_0\left(\frac{\rho+t}{2}\right)\,,\end{aligned} \tag{5.9}$$

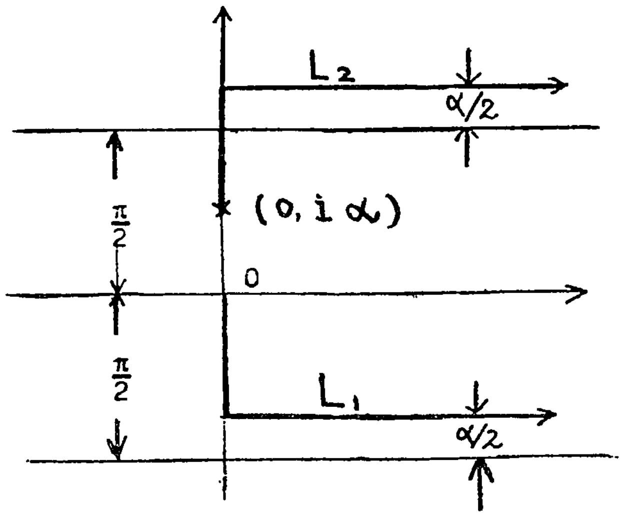

Fig. 4. u-plane

Here

$$E_0(z)=\frac{e^{-z}}{\sqrt{z}}\int_0^{\sqrt{z}}e^{u^2}du=\Gamma\left(\frac{3}{2}\right)\sum_{n=0}^{\infty}\frac{(-z)^n}{\Gamma(n+3/2)}\,. \tag{5.10}$$ [22][23]

In the same way but after long calculations, we have also

$$Q_0(0,y,t)=-\frac{1}{2}\log[2\gamma(\rho+t)]+\int_0^{\frac{\rho+t}{2}}E_0(z)dz\,. \tag{5.11}$$

Finally, if y vanishes too, we have directly from the above

$$\left.\begin{aligned} &Q_0(0,\ 0,\ t)=-\frac{1}{2}\log(4\gamma t)+\int_0^t E_0(z)dz\ ,\\ &Q_{-2}(0,\ 0,\ t)=E_0(t)\ , \end{aligned}\right\} \tag{5.12}$$

and by partial integration of the definition formula

$$\begin{aligned} &nQ_{2n}(0,\ 0,\ t)+\left(n-\frac{1}{2}\right)Q_{2n-2}(0,\ 0,\ t)\\ &=t[Q_{2n-2}(0,\ 0,\ t)+Q_{2n-4}(0,\ 0,\ t)+q_{2n-2}(0,\ 0,\ t)+q_{2n-4}(0,\ 0,\ t)]\ , \end{aligned} \tag{5.13}$$

where

$$\left.\begin{aligned} &q_{-n}(0,\ 0,\ t)=0 \ \text{ for } \ n\geqq 3\ ,\quad q_{-2}(0,\ 0,\ t)=-\frac{1}{2t}\ ,\\ &q_0(0,\ 0,\ t)=\frac{1}{4t}\ ,\ \text{etc.} \end{aligned}\right\} \tag{5.14}$$

6. Neighbourhood of the Origin

Our functions have three arguments so that we may lose their general character even near the origin.

Then, let us consider the character near the origin for a moment.

Firstly, considering the function $O_n^{(1)}$, the simplest case $n=-2$, we have from (5.3) and (5.9)

$$O_{-2}^{(1)}(x,\ 0,\ 0)\xrightarrow[x\to 0]{}\frac{1}{2}+\frac{x}{4}\log(\gamma x/2)+\cdots\ , \tag{6.1}$$

$$O_{-2}^{(1)}(0,\ y,\ t)\xrightarrow[y,t\to 0]{}\cos^2\frac{\alpha}{2}\left[1-\frac{(\rho+t)}{3}+\cdots\right], \tag{6.2}$$

Namely, it may be finite near the origin because it does along three axises as we see and this is confirmed in fact by R. Guilloton's table[9] (See § 9 also).

If so, it is smaller and negligible compared with $q_{-2}=-1/2r$ near the origin so that we may conclude from (4.8) and the definition that

$$\begin{aligned} O_{-1}^{(1)}(x,\ y,\ t)&=O_{-1}^{(1)}(0,\ y,\ t)+\int_0^x[O_{-2}(x,\ y,\ t)+q_{-2}(x,\ y,\ t)]dx\\ &\fallingdotseq\frac{1}{2}\log(4/\gamma\rho)-\frac{1}{2}\log\left(\frac{r+x}{\rho}\right)=-\frac{1}{2}\log\left[\frac{\gamma(r+x)}{4}\right], \end{aligned} \tag{6.3}$$

and then, differentiating it,

$$O_{-3}^{(1)}(x,\ y,\ t)=\frac{\partial}{\partial t}O_{-1}^{(1)}(x,\ y,\ t)-q_{-3}(x,\ y,\ t)\fallingdotseq-\frac{t}{2r(r+x)}-\frac{tx}{2r\rho^2}=-\frac{t}{2\rho^2}\ , \tag{6.4}$$

Secondly, let us consider P_{-1} of (4.2) near the origin.

Since we have[28)]

$$K_0\left(\frac{1}{2}\rho\right)\doteqdot\log(4/\gamma\rho)\,,\quad K_n\left(\frac{1}{2}\rho\right)\doteqdot\frac{(n-1)\,!}{2}\left(\frac{4}{\rho}\right)^n,\ \text{and}\ J_{2n}(x)\doteqdot\frac{(x/2)^{2n}}{(2n)!}\,,$$

its series equals nearly

$$\begin{aligned}P_{-1}(x,\,y,\,t)&\doteqdot\frac{1}{2}\log(4/\gamma\rho)+\sum_{n=1}^{\infty}(-1)^n\frac{(n-1)\,!}{(2n)\,!}\left(\frac{x}{\rho}\right)^n\cos n\alpha\\&=\frac{1}{2}\log(4/\gamma\rho)-\int_0^{\frac{x^2}{4\rho}e^{i\alpha}}E_0(z)dz\,.\end{aligned}\tag{6.5}$$

Here its real part is to be taken.

In this formula, if $(x^2/4\rho)$ is small, the integral in the right hand side is also small, but this is not the general case.

Namely, when x and ρ are small but $(x^2/4\rho)$ is very large, this integral increases logarithmically as

$$\int_0^z E_0(z)\,dz\doteqdot\frac{1}{2}\log(4\gamma z)\,,$$

so that we may obtain

$$P_{-1}(x,\,y,\,t)\underset{\substack{x,\rho\to0,\\(x^2/4\rho)>1}}{\doteqdot}\log(2/\gamma x)\,.\tag{6.6}$$

This is coincident with the predominant term of (4.3) which may be valid for such range.

Lastly, we can obtain in the same way as the above

$$P_{-2}(x,\,y,\,t)\doteqdot\frac{x}{4}\log(\gamma\rho/4)-\frac{x}{\rho}e^{i\alpha}E_0\left(\frac{x^2}{4\rho}e^{i\alpha}\right),\tag{6.7}$$

where the real part only is to be taken.

7. Asymptotic Property of $O_n^{(1)}$

Although the function $O_n^{(1)}$ is complicated in nature as we have seen, but it is fairly smaller compared with the P_n function, and that it decreases monotonically and has an asymptotic expansion when its arguments tend towards infinity.

Now, let us consider that $r=\sqrt{x^2+y^2+t^2}$ is very much larger than the unity.

Returning to the integral (1.1) and expanding its dominator of the integrand as

$$\frac{1}{k\cos^2u-1}=-(1+k\cos^2u+k^2\cos^4u+\,,\,\cdots)\,,$$

Let us integrate term by term and use the definition (1.11); then we have

$$O_n^{(1)}(x, y, t) \fallingdotseq -\left[q_n(x, y, t) + \frac{\partial}{\partial t} q_{n+2}(x, y, t) + \frac{\partial^2}{\partial t^2} q_{n+4}(x, y, t) + \cdots \right], \tag{7.1}$$

For example, making use of (1.13) we see that

$$O_0^{(1)}(x, y, t) \fallingdotseq -\frac{rt+\rho^2}{2r(r+t)^2} - \cdots, \quad \rho=\sqrt{t^2+y^2}, \tag{7.2}$$

$$O_{-1}^{(1)}(x, y, t) \fallingdotseq \frac{x}{2r(r+t)} + \frac{x[3\rho^2(r+t)+2tx^2]}{2r^3(r+t)^3} + \cdots, \tag{7.3}$$

$$O_{-2}^{(1)}(x, y, t) \fallingdotseq \frac{1}{2r} + \frac{\rho^2(r+t)-rx^2}{2r^3(r+t)^2} + \cdots, \tag{7.4}$$

which are coincident with the asymptotic characters in the degenerate cases of § 5.

The first term of the right hand side of (7.4) is a well-known mirror image term and was given by R. Guilloton from the observation of his table[9] (see § 9 also).

8. Asymptotic Property of P_{-1}

In contrast to $O_n^{(1)}$ function, P_n function takes a comparatively larger value even at the point far from the origin, and it shows a well-known Kelvin's wave pattern which has been studied by many authors[14][26][27].

Here we consider the asymptotic expansion of our simplest function P_{-1} following their methods.

Let us rewrite (3.1) as follows,

$$P_{-1}(x, y, t) = \frac{1}{2}\int_{-\infty}^{\infty} e^{ixf(u)} du, \tag{8.1}$$

that is,

$$f(u) = \cosh u - \frac{y}{2x}\sinh 2u + \frac{it}{x}\cosh^2 u, \tag{8.2}$$

and apply the saddle point method[28].

To obtain the saddle point from (8.2), the equation

$$f'(u) = \sinh u - \frac{y}{x}\cosh 2u + \frac{it}{x}\sinh 2u = 0, \tag{8.3}$$

must be solved, but it is easily found that this goes to the equation of the fourth degree and its solution is very complicated.

Hence, keeping out of confusion we will take up much simpler cases.

A) The case $y=0$

The equations (8.2) and (8.3) become

$$f(u)=\cosh u+\frac{it}{x}\cosh^2 u\ , \tag{8.4}$$

$$f'(u)=\sinh u+\frac{2it}{x}\sinh 2u=0\ , \tag{8.5}$$

so that the saddle point to be used is the origin.

Now, putting

$$p=f(u)-f(0)=\frac{u^2}{2!}f''(0)+\frac{u^4}{4!}f^{(4)}(0)+\cdots\ , \tag{8.6}$$

where
$$f''(0)=1+\frac{2it}{x}\ , \quad f^{(4)}(0)=1+\frac{8it}{x}\ , \ \cdots \tag{8.7}$$

we may write (8.1) as

$$P_{-1}(x,\ 0,\ t)=e^{ixf(0)}\int_0^\infty e^{ixp^2}\left(\frac{du}{dp}\right)dp\ . \tag{8.8}$$

Then, if we expand as

$$\frac{du}{dp}=\sqrt{\frac{2}{f''(0)}}\sum_{n=0}^\infty a_{2n}p^{2n}\ ,$$

namely,
$$a_{2n}=\frac{1}{2\pi i}\oint\frac{\sqrt{f''(0)/2}}{p^{2n+1}}dp\ , \tag{8.9}$$

the integration of (8.8) can be carried out term by term and we have

$$P_{-1}(x,\ 0,\ t)=\frac{\exp.(-t+ix+\frac{1}{4}\pi i)}{\sqrt{2x+4it}}\sum_{n=0}^\infty a_{2n}\frac{\Gamma(n+\frac{1}{2})}{(-it)^n}\ , \tag{8.10}$$

where
$$a_0=1\ , \quad a_2=-\frac{f^{(4)}(0)}{2[f''(0)]^2}\ , \tag{8.11}$$

as we get from (8.6) and (8.9).

B) The case $t=0$

The equation (8.2) and (8.3) go to

$$f(u)=\cosh u-\frac{y}{2x}\sinh 2u\ , \tag{8.12}$$

$$f'(u)=\sinh u-\frac{y}{2x}\cosh 2u\ , \tag{8.13}$$

Then, the saddle points are two, that is,

$$\sinh\begin{bmatrix}u_1\\u_2\end{bmatrix}=\frac{e^{\pm\varepsilon}}{\sqrt{2}}\ , \qquad \varepsilon=\cosh^{-1}\left(\frac{x}{\sqrt{8y}}\right)\ , \tag{8.14}$$

so that there may be three cases if x is smaller, larger than or nearly equal to $\sqrt{8}\,y$.

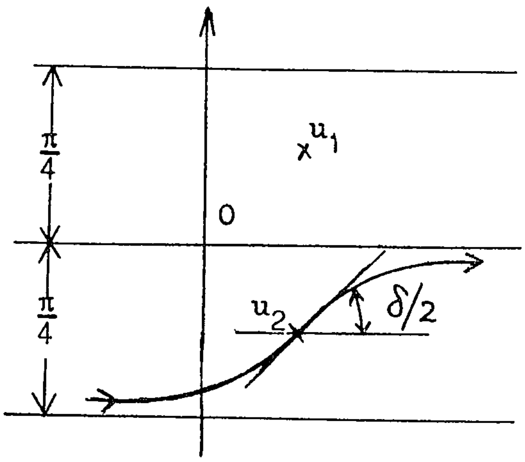

Fig. 5a. u-plane

i) If x is smaller than $\sqrt{8y}\,\varepsilon'$ in (8.14) is purely imaginary and the saddle points are as shown in Fig. 5a and the point u_2 may be used after some consideration.

Putting $\varepsilon=i\varepsilon'$ and

$$\left.\begin{aligned} &\cosh u_2=\frac{1}{\sqrt{2}}\sqrt{2+e^{-2i\varepsilon'}}=\frac{\rho\, e^{-i\delta}}{\sqrt{2}}\,, \\ &\text{namely,}\quad \rho=\sqrt{5+4\cos 2\varepsilon'}\,, \\ &\tan 2\delta=\frac{\sin 2\varepsilon'}{2+\cos 2\varepsilon'}\,, \end{aligned}\right\} \tag{8.15}$$

we may write $f(u)$ and its derivatives at this point as follows,

$$\left.\begin{aligned} &f(u_2)=\frac{\rho\, e^{-i\delta}}{4\sqrt{2}}(3+i\tan\varepsilon')\,, \\ &f''(u_2)=\frac{i\rho}{\sqrt{2}}\tan\varepsilon' e^{-i\delta}\,,\quad f'''(u_2)=-\frac{3}{\sqrt{2}}e^{-i\varepsilon'}\,, \\ &f^{(4)}(u_2)=\frac{\rho e^{i(\varepsilon'-\delta)}}{\sqrt{8}\cos\varepsilon'}(1-7e^{-2i\varepsilon'})\,. \end{aligned}\right\} \tag{8.16}$$

Now, let us take the path of integration as shown in Fig. 5a and integrate as

$$P_{-1}(x,\,y,\,0)=\frac{1}{2}e^{ixf(u)}\int_{-\infty}^{\infty}e^{-xp^2}\left(\frac{dv}{dp}\right)dp\,, \tag{8.17}$$

where

$$\left.\begin{aligned} &ip^2=f(u)-f(u_2)=\frac{v^2}{2!}f''(u_2)+\frac{v^3}{3!}f'''(u_2)+\cdots \\ &v=u-u_2 \end{aligned}\right\} \tag{8.18}$$

Expanding as

$$\left(\frac{dv}{dp}\right)=\sqrt{\frac{2i}{f''(u_2)}}\sum_{n=0}^{\infty}a_n p^n\,,\qquad a_n=\frac{\sqrt{f''(u_2)/2i}}{2\pi i}\oint\frac{dv}{p^{n+1}}\,, \tag{8.19}$$

that is especially

$$a_0=1\,,\quad a_2=\frac{i[f^{(3)}(u_2)]^2}{12[f''(u_2)]^3}-\frac{if^{(4)}(u_2)}{4[f''(u_2)]^2}\,,$$

we have finally

$$P_{-1}(x,\,y,\,0)\doteqdot\frac{1}{2}e^{ixf(u_2)+i\delta/2}\sqrt{\frac{\pi}{y\rho\sin\varepsilon'}}\left[1+\frac{a_2}{2x}+\cdots\right] \tag{8.20}$$

ii) When $x>\sqrt{8}y$, ε is real and the saddle points lie on the real axis as shown in Fig. 5b.

This is a well known case in which there are the diverging and transverse wave-systems inner the Kelvin angle[26].

In this case, $f(u)$ and its derivatives are all real and they are

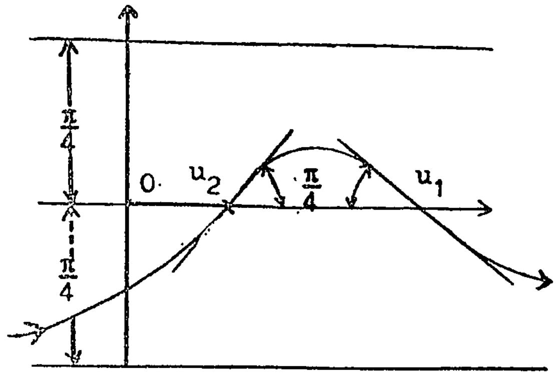

Fig. 5b. u-plane

$$\left.\begin{aligned} f(u_j)&=\frac{\cosh^3 u_j}{\cosh 2u_j}=\left(1+\frac{1}{2}e^{\pm 2\varepsilon}\right)^{3/2}e^{\mp\varepsilon}/(2\cosh\varepsilon)\,,\\ f'(u_j)&=\mp\sqrt{1+\frac{1}{2}e^{\pm 2\varepsilon}}\tanh\varepsilon\,,\\ f''(u_j)&=-\frac{3}{\sqrt{2}}e^{\pm\varepsilon}\,,\qquad f^{(4)}(u_j)=\sqrt{1+\frac{1}{2}e^{\pm 2\varepsilon}}(e^{\mp\varepsilon}-7e^{\pm\varepsilon})/(2\cosh\varepsilon)\,,\end{aligned}\right\}\tag{8.21}$$

where $j=1$ or 2 and the double sign is taken as the upper one for $j=1$ and the lower for $j=2$.

In a usual way, let us integrate along the paths shown in Fig. 5b, namely,

$$\left.\begin{aligned} &P_{-1}(x,y,0)=\sum_{j=1,2}\frac{1}{2}e^{ixf(u_j)}\int_{-\infty}^{\infty}e^{-xp^2}\frac{du}{dp}dp\,,\\ \text{where}\quad &ip^2=f(u)-f(u_j)=\frac{v^2}{2!}f''(u_j)+\frac{v^3}{3!}f'''(u_j)+\cdots\,,\\ &v=u-u_j\,,\end{aligned}\right\}\tag{8.22}$$

expanding as

$$\left.\begin{aligned} &\frac{du}{dp}=e^{i\pi/4}\sqrt{\frac{2}{f''(u_j)}}\sum_{n=0}^{\infty}a_np^n\,,\\ &a_0=1\,,\quad a_2=\frac{i[f'''(u_j)]^2}{12[f''(u_j)]^3}-\frac{if^{(4)}(u_j)}{4[f''(u_j)]^2}\,,\end{aligned}\right\}\tag{8.23}$$

then we have

$$P_{-1}(x,y,0)\fallingdotseq\sum_{j=1,2}e^{ixf(u_j)+i\pi/4}\sqrt{\frac{\pi}{2xf''(u_j)}}\left(1+\frac{a_2}{2x}+\cdots\right).\tag{8.24}$$

iii) When x is nearly equal to $\sqrt{8}y$, the preceding two formulas give wrong approximation.

In this case, the most reasonable formula owes to F. Ursell[27] and we will follow his analysis.

Let us consider $\mathcal{E}$ as real and put (8.12) hs

$$\left.\begin{aligned} f(u)&=-\frac{v^3}{3}+v\mu(\mathcal{E})+\nu(\mathcal{E})\,, \\ \text{namely ,}\quad \mu(\mathcal{E})&=\left[\frac{3}{4}\{f(u_1)-f(u_2)\}\right]^{2/3},\ \ \nu(\mathcal{E})=\frac{1}{2}[f(u_1)\mp f(u_2)] \end{aligned}\right\} \tag{8.25}$$

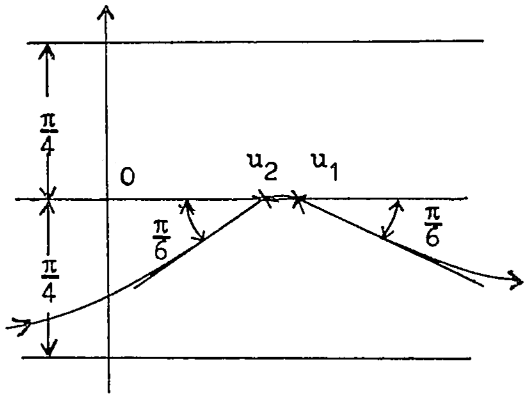

Fig. 5c. u-plane

then the saddle points in v-plane are

$$\left.\begin{matrix} v_1 \\ v_2 \end{matrix}\right\}=\pm\sqrt{\mu(\mathcal{E})}\,. \tag{8.26}$$

Now, if we may expand as

$$\frac{du}{dv}=\sum_{n=0}^{\infty}a_n(v^2-\mu)^n+v\sum_{n=0}^{\infty}b_n(v^2-\mu)^n\,, \tag{8.27}$$

and take the path of integration as shown in Fig. 5c, we can integrate as follows,

$$\begin{aligned} P_{-1}(x,\,y,\,0)&=\frac{1}{2}\int_{-\infty}^{\infty}\exp.\left[ix\left(-\frac{v^3}{3}+v\mu+\nu\right)\right]\frac{du}{dv}dv \\ &=\frac{\pi e^{ix\nu(\varepsilon)}}{x^{1/3}}\left[a_0A_i(-\mu x^{2/3})+\frac{ib_0}{x^{1/3}}A_i'(-\mu x^{2/3})+\cdots\right], \end{aligned} \tag{8.28}$$

where

$$A_i(z)=\frac{1}{\pi}\int_0^{\infty}\cos\left(\frac{p^3}{3}+zp\right)dp\,,$$

that is the Airy's integral[28].

The coefficients a_0 and b_0 are found to be

$$\left.\begin{matrix} a_0 \\ b_0 \end{matrix}\right\}=\mu^{1/4}\left[\frac{1}{\sqrt{f''(u_2)}}\pm\frac{1}{\sqrt{|f''(u_1)|}}\right]\Big/\sqrt{2}\,. \tag{8.29}$$ [27]

Finally, it is easily seen that the formula (8.28) is applicable for the imaginary $\mathcal{E}$, that is, outer region of Kelvin angle in which case μ of (8.25) is negative and the argument of Airy's integral changes to positive[27].

9. On the Numerical Tables

There are many tables of such functions prepared for the object to compute the wave-making resistance or the wave profile and pattern.

The followings are their list available for us in our notation.

I) *a)* $P_n(x, 0, 0)$ for $n=0(1)9$ and $x=0(0.4)4.4,5(1)40$
where the number in the parenthese means the interval of the parameter.
with 4 significant figures by T. Jinnaka[19].

b) $\frac{2}{\pi}P_0(x, 0, 0)$ and $\frac{2}{\pi}[P_1(x, 0, 0)-1]$,
for $x=0(0.1)1.0(0.2)10.0(0.4)50.0$ with 4 figs., by T. Inui[18].

c) $P_n(x, 0, 0)$ for $n=-7(1)1$, $x=0(0.5)2(1)16$ with 7 figs., by M. Bessho[6].

d) $P_n(0, 0, t)$ for $n=-7(2)5$, $t=0\sim10$, with 6 figs. and $U_n(t)=(-1)^n P_{-2n-1}(0, 0, t)$ for $n=0(1)31$, $t=0\sim6$ with 10 figs. by M. Bessho[6].

e) $\frac{1}{2\pi}Q_0(x)$ and $\frac{1}{2\pi}Q_1(x)$, for $x=0(0.1)1.0(0.2)10.0(0.4)50$ with 4 figs. by T. Inui[18].

f) $O_n^{(1)}(x, 0, 0)$ for $n=-2,-1, 0$, $x=0(0.5)2(1)16$ and $\int_0^x O_{-1}^{(1)}(x, 0, 0)dx$ and $\left(\frac{d}{dx}\right)^n O_{-1}^{(1)}(x, 0, 0)$ for $n=1(1)4$, $x=0(0.5)2(1)16$ with 7 figs. by M. Bessho[6].

g) $Q_{2n}(0, 0, t)$ for $2n=-6(2)4$, $t=0(0.1)1.0(0.2)10$ with 6 figs.
$E_0(t)$, $\int_0^t E_0(t)dt$ for $t=0(0.1)1(1)10$ with 10 figs. and $E_n(t)=(-1)^n Q_{-2n-2}(0, 0, t)$ for $n=0(1)9$, $t=0\sim3.2$ with 8 figs. by M. Bessho[6].

II) *a)* $P_{-3}(x, 0, t)$ for $x=0\sim60$, $\sqrt{t}=0\sim1.0$ with 4 figs. by National Physical Laboratory, *Ma*/16/1502[25].

b) $P_{-1}(x, 0, t)$ and $O_{-1}^{(1)}(x, 0, t)$ for $x=0\sim19.5$, $t=0\sim1.0$, with 4 figs. by T. Takahei[30].

c) $O_{-4}^{(1)}(x, 0, t)$ and the wave elevation by a point doublet along its path, with 4 figs. by Tokyo University[31].

d) $P_{-1}(x, 0, t)$ and $O_{-1}^{(1)}(x, 0, t)$ for $x=0\sim27$, $t=0\sim5.2$ with 8 figs. by T. Iwata[32].
These four tables are prepared for the computation of the wave-making resistance and the wave profile.

e) $P_n(x, 0, t)$ for $n=-7(1)2$, $x=0\sim16$, $t=0\sim6$ and $O_n^{(1)}(x, 0, t)$ for $n=-3(1)-1$, $x=0\sim16$, $t=0\sim6$ are prepared for the computation of the submerged body problem by M. Bessho[6].

III) *a)* $\frac{S}{n^2}(x, y, a+z)=-\frac{g}{\pi V^2}O_{-2}^{(1)}\left[\frac{g|x|}{V^2}, \frac{gy}{V^2}, \frac{-g(a+x)}{V^2}\right]$ and
$\frac{A}{n^2}(x, y, a+z)=\frac{g}{\pi V^2}P_{-2}\left[\frac{g|x|}{V^2}, \frac{gy}{V^2}, \frac{-g(a+z)}{V^2}\right]$,

setting $g/V^2=0.4$, for $gx/V^2=0\sim20$, $gy/V^2=0\sim4$ and $-g(a+z)/V^2=0\sim0.8$ with 3 figs. by R. Guilloton[8)9)].

b) $\frac{1}{2\pi}Z_l(q,\theta)-Z_w(q,\theta)=\frac{1}{2}\log(2\gamma q)+O_0{}^{(1)}(x,y,0)$, for $q=\sqrt{x^2+y^2}=0\sim 20$, $\theta=\tan^{-1}(y/x)=0\sim 180°$ given almost by figures by T. Jinnaka[20].

10. Conclusion

The preceding analysis shows that

1. the function considered is represented by single integral instead of double integral so that the computation may become simpler.
2. the various limits of the function are considered and related to the known functions as far as possible so that the general feature may be elucidated.

We have the similar work by R. Guilloton in which he showed heuristically and numerically its property and the extraordinary way of computing the various quantities of the velocity field around the ship with the aid of his tables, but, mathematically speaking, his method has some difficulties which we hesitate to proceed with.

Our final outcome must be the same as his and this work might be the second step in the attack on this problem.

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