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"On the Two-Dimensional Theory of the Rolling Motion of Ships"

(To Prof. T. Itoh this paper is dedicated.)

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Abstract

The author treats with the problem of the rolling motion of ships among waves in view of the two-dimensional theory of the water wave.

Firstly, he constructs the integral equations to determine the velocity potentials neccessary to solve this problem and shows that the diffraction potentials can be deduced from other potentials. From this fact, Haskind-Hanaoka's relation with respect to the exciting forces of the wave can be understood easily.

Secondly, he solves the equation of motion and shows that the roll-exciting moment of the wave can be represented by making use of the coefficient like as the effective wave slope one in the classical theory but this coefficient is deduced from the concept other than in the classical theory.

Thirdly, he gives the approximate values of all necessary quantities when the wave length is very much larger than the ship breadth, and considers the possibility to make zero or reduce the roll-exciting moment, he touches the experimental methods to investigate this problem, especially Motora's method to measure the roll-exciting moment of the wave.

1. Introduction.

The classical theory of rolling motion of ships developed by Froude, Kryloff and Watanabe seems reliable in most cases at present [1]. It has, as the basis, a sound physical interpretation of that phenomena but its theory has a serious weak point in view of the recent theory of the water waves, and many authors attack this problem in this stand point but there remain some points to be elucidated more clearly as Ir. G. Vossers says in his lecture [14].

In these research works, F. Ursell finds out that there is two dimensional cylindrical section shape which emits no wave in the rolling oscillation about an axis [11, 17] and T. Hishida also independently states that there is an axis of any section shape the oscillation about which emits no wave [8, 9, 10].

In another respect, we have Haskind-Hanaoka's relation with regard to the wave-ex-

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citing force that it is proportional to the square root of the wave-damping [12, 15].

Hence, we arrive easily the idea that it may be possible to reduce the wave exciting moment and then the rolling oscillation by selecting a suitable center of the rolling motion. This possibility, however, can not realize successfully although in the heaving-dipping motion it does. [21, 22]. This is mainly because of the existence of the swaying oscillation.

Thus, we feel the necessity to verify the theory once more especially in its physical meaning and validity.

2. Velocity Potentials. [5, 25, 26, 28]

Consider the water motion around an infinitite cylinder floating on the water and assume that its cylinder is symmetric about its vertical center plane and stands for some section of a ship.

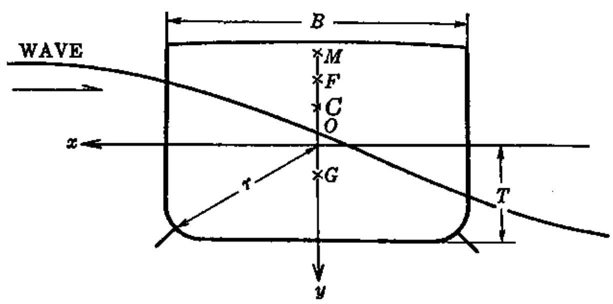

Fig. 1. Co-ordinate system.

Taking the coordinate system as shown in Fig. 1, and assuming that all motions are periodic in time with circular frequency $\omega=2\pi/\tau$, τ is the period, we can introduce the velocity potential $\Phi(x, y, t)$ as

$$\Phi(x, y, t)=Re[\varphi(x, y)e^{i\omega t}]=\varphi_c \cos\omega t-\varphi_s \sin\omega t \tag{2.1}$$

where the suffix C and S stand for the real and imaginary part, the pressure $P(x, y, t)$ except the statical buoyancy equals, for the first approximation,

$$P(x, y, t)=Re[p(x, y)e^{i\omega t}]\risingdotseq\frac{\partial}{\partial t}\Phi(x, y, t)\,;\; p(x, y)=i\omega\varphi(x, y)\,, \tag{2.2}$$

and the surface dislocation downwards $\eta(x, t)$ is

$$\frac{\partial}{\partial t}\eta(x, t)\risingdotseq-\frac{\partial\Phi}{\partial y}(x, 0, t)\eta(x, t)=Re\left[\frac{\omega}{ig}\varphi(x, 0)e^{i\omega t}\right], \tag{2.3}$$

where g is the gravity constant.

The pressure must be constant on the water surface and this condition becomes for the velocity potential as follows:

$$K\varphi(x, 0)+\frac{\partial}{\partial y}\varphi(x, 0)=0\,, \tag{2.4}$$

where

$$K=\omega^2/g\,.$$

Now, in the stand point of the linear theory, the whole velocity potential can be superposed of each velocity potential corresponding to all component motions.

Let the suffix stand for the related quantities of each component motion as follows:

0 .... the incident wave
1 .... the swaying motion (x)
2 .... the heaving-dipping motion (y)
3 .... the rolling motion about the origin
4 .... the diffraction of the incident wave
5 .... the rolling motion about the center of gravity and of course this is completely dependent with the motion 1 and 3.

The boundary condition of each velocity potential over the immersed cylinder surface C is

$$\frac{\partial}{\partial n}\varphi_j(x, y)=i\omega X_j\frac{\partial x_j}{\partial n}, \quad j=1, 2, 3, \tag{2.5}$$

where n is the outward normal of the curve C, X_j is the amplitude of the oscillation and, for the convenience, to be understood that it is also written as

$$X_1\equiv X,\ X_2\equiv Y,\ X_3\equiv\theta,\ X_5\equiv\theta,\ X_4=Ka=\theta_w\ , \tag{2.6}$$

where a is the amplitude of the incident wave, and so θ_w is the maximum wave slope, and $x_1\equiv x$, $x_2\equiv y$ and x_3 is a harmonic function regular inside the cylinder and has the boundary value

$$\left.\begin{aligned}\frac{\partial}{\partial n}x_3&=y\frac{\partial x}{\partial n}-x\frac{\partial y}{\partial n}\ ,\\ \text{similarly}\ x_5&=x_3-lx_1=x_3-lx\end{aligned}\right\} \tag{2.7}$$

It is also convenient for the following treatment to normalize quantities, namely,

$$\left.\begin{aligned}\varphi_j(x, y)&=i\omega X_j\phi_j(x, y)\ , \quad j=1, 2, 3, 5\ ,\\ \text{then}\ \frac{\partial}{\partial n}\phi_j(x,y)&=-\frac{\partial x_j}{\partial n}\ , \quad j=1, 2, 3, 5\ ,\end{aligned}\right\} \tag{2.8}$$

In these potentials, ϕ_5 is clearly given as $\phi_3-l\phi_1$.

Similarly, being a the amplitude of the incident wave coming from the positive direction of the x—axis, we have

$$\varphi_0(x, y)=\frac{iga}{\omega}\phi_0(x, y),\ \phi_0(x, y)=e^{-Ky+iKx} \tag{2.9}$$

$$\frac{\partial}{\partial n}(\phi_0+\phi_4)=0 \tag{2.10}$$

Moreover, we can divide the incident wave into the stationary wave, that is,

$$\left.\begin{aligned}&\phi_0=\phi_{0A}+i\phi_{0B}\\&\phi_{0A}=e^{-Ky}\cos Kx,\ \phi_{0B}=e^{-Ky}\sin Kx\ ,\end{aligned}\right\}\tag{2.11}$$

Accordingly, the diffraction potentials also can be done as

$$\phi_4=\phi_{4A}+i\phi_{4B}\ ,\tag{2.12}$$

$$\frac{\partial}{\partial n}(\phi_{4A}+\phi_{0A})=\frac{\partial}{\partial n}(\phi_{4B}+\phi_{0B})=0\ ,\tag{2.13}$$

Thus, considering the curve C symmetric, it is clear that

$$\left.\begin{aligned}&\phi_j(x,y)=-\phi_j(-x,y)\ \text{ for }\ j=1,3,5,4B\ .\\&\phi_j(x,y)=\phi_j(-x,y)\ \text{ for }\ j=2,4A\ .\end{aligned}\right\}\tag{2.14}$$

Meanwhile, all the velocity potentials having radiating waves can be represented as

$$\phi(x,y)=\frac{1}{2\pi}\int_c\left(\frac{\partial\phi}{\partial n}G-\phi\frac{\partial}{\partial n}G\right)dS(x',y'),\tag{2.15}$$

where G is the unit sink potential and

$$G=G_c+iG_s$$

$$\left.\begin{aligned}&G_c(x,y;x',y')=\frac{1}{2}\log\left[\frac{(x-x')^2+(y-y')^2}{(x-x')^2+(y+y')^2}\right]-2P.V.\int_0^\infty\frac{e^{-K(y+y')}\cos k(x-x')\,dk}{k-K}\\&G_s=2\pi e^{-K(x-x')}\ ,\end{aligned}\right\}\tag{2.16}$$

Since

$$G(x,y;x',y')\xrightarrow[K|x-x'|\gg1]{}2\pi ie^{-K(y+y')-iK|x-x'|}\tag{2.17}$$

then we have

$$\left.\begin{aligned}\phi&\xrightarrow[Kx\gg1]{}iH^+(K)e^{-Ky-iKx}\ ,\\&\xrightarrow[Kx\ll-1]{}iH^-(K)e^{-Ky+iKx}\ ,\end{aligned}\right\}\tag{2.18}$$

where

$$H^{\pm}(K)=\int_c\left(\frac{\partial\phi}{\partial n}-\phi\frac{\partial}{\partial n}\right)e^{-Ky\pm iKx}dS\ ,\tag{2.19}$$

Moreover, since there is the relation (2.14), we have

$$\left.\begin{aligned}&H_j^+(K)=H_j^-(K)=\int_c\left(\frac{\partial}{\partial n}\phi_j-\phi_j\frac{\partial}{\partial n}\right)e^{-Ky}\cos Kx dS\\&\qquad\qquad\qquad\qquad\text{for } j=2,4A\ .\\&H_j^+(K)=-H_j^-(K)=i\int_c\left(\frac{\partial}{\partial n}\phi_j-\phi_j\frac{\partial}{\partial n}\right)e^{-Ky}\sin Kx dS\ ,\\&\qquad\qquad\qquad\qquad\text{for } j=1,3,5,4B\ .\end{aligned}\right\}\tag{2.20}$$

We can write (2.15) as two formulas, that is, the real and imaginary part respectively, putting the boundary condition (2.8),

$$\left.\begin{aligned}\phi_{jc}(x,y)&=\frac{-1}{2\pi}\int_c\left(\frac{\partial}{\partial n}x_j+\phi_{jc}\frac{\partial}{\partial n}\right)G_c dS-e^{-Ky}\begin{pmatrix}\cos Kx\\-i\sin Kx\end{pmatrix}H_{js}^{+}(K)\,,\\\phi_{js}(x,y)&=\frac{1}{2\pi}\int_c\left(-\phi_{js}\frac{\partial}{\partial n}\right)G_c dS+e^{-Ky}\begin{pmatrix}\cos Kx\\-i\sin Kx\end{pmatrix}H_{jc}^{+}(K)\,,\end{aligned}\right\}\tag{2.21}$$

cosine term for $j=2$,

sine term for $j=1, 3, 5$,

where

$$\left.\begin{aligned}H_{jc}^{+}\\H_{js}^{+}\end{aligned}\right\}=\int_c\left\{\frac{\partial}{\partial n}\begin{pmatrix}\phi_{jc}\\\phi_{js}\end{pmatrix}-\begin{pmatrix}\phi_{jc}\\\phi_{js}\end{pmatrix}\frac{\partial}{\partial n}\right\}e^{-Ky+iKx}dS\,,\tag{2.22}$$

Especially, for the diffraction potential, remembering (2.10) and (2.13), and since

$$\frac{1}{2\pi}\int_c\left(\frac{\partial}{\partial n}\phi_0-\phi_0\frac{\partial}{\partial n}\right)GdS=0\,,$$

we have

$$\left.\begin{aligned}\phi_4\\\phi_{4A}\\\phi_{4B}\end{aligned}\right\}=-\frac{1}{2\pi}\int_c\left\{\begin{aligned}(\phi_0+\phi_4)\\(\phi_{0A}+\phi_{4A})\\(\phi_{0B}+\phi_{4B})\end{aligned}\right\}\frac{\partial}{\partial n}GdS\,,\tag{2.23}$$

and

$$H_4^{+}(K)=-\int_c(\phi_0+\phi_4)\frac{\partial}{\partial n}e^{-Ky+iKy}dS\,,\tag{2.24}$$

Hence, similarly as the above, we can write down as

$$\left.\begin{aligned}&\left.\begin{aligned}\phi_{4AC}(x,y)\\\phi_{4BC}(x,y)\end{aligned}\right\}=-\frac{1}{2\pi}\int_c\begin{pmatrix}\phi_{0A}+\phi_{4AC}\\\phi_{0B}+\phi_{4BC}\end{pmatrix}\frac{\partial}{\partial n}G_c dS+e^{-Ky}\begin{pmatrix}-\cos KxH_{4AS}^{+}\\i\sin KxH_{4BS}^{+}\end{pmatrix},\\&\left.\begin{aligned}\phi_{4AS}(x,y)\\\phi_{4BS}(x,y)\end{aligned}\right\}=-\frac{1}{2\pi}\int_c\begin{pmatrix}\phi_{4AS}\\\phi_{4BS}\end{pmatrix}\frac{\partial}{\partial n}G_c dS+e^{-Ky}\begin{pmatrix}\cos KxH_{4AC}^{+}\\-i\sin KxH_{4BC}^{+}\end{pmatrix},\end{aligned}\right\}\tag{2.25}$$

All equations of (2.21) and (2.25) along the boundary C can be understood as the integral equations to determine the doublet strength ϕ on that boundary. Each pair of these are integral equations of Fredholm's second type, and might have unique solution.
As easily seen, however, all of them are not independent with each other and there are the same functions as others except a constant multiplier. Thus, we can find

$$\phi_{1S}(x,y)H_{3C}^{+}(K)=\phi_{3S}(x,y)H_{1C}^{+}(K),\ \text{on } C\,.\tag{2.26}$$

$$\left.\begin{aligned}(\phi_{0A}+\phi_{4AC})H_{4AC}^{+}&=i\phi_{4AS}(1+iH_{4AS}^{+})\,,\\(\phi_{0B}+\phi_{4BC})H_{4BC}^{+}&=i\phi_{4BS}(1+iH_{4BS}^{+})\,,\end{aligned}\right\}\text{ on } C\,. \tag{2.27}$$

$$\phi_{4AS}H_{2C}^{+}=\phi_{2S}H_{4AC}^{+}\,,\ \phi_{4BS}H_{1C}^{+}=\phi_{1S}H_{4BC}^{+}\,,\ \text{ on } C\,, \tag{2.28}$$

These relations reduce very much the labour to solve the integral equations. For example, if we know ϕ_1 and ϕ_2 already, (2.27) and (2.28) show that we have a sufficient knowledge of the diffraction potential. Haskind-Hanaoka's formula is easily understood from this fact. In the form of H-function, these relations get to the following formulas

$$\frac{H_{1S}^{+}(K)}{H_{1C}^{+}(K)}=\frac{H_{3S}^{+}(K)}{H_{3C}^{+}(K)}=\frac{H_{5S}^{+}(K)}{H_{5C}^{+}(K)}\,, \tag{2.29}$$

$$\left.\begin{aligned}H_{4AS}^{+}&=(H_{4AC}^{+})^2+(H_{4AS}^{+})^2\,,\\iH_{4BS}^{+}&=(H_{4BC}^{+})^2+(H_{4BC}^{+})^2\,,\end{aligned}\right\} \tag{2.30}$$

$$\frac{H_{4AS}^{+}}{H_{4AC}^{+}}=\frac{H_{2S}^{+}}{H_{2C}^{+}}\,,\qquad \frac{H_{4BS}^{+}}{H_{4BC}^{+}}=\frac{H_{1S}^{+}}{H_{1C}^{+}}\,, \tag{2.31}$$

3. Forces and Moments. [15, 25, 27, 28]

Let F_{ij} be the j-force or moment by the i-motion, then, by (2.2)

$$F_{ij}=-\rho\int_c p(x,y)\frac{\partial}{\partial n}x_j dS=-\rho i\omega\int_c \varphi(x,y)\frac{\partial}{\partial n}x_j dS\,,$$

in the normalized form,

$$f_{ij}=\frac{F_{ij}}{\rho\omega^2 X_j}=\int_c \phi_i\frac{\partial}{\partial n}x_j dS=-\int\phi_i\frac{\partial}{\partial n}\phi_j dS\,, \tag{3.1}$$

In the same way, the exciting force or moment E_j is represented as

$$e_j\equiv\frac{E_j}{\rho g a}=-\int_c(\phi_0+\phi_4)\frac{\partial}{\partial n}\phi_j dS\,, \tag{3.2}$$

Then we can easily find the well-known relations by applying Green's theorem.

$$f_{ij}=f_{ij} \tag{3.3}$$

$$e_j=-H_j^{+}(K) \tag{3.4}$$

The latter is Haskind's formula and its generalized form to more general cases is derived by T. Hanaoka independently and he suggests the possibility to obtain the wave exciting force from the forced oscillation tests.

The imaginary part of (3.1), that is, the damping is also evaluated by Green's theorem, and

$$Im\{f_{ij}\}\equiv f_{ijs}=-H_i^+(K)\overline{H}_j^+(K)\ , \tag{3.5}$$

Nextly, since the real part of (3.1) is the added mass term, we put it as follows for the simplicity;

$$\left.\begin{aligned} &f_{11C}=k_1(K)\nabla\ ,\quad f_{22C}=k_2(K)\nabla\ ,\\ &f_{31C}=f_{13C}=k_1(K)l_1(K)\nabla\ ,\\ &f_{33C}=k_1(K)\kappa_3^2(K)\nabla\ ,\end{aligned}\right\} \tag{3.6}$$

where ∇ is the area of the cylinder.

Then, k_1 and k_2 are the added mass coefficients, κ_3 is the radius of added inertia and l_1 is the distance from the origin to the center of the added mass for the swaying motion. Lastly, if we put

$$\left.\begin{aligned} &e_3/e_1=H_3^+(K)/H_1^+(K)=l_w\ ,\\ &e_5/e_1=l_w-l\ ,\end{aligned}\right\} \tag{3.7}$$

we know by (2.29) that l_w is real, namely, the exciting motion of the incident wave is in phase with the exciting force in the x-direction. This is confirmed by the numerical calculation by K. Tamura [20].

From (3.5) and (3.7) we have for the damping the relations:

$$\left.\begin{aligned} &f_{33S}/f_{13C}=f_{13S}/f_{11S}=l_w\ ,\\ &f_{55S}/f_{15S}=f_{15S}/f_{11S}=l_w-l\ ,\end{aligned}\right\} \tag{3.8}$$

which is also confirmed by the calculations [20].

All of these quantities for so-called Lewis forms are calculated by K. Tamura and F. Tasai so that we could solve the equation of the motion [18, 20].

However, it is convenient to show their approximate values for small K value, that is, when the wave length is much larger than the characteristic length of the cylinder in the present problem. In such case, the velocity potentials equal nearly of the limit case $K=0$, namely, when the water surface is assumed as rigid [8, 9, 11, 13].

Thus, the appoximation for (3.6) is

$$\left.\begin{aligned} &f_{11C}\fallingdotseq k_1(0)\nabla\ ,\quad f_{33C}\fallingdotseq k_1(0)\kappa_3^2(0)\nabla\ ,\\ &f_{31C}\fallingdotseq k_1(0)l_1(0)\nabla\ ,\end{aligned}\right\} \tag{3.9}$$

Here, since $k_2(0)$ becomes usually infinite, it must be considered always as dependent on K.

For the wave exciting forces, expanding the exponential term of (2.19) and taking its first term, we have

$$H_1^+(K) \doteqdot iK\int_c \left(x\frac{\partial}{\partial n}\phi_1 - \phi_1\frac{\partial x}{\partial n}\right)dS = -iK\nabla(1+k_1)\,, \tag{3.10}$$

$$H_2^+(K) \doteqdot \int_c \frac{\partial\phi_2}{\partial n}dS - K\int_c \left(y\frac{\partial\phi_2}{\partial n} - \phi_2\frac{\partial y}{\partial n}\right)dS = K(1+k_2)\nabla - B\,, \tag{3.11}$$

$$H_3^+(K) \doteqdot iK\int_c \left(x\frac{\partial\phi_3}{\partial n} - \phi_3\frac{\partial x}{\partial n}\right)dS = -iK\nabla(\overline{OM}+k_1l_1)\,, \tag{3.12}$$

where B is the breadth of the cylinder and $\overline{OM}$ is the distance between the origin O and the metacenter M measured as positive when M lies below the origin, that is,

$$\begin{aligned}\overline{OM} &= -\frac{1}{\nabla}\int_c x\frac{\partial\phi_3}{\partial n}dS = \frac{1}{\nabla}\int_c \left(xy\frac{\partial x}{\partial n} - x^2\frac{\partial y}{\partial n}\right)dS \\ &= \overline{OB} - \overline{BM},\ \overline{BM} = B^3/12\,,\end{aligned} \tag{3.13}$$

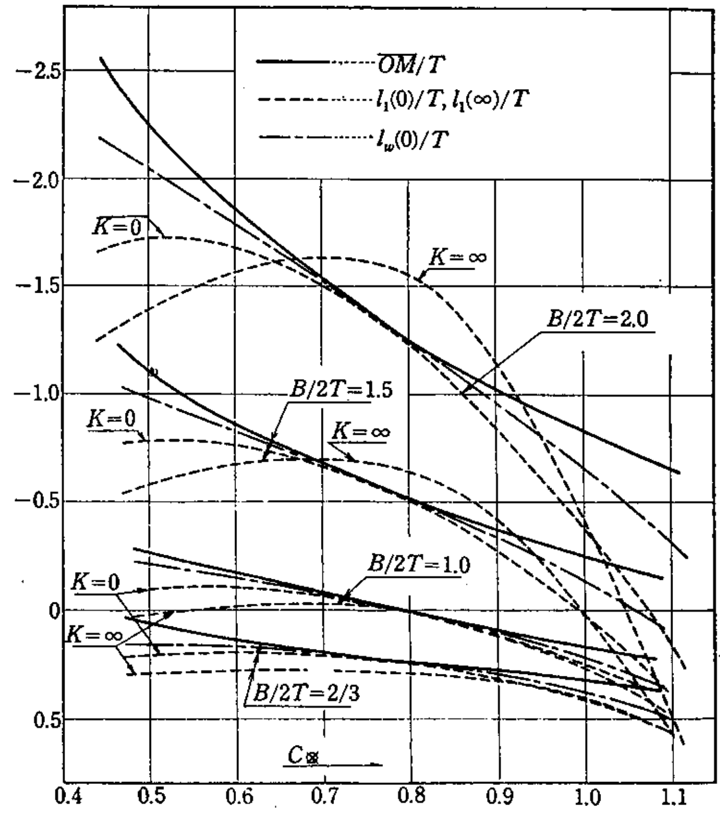

Fig. 2. Metacenter, center of added mass and point on which wave-exciting Force acts.

Since the statical inclining moment proportional to the displacement acts on the metacenter M, (3.10) and (3.12) show that the exciting force of the wave acts on the

point F which lies between M and the center of the added mass as shown in Fig. 1, where B means the center of buoyancy.
Making use of (3.10) and (3.12), (3.7) becomes

$$l_w \doteqdot (\overline{OM}+k_1 l_1)/(1+k_1)\ , \tag{3.14}$$

Hence, for small K-value we know that:

i) The wave exciting force for x-direction, (3.10), does not vanish generally as far as the velocity potential is of single valued [24].

ii) The force for y-direction (3.11) vanishes when

$$B=K(1+k_2)V\ , \tag{3.15}$$

This is the fundamental idea followed by S. Motora [21].

iii) The moment about the origin, (3.12), vanishes when

$$\overline{OM}+k_1 l_1=0,\ \text{or}\ l_w=0\ , \tag{3.16}$$

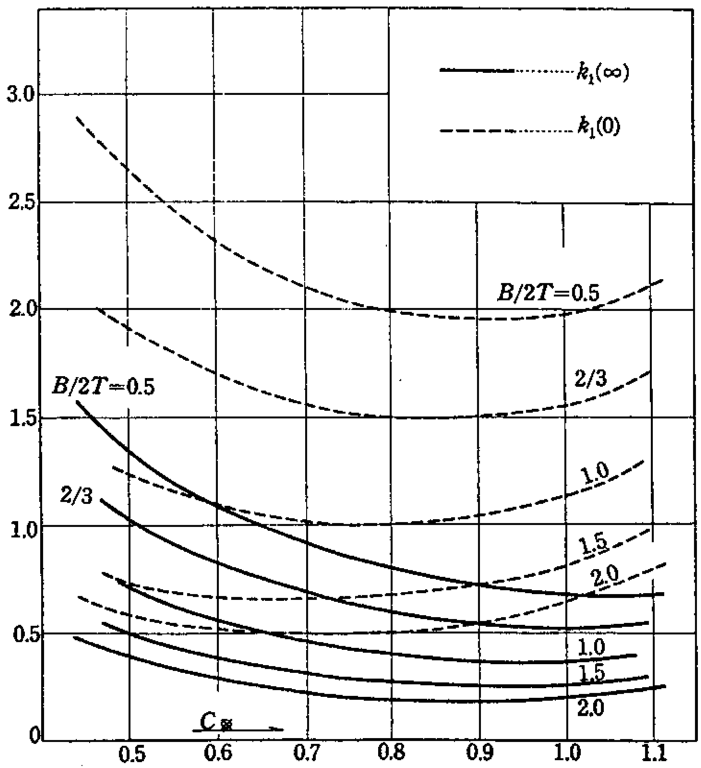

Fig. 3. Added mass coefficient.

Thus, Ursell's cylinder having no wave-damping is the section form which satisfy the relation (3.16).

iv) The exciting moment of the wave about the center of gravity vanishes by (3.7), when

$$l=l_w \fallingdotseq (\overline{OM}+k_1 l_1)/(1+k_1)\ ,$$

Hence, there always exists such point as T. Hishida says, and Isshiki et al. reaffirmed this fact [9, 22].

In the same way as the above, we have the approximation for $K=\infty$ in the same form as (3.9) and, for the exciting force, since we can assume that the wave emitted from the one side can not go to the other side, considering the ship' side is vertical near the water surface, we have by the formula of T. H. Havelock [4].

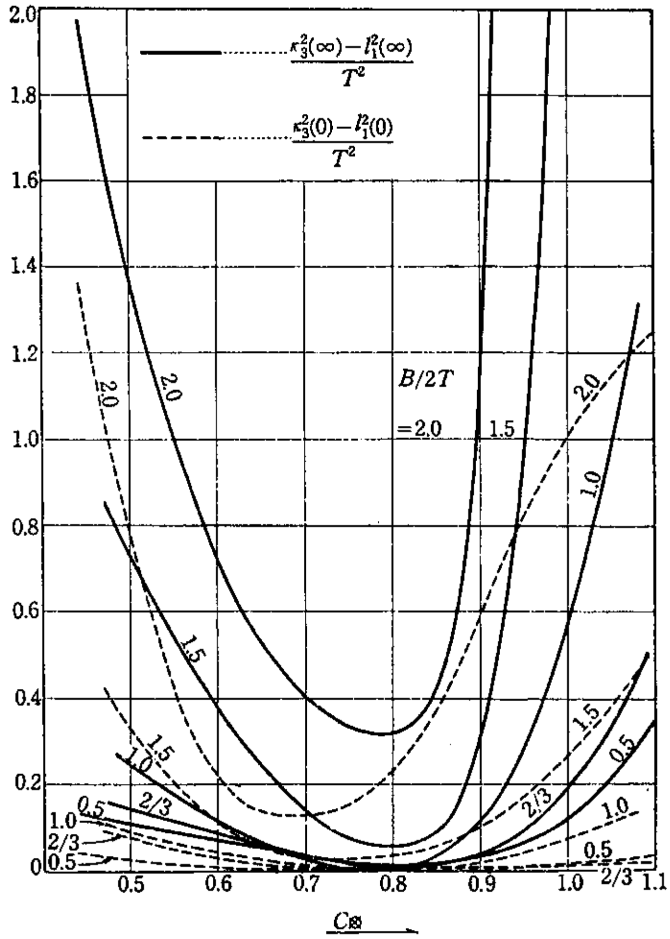

Fig. 4. Least Radius of gyration of added inertia.

$$H_1^+(K) \fallingdotseq 2\int_0^\infty \frac{\partial\phi_1}{\partial x} e^{-Ky+\frac{iKB}{2}} dy \fallingdotseq -\frac{2}{K} e^{\frac{iKB}{2}}\ , \tag{3.17}$$

$$H_3^+(K) \fallingdotseq 2\int_0^\infty \frac{\partial\phi_3}{\partial x} e^{-Ky+\frac{i}{2}KB} dy \fallingdotseq -\frac{2}{K^2} e^{\frac{iKB}{2}}\ , \tag{3.18}$$

Accordingly,

$$l_w \doteqdot 1/K\ , \tag{3.19}$$

This value is an accurate approximation for large K compared with the calculated one by J. Kotik [16].

In Fig. 2, 3 and 4, these approximate values for Lewis form are shown [28].

4. The Equation of Motion. [2, 25, 28]

The equations of motion or more accurately, the balance equations of the linear forces and moments when the cylinder moves stationarily the harmonic motion among waves, are, except the time factor,

$$\left.\begin{aligned} &-\frac{\omega^2}{g}WX=F_{11}+F_{15}+E_1\ ,\\ &-\frac{\omega^2}{g}WY-\rho gBY=F_{22}+E_2\ ,\\ &-\frac{\omega^2}{g}I\theta+W\overline{GM}\theta=F_{51}+F_{55}+E_5\ , \end{aligned}\right\} \tag{4.1}$$

where $W=\rho g\nabla$, $I=W\kappa^2$ and κ is the radius of gyration.

Introducing the definition of the preceding paragraph, we can rewrite then as follows:

$$(\nabla+f_{11})\chi+f_{15}\theta=\frac{a}{K}H_1^+(K)\ , \tag{4.2}$$

$$\left(\nabla\kappa^2+f_{55}-\frac{\nabla}{K}\overline{GM}\right)\theta+f_{15}X=\frac{a}{K}H_5^+(K)\ , \tag{4.3}$$

$$\left(\nabla+f_{22}-\frac{B}{K}\right)Y=\frac{a}{K}H_2^+(K)\ , \tag{4.4}$$

Since we assume the symmetry of the cylinder, the heaving-dipping oscillation is independent on the other ones and the solution is

$$Y=\frac{aH_2^+(K)}{K(\nabla+f_{22})-B}\ , \tag{4.5}$$

If we put the approximation (3.11) into the above, we have

$$Y\doteqdot a\ , \tag{4.6}$$

that is, the cylinder moves with the wave.

But, at the resonance, that is, when

$$K\nabla(1+f_{22c})=K\nabla(1+k_2)=B\ , \tag{4.7}$$

we have from (3.5)

$$Y = ia/K\overline{H}_2^+(K) , \tag{4.8}$$

thence, if the exciting force of the wave vanishes, comparing (4.7) with (3.15) we must conclude that the heaving amplitude becomes infinite at the resonance as far as we consider only the wave-damping.

This is a natural consequence of Haskind's formula in two dimensional problem which says that the wave-damping is proportional to the square of the exciting force. But, actually, there is another damping, say, the frictional and eddy one, so that we may expect to reduce the oscillation by reducing the exciting force.

On the other hand, the swaying and rolling oscillation are dependent with each other by the term f_{15}.

The solution is easily found as follows:

$$X/a = D_1/\Delta , \qquad \theta/\theta_w = D_5/\Delta , \tag{4.9}$$

where

$$\left.\begin{aligned} D_1 &= \frac{\nabla}{K}\left\{\left(\chi^2 + k_1\kappa_5^2 - \frac{\overline{GM}}{K}\right) - k_1(l_1 - l)(l_w - l)\right\} H_1^+(K) , \\ D_5 &= \frac{\nabla}{K^2}\{(1+k_1)l_w - (l + k_1 l_1)\} H_1^+(K) , \end{aligned}\right\} \tag{4.10}$$

$$\left.\begin{aligned} Re\{\Delta\} &= \left\{(1+k_1)\left(\kappa^2 + k_1\kappa_5^2 - \frac{\overline{GK}}{K}\right) - k_1^2(l_1 - l)^2\right\}^2 \nabla^2 , \\ -Im\{\Delta\} &= \left\{\left(\kappa^2 + k_1\kappa_5^2 - \frac{\overline{GM}}{K}\right) + (1+k_1)(l - l_w)^2 - 2k_1(l_1 - l)(l_w - l)\right\} \nabla |H_1^+|^2 , \end{aligned}\right\} \tag{4.11}$$

$$k_1\kappa_5^2\nabla = f_{55C} = \{\kappa_3^2 - l_1^2 + (l - l_1)^2\} k_1 \nabla , \tag{4.12}$$

At the resonance, since there is the condition,

$$Re\{\Delta\} = 0 \tag{4.13}$$

we have from (4.11)

$$\Delta = \frac{K^4 |D_5|^2}{i(1+k_1)} , \qquad D_1 = -\frac{K k_1 (l_1 - l)}{(1+k_1)} D_5 , \tag{4.14}$$

Putting these into (4.9), we have

$$\left.\begin{aligned} \theta/\theta_w &= i(1+k_1)\nabla/(K^4 \overline{D}_5) , \\ X/a &= -ik_1(l_1 - l)\nabla/(K^3 \overline{D}_5) , \\ \theta/(KX) &= -(1+k_1)/\{K k_1 (l_1 - l)\} , \end{aligned}\right\} \tag{4.15}$$

In another way, we have the exciting moment of the wave from the first and the third equation of (4.1) as follows:

$$\frac{M}{\rho g \Delta \theta_w}=\frac{-iKD_5}{(1+k_1)\nabla^2}=\frac{1}{iK\nabla}\left(l_w-\frac{l+k_1 l_1}{1+k_1}\right)H_1^+(K)\,, \tag{4.16}$$

Thus, we see in (4.14), (4.15) and (4.16) the same conclusion as in the heaving oscillation.

Moreover, if we introduce the coefficient of the effective wave-slope as usual as [1, 5, 6]

$$M/(\rho g \nabla \theta_w)=\gamma \overline{GM}\,, \tag{4.17}$$

Comparing this with the above, it is given as

$$\gamma=\frac{i}{K\overline{GM}\nabla}\left(l_w-\frac{l+k_1 l_1}{1+k_1}\right)H_1^+(K)\,, \tag{4.18}$$

When K is small, putting the approximation (3.10), (3.13) and (3.14), we have

$$\gamma \doteqdot 1 \tag{4.19}$$

This may be the correct conclusion of the present order of approximation [2] and this means that the exciting moment of the wave can not vanish in free rolling motion, although there exists always a point about which the exciting moment vanishes.

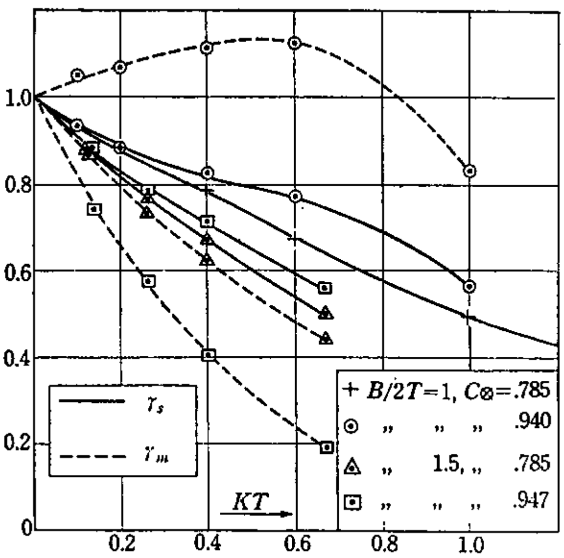

Fig. 5. Calculated examples of γ_s and γ_m.

This is because the reaction of the sway motion cancells out the exciting force of the wave as if there were no dynamical effect.

In the definition (4.18) of γ we can also divide it as

$$\gamma\overline{GM}=\gamma_s\overline{OG}-\gamma_m\overline{OM}=\gamma_s\overline{GM}-(\gamma_s-\gamma_m)\overline{OM}\,, \tag{4.20}$$

$$\gamma_s=iH_1^+(K)/\{K(1+k_1)\nabla\}\,, \tag{4.21}$$

$$\gamma_m=\gamma_s\{(1+k_1)l_w-k_1l_1\}\,, \tag{4.22}$$

Then, the effect of the center of gravity upon γ is confined itself on the first term of (4.20).

Some examples calculated from F. Tasai's tables are shown in Fig. 5. [18, 28].

Since we have, expanding the exponential term of H_j to the next,

$$\left.\begin{aligned} iH_1^+(K)/K &\fallingdotseq (1+k_1)\Delta-K\int_c\left(\phi_1\frac{\partial}{\partial n}+\frac{\partial x}{\partial n}\right)xydS\,,\\ iH_3^+(K)/K &\fallingdotseq (\overline{OM}+k_1l_1)\Delta-K\int_c\left(\phi_3\frac{\partial}{\partial n}+\frac{\partial x_3}{\partial n}\right)xydS\,, \end{aligned}\right\} \tag{4.23}$$

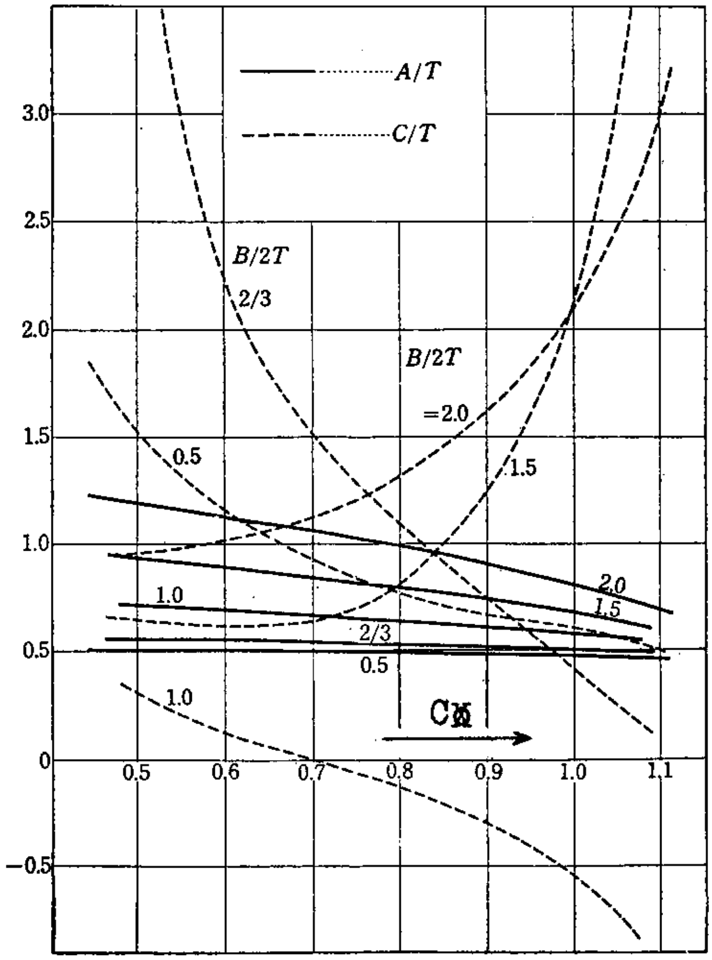

Fig. 6. Function A and C.

putting these into (4.21) and (4.22), we have the following approximation.

$$\gamma_s \doteqdot 1 - KA, \qquad A = \frac{1}{(1+k_1)V}\int_c \left(\phi_1 \frac{\partial xy}{\partial n} + xy\frac{\partial x}{\partial n}\right) dS, \tag{4.24}$$

$$\gamma_m \doteqdot 1 - KC, \quad \overline{OM}C = -k_1 l_1 A + \int_c \left(\phi_3 \frac{\partial xy}{\partial n} + xy\frac{\partial x_3}{\partial n}\right) dS, \tag{4.25}$$

where ϕ_1 and ϕ_3 are considered as the limit potential of $K=0$.

Then A, C and approximate γ are easily calculated for Lewis form as shown in Fig. 6 and 7. Comparing these values with the exact values in Fig. 5, we can see that they are fairly good approximation for small K.

If we neglect ϕ_1, ϕ_3 and k_1 in (4.24) and (4.25), we have the classical theory by Froude-Kriloff-Watanabe and

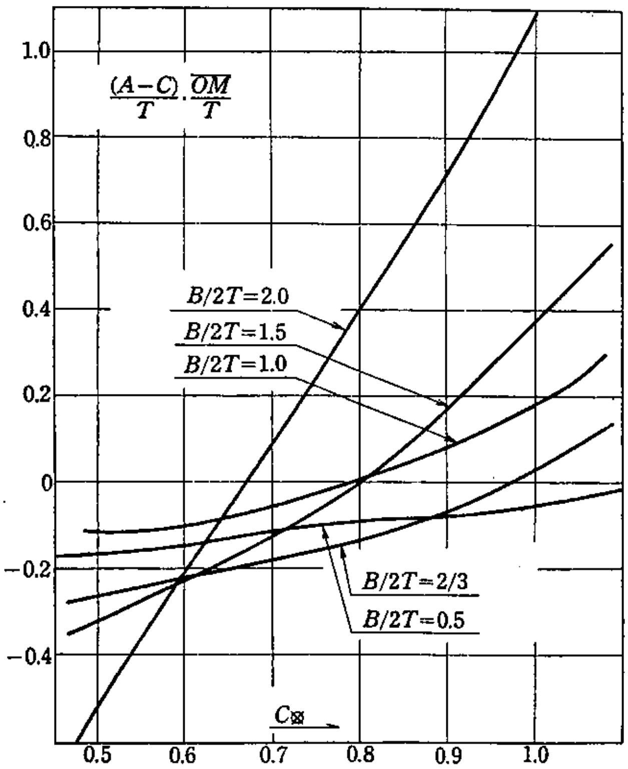

Fig. 7. Approximate value of $\overline{OM}(\gamma_m - \gamma_s) = K\overline{OM}(A-C)$.

$$\gamma' \overline{GM} = \gamma_s' \overline{OG} - \gamma_m' \overline{OM}, \tag{42.6}$$

$$\gamma_s' \doteqdot 1 - KA', \quad A' = \frac{1}{V}\int_c xy\frac{\partial x}{\partial n} dS = \overline{OB}, \tag{4.27}$$

$$\gamma_m' \doteqdot 1 - KC', \quad C' = -\frac{1}{\overline{OM}V}\iint (x^2 - y^2)\, dx\, dy, \tag{4.28}$$

These formulas are not exact theoretically, but they are analogous as (4.26) and (4.27). Hence, the present theory may not be so different with the classical one in this character.

Now, let us consider once more the problem to reduce γ to zero.
By (4.20), (4.21) and (4.22), at first,

$$\gamma=0 \qquad \text{for } \gamma_s=0\,, \tag{4.29}$$

but this is not realistic because the sway-exciting force does not vanish in usual [24].

Secondly, puting (4.24) and (4.25) into (4.20) and considering that the added moment of inertia is very small in usual and K-value at the resonance is nearly

$$K \fallingdotseq \overline{GM}/\kappa^2\,, \tag{4.30}$$

we have

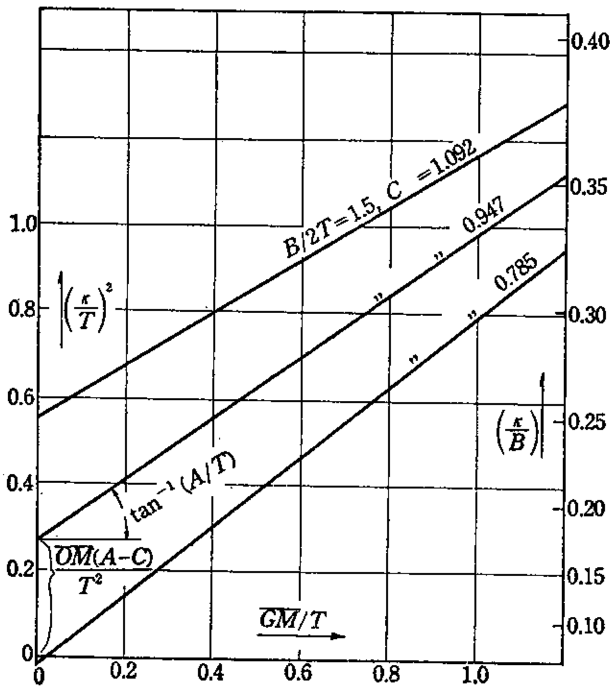

Fig. 8. Combination of the metacentric height and the radius of gyration which has no roll-exciting moment of wave.

$$\gamma \fallingdotseq 0 \qquad \text{for } \kappa^2 \fallingdotseq \frac{GM}{K} \fallingdotseq A\overline{GM}-(C-A)\overline{OM}\,, \tag{4.31}$$

Thus, as shown in Fig. 8 for Lewis form, if we select a pair of $\overline{GM}$ and κ on the straight line, γ becomes nearly zero. However, such pair of values means generally a very short period of the resonance.

5. Experiments. [28]

Many authors have carried out the experiments with respect to the verification of the theory of waves and reports that the theory is very accurate and F. Tasai describes also that the rolling motion of a ship is predicted with fairly good approximation [19] but K. Tamura tells us that theoretical prediction of the rolling motion is not very good with the two-dimensional cylinder [20].

Now, the equations of free motion without the incident wave are from (4.2) and (4.3) except the heaving,

$$\left.\begin{aligned}&(\nabla+f_{11})X+f_{15}\theta=0\ ,\\&\left(\nabla\kappa^2+f_{55}-\frac{\nabla}{K}\overline{GM}\right)\theta+f_{15}X=0\ ,\end{aligned}\right\}\tag{5.1}$$

Of course, they are not true equations of motion but merely linearized and neglected the external forces to keep the harmonic oscillation.
The first equation of them gives

$$X/\theta=-f_{15}/(\nabla+f_{11})\fallingdotseq k_1(l-l_1)/(1+k_1)\ ,\tag{5.2}$$

neglecting the imaginary parts to be considered as small. This is the same as the third of (4.15), and that, putting this into the second equation of (5.1), we have

$$\overline{GM}/K\fallingdotseq\kappa^2+f_{55c}/\nabla-k_1^2(l_1-l)^2/(1+k_1)\ ,\tag{5.3}$$

which is the same as the resonance condition (4.13).

Thence, the free motion of the cylinder is similar as at the resonance among wave, and the energy loss of the cylinder in both conditions must be the same. Since the energy loss by the emitting wave per one swing is

$$\frac{\rho gK}{4}|\theta H_5^+ + XH_1^+|^2\ ,$$

and the loss of the potential energy is

$$\rho g\nabla\overline{GM}\theta\Delta\theta\ ,$$

where $\Delta\theta$ is the decrement of the inclining angle per one swing, equating above two loss and using (5.2) and (4.18), we have

$$\frac{\Delta\theta}{\theta}=\frac{\pi}{2}K^3|\gamma|^2\overline{GM}\nabla\ ,\tag{5.4}$$

At first, this value for the usual K-value is very small, and it seems impossible to explain the damping of usual ships.

Secondly, it seems to contradict the fact that the upper the center of gravity the greater the roll-damping is [29].

Hence, in two-dimensional problem, the theory does not seem to be able to predict the actual motion as K. Tamura observes, but, in three dimension, it may success as Tasai does because the fore and aft end parts of a ships emit comparatively large wave as Hishida says [10]. In any way, since the wave damping is merely a part of the whole damping, it is more important to study the frictional or eddy damping in this respect. It is actually impossible to obtain the exciting force of the wave from the damping measurement although it is possible by Haskind's formula if there would be no viscosity.
Of course, here is yet a possibility by measuring the diverging wave amplitude of the cylinder forced the harmonic oscillation, but its measurement is very difficult [17, 22].
Hence, we may measure independently the exciting force of the wave.
S. Motora et al. proposed a method to measure the inclining moment M among waves restricting only the angular displacement [1].

The equations of motion in this case are

$$\left.\begin{aligned} (\nabla+f_{11})X&=\frac{a}{K}H_1^+(K)\,, \\ f_{15}X&=\frac{a}{K}H_5^+(K)+\frac{M}{\rho g K}\,, \end{aligned}\right\} \tag{5.5}$$

The swaying amplitude is given by the first equation as

$$\frac{X}{a}=\frac{H_1^+(K)}{K(\nabla+f_{11})}\,, \tag{5.6}$$

and, comparing with the definition (4.21), this is

$$\frac{X}{a}\doteqdot -i\gamma_s\,. \tag{5.7}$$

Putting (5.6) into the second equation, we have

$$\frac{M}{\rho g\nabla\theta_w}=-\frac{(1+k_1)H_1^+(K)}{(\nabla+f_{11})K}\left(l_w-\frac{l+k_1l_1}{1+k_1}\right)\,, \tag{5.8}$$

which equals (4.16) neglecting the imaginary part of f_{11} in the denominator.

This formula gives us the theoretical basis of Motora's test. Moreover, the formula (5.7) permits us to estimate the coeffictent γ_s from the swaying amplitude and also γ_m by (4.20). These tests have been carried out successfully at Tokyo University and the present theory seems reliable at least up to this day [23].

In the experimental point of view, the present theory takes the so-called active resistance into account but its practical treatment is similar to the classical method as we see except the physical explanation of the coefficient γ which is called the effective wave-slope one classically, because it is verified experimentally that the effect of sway oscillations does not serious for the rolling motion as a whole classically [7].

Namely, as explained in the preceding, it is more realistic to understand it as the reducing coefficient of the statical metacentric height, when a ship moves among waves receiving dynamical forces and the statical buoyancy from the water. Hence, we may call $\gamma\overline{GM}$ the dymanical metacentric height.

Another one of new points of view in this respect is of roll-reducing action of the bilge keel. Namely, it acts as an eddy-making damper and has no influence on the exciting force in the classical theory [1], but taking its added mass into account, it has also an effect to reduce it in the present theory, because the bilge keel with its added mass acts as if the cylinder has a fuller section coefficient than the true geometric one, and the fuller the section, the smaller γ is as seen in Fig. 7.

The experiments verify this fact [23].

6. Conclusion.

The theory of water wave of the two-dimensional oscillatory cylinder is one of the most advanced in this field, and explains successfully many experiments but the direct application of it to the rolling motion seems hopeless as K. Tamura's experiment because of the effect of the viscosity, especially the bilge keel, can not be neglected.

On the other hand, the classical Froude-Kryloff-Watanabe theory explains well the experimental result as a whole. Thence, in this paper, the author tries to abridge this gap in the one side and, in the other side, studies the problem to reduce the exciting moment of the wave, making use of Haskind-Hanaoka's relation.

The conclusions are as follows;

i). Some of all necessary velocity potentials are dependent with each other, especially, the diffraction potentials are all derived from other potentials.
From this fact Haskind's formula is easily understood.
The inclining moment and the swaying force of the wave excitation is in phase.

ii). When K is very small, this is usual in the rolling motion, the center C of the added mass for the swaying lies under the metacenter M but just on it in the elliptic section.
The lateral center of the wave-exciting force lies in the segment $\overline{CM}$ with the ratio $1: k_1$, k_1 is the added mass coefficient to the displacement volume, but the exciting moment of the wave in the free motion equals the statical one because the inertial reaction of the swaying cancells out the wave force corresponding to the added mass force acting on the point C.

iii). The wave-damping of the two-dimensional cylinder seems so small that the rolling motion cannot be explained theoretically in general.

iv). The inclining moment of the wave is represented in the form

$$M/\rho g V\theta_w=\gamma\overline{GM}$$

as like as in thh classical theory.
It seems convenient to divide it as

$$\gamma\overline{GM}=\gamma_s\overline{OG}-\gamma_m\overline{OM}=\gamma_s\overline{GM}+(\gamma_s-\gamma_m)\overline{OM},$$

because γ_s and γ_m are independent upon the center of gravity by this substitution.
Experimentally, these coefficients are measured by Motora's method.

v). There may be possibilities to make zero the rolling moment of the wave, but they seem difficult to realize for the practical case at the present stage of knowledge.

vi). The bilge keel has a function to reduce the exciting moment of the wave.
The recent experiments show that the present theory is moderately reliable.

We could a parallel analysis in the three dimensional case, but the problem is very difficult in the numerical analysis and in the point what it can not be assumed usually that the incident wave from any direction is much larger than the ship length. Of course, the strip method will be a possible way to solve the problem, and F. Tasai says that the theory fits well to the experiment. However, otherwise the effect of the viscosity were taken account into the theory, especially of the function of the bilge keel, it will be difficult to understand the motion of ships.

Lastly, the author thanks with his heart to Prof. Motora for his many kind and suggestfull discussions through the present research.

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