Reprinted from

Eighth Symposium

NAVAL HYDRODYNAMICS

HYDRODYNAMICS IN THE OCEAN ENVIRONMENT

ARC-179

Office of Naval Research
Department of the Navy

VARIATIONAL APPROACHES TO STEADY SHIP WAVE PROBLEMS

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INTRODUCTION

Although there have been many fruitful engineering applications of the theory of the wave-making resistance of ships, it is still not possible to completely explain the wave resistance of the usual surface-piercing ships. The so-called order theory gives us insight into the structure and composition of our approximate theory; however, we do not yet have a consistent and practical theory which is universally acceptable.

The author has speculated on what would be the best approximation to our boundary value problem. In this connection, is there a useful principle which corresponds to the Rayleigh-Ritz principle in the theory of elasticity? The present paper will provide a partial answer.

Our first aim is to introduce a variational principle which corresponds to the linearized boundary value problem. This is accomplished by introducing Flax's expression from wing theory. [6]

Our second aim is to find an alternate expression which will enable us to treat blunt bodies, since Flax's method is useful only for thin wings. Gauss' variational expression [24,25] for the boundary problem of a harmonic function is introduced for this purpose. This is shown to be equivalent to extremizing the Lagrangian or kinetic potential. The resulting dynamical interpretation of the boundary value problem is similar to the approaches of many other authors who have studied free surface problems by using the Lagrangian [3,12,13,14].

I. FLAX'S VARIATIONAL PRINCIPLE

The variational principle introduced by A. H. Flax in wing theory [6] may be directly applied to our problem. Those unfamiliar with this principle are directed to Appendix A.

If the Kutta-Joukowski condition [6,7,8] is satisfied at the trailing edge, we have the reciprocity relation

$$\iint_S p\tilde{w}\,dx\,dy = \iint_S \tilde{p}w\,dx\,dy \tag{1.1}$$

by (A.8) and (A.24), where p is the pressure, w is the vertical velocity component, and tildas denote reverse flow quantities. The integration is over the wetted portion of the ship hull S.

Let $\zeta(x,y)$ be the free surface elevation. The variation of the integral

$$I = \iint_S [(p - \tilde{p})\zeta_x - \tilde{p}w]\,dx\,dy \tag{1.2}$$

due to variations of p and $\tilde{p}$ takes the form

$$\delta I = \iint_S [\delta p(\zeta_x - \tilde{w}) - \delta\tilde{p}(\zeta_x + w)]\,dx\,dy. \tag{1.3}$$

Since the variations δp and $\delta\tilde{p}$ are arbitrary, the pressure which extremizes the integral I is equivalent to the solution of the boundary value problem (A.25) and (A.26); that is, the problem for the perturbation potential ϕ with the conditions

$$\begin{aligned} \zeta_x &= -w = \phi_z \\ \zeta_x &= \tilde{w} = -\tilde{\phi}_z \end{aligned} \tag{1.4}$$

on the free surface. The stationary value of I is the drag; namely,

$$[I] = \iint_S p_0\zeta_x\,dx\,dy, \tag{1.5}$$

where p_0 denotes the correct solution. [6,24,26] Thus, the boundary value problem is converted to a variational problem, the solution of which is suggested by various methods of approximation. [6]

If we introduce the error integral,

$$E^* = \iint_S (p - p_0)(\tilde{w} - \zeta_x)\,dx\,dy, \tag{1.6}$$

we see from (1.1), (1.4), and (1.5) that

$$E^* = D - I. \tag{1.7}$$

Therefore, Flax's principle produces an approximate solution which makes the error integral (1.6) stationary. [23]

This method suggests powerful means for obtaining approximate solutions, but unfortunately it has been applied only to thin hydroplanes and wings. [7]

II. GAUSS' VARIATIONAL PRINCIPLE

In this section, we assume there is no free surface. Then the velocity potential has the following representations for the source-sink and doublet distributions:

$$\phi_i(P) = \frac{1}{4\pi}\iint_S \frac{\sigma_i(Q)}{r(P,Q)}\, dS(Q), \quad i = 0,1,2,\ldots \tag{2.1}$$

and

$$\phi_i(P) = \frac{1}{4\pi}\iint_S \mu_i(Q)\,\frac{\partial}{\partial\nu}\,\frac{1}{r(P,Q)}\, dS(Q), \quad i = 0,1,2,\ldots \tag{2.2}$$

Here, quantities with the suffix zero stand for the correct solutions while those with other suffices are not necessarily correct. For these potentials we have the following reciprocity relations:

$$\iint_S \phi_1\sigma_2\, dS = \iint_S \sigma_1\phi_2\, dS, \tag{2.3}$$

$$\iint_S \mu_2\phi_{1\nu}\, dS = \iint_S \mu_1\phi_{2\nu}\, dS, \tag{2.4}$$

and

$$\iint_S \phi_1\phi_{2\nu}\, dS = \iint_S \phi_2\phi_{1\nu}\, dS. \tag{2.5}$$

Gauss's variational principle for the Dirichlet problem states that if we consider the functional

$$G = \tfrac{1}{2}\iint_S (\phi - 2f)\sigma\, dS, \tag{2.6}$$

where

$$f = \phi_0 \text{ is given on } S, \tag{2.7}$$

then the function ϕ which gives the maximum value to G is the solution of the Dirichlet problem. [9,10] This is easily verified by making use of the reciprocity (2.3).

In the same way, we may construct a variational principle for the Newmann problem as follows: Let us consider the extremum problem for the functional

$$H = \frac{1}{2}\iint_S (\phi_\nu - 2f_\nu)\mu \, dS, \tag{2.8}$$

where

$$f_\nu = \phi_{o\nu} \text{ is given on } S. \tag{2.9}$$

This problem is seen to be equivalent to the present boundary value problem by making use of (2.4).

Alternately, we may construct a variational problem by making use of (2.5); namely, by introducing the functional

$$J = \frac{1}{2}\iint_S \phi(2f_\nu - \phi_\nu) \, dS, \tag{2.10}$$

and taking the variation, we have

$$\delta J = \iint_S \delta\phi(f_\nu - \phi_\nu) \, dS. \tag{2.11}$$

From this we see the equivalence to the boundary value problem. [24,25]

Now, since

$$\iint_S \phi_1 \phi_{2\nu} dS = \iiint_D \nabla\phi_1 \nabla\phi_2 \, d\tau, \tag{2.12}$$

where D is the entire water domain and $d\tau$ is a volume element, a natural measure of the error of an approximate solution ϕ is

$$E = \frac{1}{2}\iiint_D [\nabla(\phi - \phi_o)]^2 \, d\tau, \tag{2.13}$$

which becomes

$$E = \tfrac{1}{2}\int\int_S (\phi - \phi_0)(\phi_\nu - \phi_{0\nu})\, dS = J_0 - J, \qquad (2.14)$$

by Green's theorem. Here,

$$J_0 = \tfrac{1}{2}\int\int_S \phi_0\phi_{0\nu}\, dS \qquad (2.15)$$

is the correct value. We see clearly that

$$\delta E = -\, \delta J. \qquad (2.16)$$

Since E is non-negative, we have the inequality [10]

$$J_0 \geq J. \qquad (2.17)$$

It is well-known that among all functions ϕ having a finite energy integral,

$$T = \tfrac{1}{2}\int\int\int_D [\nabla\phi]^2\, d\tau, \qquad (2.18)$$

and a given normal derivative on S, the one which minimizes T is a harmonic function [1,4]. Accordingly, if we solve this minimization problem, say by the relaxation method, we have the inequality

$$T \geq J_0. \quad [1,4] \qquad (2.19)$$

This is the dual of (2.17) and we now have the variational problem (2.7) as an involutory transformation of the latter minimization problem. (See, for example, the textbook on variational calculus [11].)

III. A VARIATIONAL PROBLEM FOR THE LAGRANGIAN

The preceding principle can be easily extended to flow in a gravitational field. Let us consider the functional

$$L = T - V, \qquad (3.1)$$

where

$$T = \frac{1}{2}\iiint_D [\nabla\phi]^2 \, d\tau \qquad (3.2)$$

and

$$V = \frac{g}{2}\iint_F \zeta^2 \, dx\, dy \qquad (3.3)$$

are the total kinetic and potential energies, respectively. L is just the kinetic potential or Lagrangian. [5] Assume that the function ϕ has a given normal derivative

$$\phi_\nu = -x_\nu \quad \text{on} \quad S \quad \text{and} \quad F. \qquad (3.4)$$

Taking the variation of L, we have

$$\delta L = -\iiint_D \phi\nabla^2\delta\phi \, d\tau + \iint_S \phi\delta\phi_\nu \, dS +$$

$$\iint_F [\phi\delta\phi_\nu + \{(\tfrac{1}{2}\nabla\phi)^2 - g\zeta\}\delta\nu] \, dS.$$

Making use of (3.4), which is also true for the new deflected free surface, we have

$$\delta L = -\iiint_D \phi\nabla^2\delta\phi \, d\tau + \frac{1}{\rho}\iint_F p\delta\nu \, dS, \quad [3,14] \quad (3.5)$$

where

$$p/\rho = -\phi_x - \tfrac{1}{2}(\nabla\phi)^2 - g\zeta. \qquad (3.6)$$

Hence, if the pressure at the free surface vanishes, the stationary value of L will be attained when $\delta\phi$ is harmonic. This is just an extension of Kelvin's minimum energy principle. [1,4]

On the other hand, if $\delta\phi$ is harmonic, then the stationary value of L is attained when the free surface pressure is constant and zero. The latter is an extension of Riabouchinsky's principle of minimum added mass. [3,14]

The variational problem can be transformed so that the constraint condition is converted to a natural condition. Let us add a term which is zero at the stationary point. Consider the functional

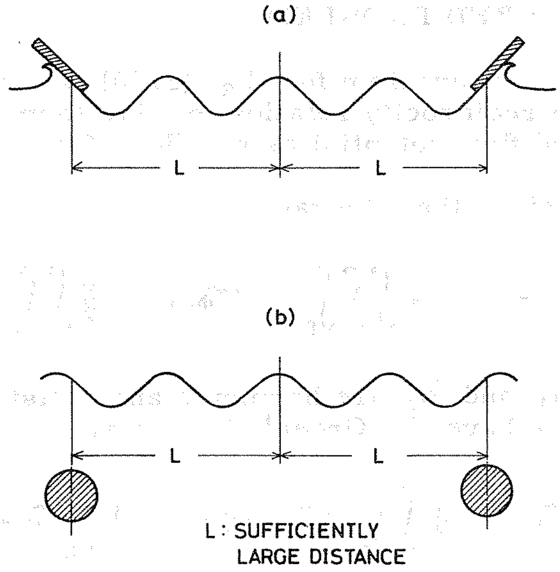

Fig. 2. Riabouchinsky Models

$$P = P_F + P_S , \tag{3.19}$$

where

$$P_S = -\iint_S \phi(x_\nu + \tfrac{1}{2}\phi_\nu)\, dS, \tag{3.20}$$

and

$$P_F = -\frac{1}{2g}\iint_F \phi(\phi_{xx} + g\phi_z)\, dS. \tag{3.21}$$

Accordingly, if we set

$$\phi_{xx} + g\phi_z = 0 \quad \text{on} \quad F, \tag{3.22}$$

which is just the dynamic boundary condition, then

$$P_F = 0, \tag{3.23}$$

and we are left with a variational calculus problem for P_S.

IV. THE LINEARIZED PROBLEM

The variational problem for P_S (3.20) is not satisfactory since there is no reciprocity relation for this form. We must introduce the reversed flow potential as was done for Flax's principle.

Let us consider the integral

$$L^*(\phi_1, \tilde{\phi}_2) = L^*(\tilde{\phi}_2, \phi_1) = -\tfrac{1}{2}\iiint_D \nabla\phi_1 \nabla\tilde{\phi}_2 \, d\tau - \frac{g}{2}\iint_F \zeta_1 \tilde{\zeta}_2 \, dx\, dy. \quad (4.1)$$

Assuming that ϕ_1 and $\tilde{\phi}_2$ are harmonic and satisfy the free surface condition, we have, by Green's theorem,

$$L^*(\phi_1, \tilde{\phi}_2) = -\tfrac{1}{2}\iint_S \phi_1 \tilde{\phi}_{2\nu}\, dS = -\tfrac{1}{2}\iint_S \tilde{\phi}_2 \phi_{1\nu}\, dS, \quad (4.2)$$

where S is the surface of a submerged body. This is the reciprocity theorem for a submerged body. [8]

If $\tilde{\phi}_\nu = -\phi_\nu$, then

$$L^*(\phi, \tilde{\phi}) = \tfrac{1}{2}\iint_S \phi\phi_\nu \, dS = L(\phi, \phi), \quad (4.3)$$

where

$$L(\phi, \phi) = \tfrac{1}{2}\iiint_D (\nabla\phi)^2 \, d\tau - \frac{g}{2}\iint \zeta^2 \, dx\, dy. \quad (4.4)$$

L^* is called the modified Lagrangian integral [5]. Note that $L(\phi, \phi)$ has a finite value in the linearized case but not in the finite amplitude case.

If S is the wetted part of a surface-piercing body which is under the waterline before the free surface is disturbed, there is an additional term from the surface integral. [15,16,19,20,21] The reciprocity theorem, in this case, is

$$\begin{aligned} L^*(\phi_1, \tilde{\phi}_2) &= -\tfrac{1}{2}\int_L \phi_1 \tilde{\zeta}_2 \, dy - \tfrac{1}{2}\iint_S \phi_1 \tilde{\phi}_{2\nu}\, dS \\ &= \tfrac{1}{2}\int_L \tilde{\phi}_2 \zeta_1 \, dy - \tfrac{1}{2}\iint_S \tilde{\phi}_2 \phi_{1\nu}\, dS. \quad (4.5) \end{aligned}$$

When $\phi_1 = \phi$, $\tilde{\phi}_2 = \tilde{\phi}$ and $\phi_\nu = -x_\nu$, L^* becomes

$$L^*(\phi,\tilde{\phi}) = \frac{1}{2}\int_L \tilde{\phi}\zeta x_n \, dS + \frac{1}{2}\iint_S \tilde{\phi} x_\nu \, dS, \tag{4.6}$$

where n is the inner normal to the waterline curve L in the horizontal plane. Thus, the first term in the right-hand side of (4.6) is the correction for the change of the wetted surface S. [16] This is justified, on the one hand, by the dynamical meaning of the Lagrangian and, on the other hand, by the linearization procedure of the preceding section.

For the case of a pressure distribution over the water surface, we may integrate (4.5) by parts and make use of the formulas in Appendix A. This results in the expression

$$L^*(\phi_1,\tilde{\phi}_2) = -\frac{1}{2}\iint_S \phi_{1x}\tilde{\zeta}_2 \, dx\, dy = \frac{1}{2}\iint_S \tilde{\phi}_{2x}\zeta_1 \, dx\, dy$$

$$= \frac{1}{2\rho}\iint_S [p_1 + \rho g\zeta_1]\tilde{\zeta}_2 \, dx\, dy = \frac{1}{2\rho}\iint_S [\tilde{p}_2 + \rho g\tilde{\zeta}_2]\zeta_1 \, dx\, dy. \tag{4.7}$$

Thus, the reciprocity becomes [8]

$$\mathcal{L}^*(p_1,\tilde{p}_2) = \frac{1}{2\rho}\iint_S p_1\tilde{\zeta}_2 \, dx\, dy = \frac{1}{2\rho}\iint_S \tilde{p}_2\zeta_1 \, dx\, dy \tag{4.8}$$

where

$$\mathcal{L}^*(p_1,\tilde{p}_2) = L^*(\phi_1,\tilde{\phi}_2) - \frac{g}{2}\iint_S \zeta_1\tilde{\zeta}_2 \, dx\, dy. \tag{4.9}$$

Making use of these reciprocities, we may easily show the equivalence of the boundary value problem to the variational problem for the functional I^*, where

$$I^* = \frac{1}{2}\iint_S [\phi\tilde{\phi}_\nu - (\phi - \tilde{\phi})x_\nu] \, dS, \tag{4.10}$$

for a submerged body, and

$$I^* = -\frac{1}{2\rho}\iint_S [p\tilde{\zeta} - (p - \tilde{p})\zeta_0] \, dx\, dy, \tag{4.11}$$

for a pressure distribution. [24,26]

Alternate representations for these integrals are

$$I^* = L^*(\phi_0, \tilde{\phi}_0) - L^*(\phi - \phi_0, \tilde{\phi} - \tilde{\phi}_0), \quad (4.12)$$

and

$$I^* = \mathfrak{L}^*(p_0, \tilde{p}_0) - \mathfrak{L}^*(p - p_0, \tilde{p} - \tilde{p}_0), \quad (4.13)$$

where the suffix zero stands for the correct solution. These formulas show that the variational principle extremizes the Lagrangian of the error and that the stationary values are just given by the Lagrangian.

The difficulty arises in the case of a surface-piercing body. From (4.12), the functional to be extremized is

$$I^* = - L^*(\phi, \tilde{\phi}) + \frac{1}{2}\int_L (\tilde{\phi}\zeta_0 - \phi\tilde{\zeta}_0)\, dy + \frac{1}{2}\iint_S (\tilde{\phi} - \phi)x_\nu\, dS. \quad (4.14)$$

Taking the variation, we have the boundary conditions equivalent to this variational problem,

$$\phi_x = - g\zeta_0, \; \tilde{\phi}_x = g\zeta_0 \quad \text{on} \quad L, \quad (4.15)$$

$$\phi_\nu = - \tilde{\phi}_\nu = - x_\nu \quad \text{on} \quad S. \quad (4.16)$$

But we have no knowledge of the surface elevation on L, a priori, as this problem may be indeterminant. [17,23] We must remember here that the solution is unique only when the detachment points are fixed by the theory of cavitation. [3,14]

This difficulty may be avoided by introducing a homogeneous solution for the two-dimensional, linearized case (see Appendix C).

For the present case, we might proceed as follows: Let us consider the difference between a surface piercing body and the limiting case of a submerged body moving very close to the free surface as in Fig. 3. [23] The boundary condition on the water surface above the submerged body must be $\phi_z = 0$, but since the top is also the free surface, this is equivalent to

$$\phi_z = \zeta_x(x,y) = 0 \quad \text{on} \quad \overline{F}, \quad (4.17)$$

or integrating, we have

$$\phi_x(x,y,0) = - g\zeta(x,y) = \text{Const} = \text{func}\ (y) \quad \text{on} \quad \overline{F}. \quad (4.18)$$

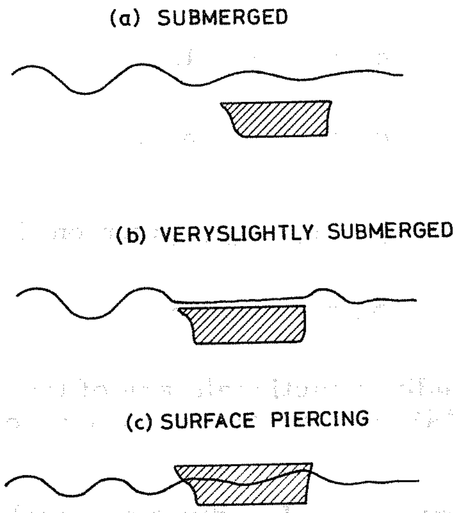

Fig. 3. Slightly Submerged Ship

This formula shows that there may be a thin layer of uniform flow over the top of the submerged body.

When this layer moves with the body,

$$\phi_x(x,y,0) = -g\zeta(x,y) = -1 \quad \text{on} \quad \overline{F}, \tag{4.19}$$

and we clearly have the case of a surface-piercing body.

On the other hand, the boundary value problem of a submerged body is equivalent to the variational problem (4.10). After solving this problem, we may calculate the surface elevation over the top water plane by (4.18), but it will differ from (4.19), in general. In this case, it might be necessary to introduce another potential which satisfies condition (4.19), in addition to the above potential. This procedure may not be practical because the treatment of the top water plane is difficult.

In this case, it would be more convenient to consider the following two boundary value problems: Let us split the velocity potential into two parts,

$$\phi = \phi_1 + \phi_2, \tag{4.20}$$

with boundary conditions

$$\phi_{1x} = 0 \quad \text{on} \quad L$$
$$\phi_{1\nu} = - x_\nu \quad \text{on} \quad S \tag{4.21}$$

$$\zeta_2 = \tilde{\zeta}_2 = \zeta_{20}, \text{ given on } L$$
$$\phi_{2\nu} = 0 \quad \text{on} \quad S \tag{4.22}$$

The corresponding functionals are of the form (4.10) for ϕ_1, and of the form (4.14), without the third term on the right-hand side, for ϕ_2.

For the present case, ζ_{20} must be equal to $1/g$ by (4.19); however, in general, it will be arbitrary and, perhaps, a constant of the form (4.18). ϕ_2 is called the homogeneous solution. [18,22, 26]

Finally, it should be noticed that the Lagrangian is closely related to the far-field potential. For a submerged body, we have, from the boundary conditions, (A.9), (A.11), and (4.3),

$$B = -\iint_S x x_\nu \, dS + 2L(\phi, \phi)$$
$$= 2L(\phi, \phi) + \nabla, \tag{4.23}$$

where ∇ is the displaced volume. For a surface-piercing body, interpreting condition (A.10) as a correction for the real wetted surface, we have

$$\iint_S x\phi_\nu \, dS + \frac{1}{g}\int_L x\phi_x \, dy = \nabla, \tag{4.24}$$

where ∇ is the displacement volume under the still waterline. Therefore, we can write (A.11) as

$$B = \nabla + 2L^*(\phi, \tilde{\phi}_1 + \tilde{\phi}_2), \tag{4.25}$$

where ϕ_1 and ϕ_2 are defined by (4.21) and (4.22), with $\zeta_{20} = 1/g$.

For a pressure distribution, we have, from (A.18) and (4.8),

$$B = 2\rho\mathfrak{L}^*(p, \tilde{p}_2), \tag{4.26}$$

where p_2 is a homogeneous solution, as is ϕ_2, and $\tilde{\zeta}_2 = 1/g$. Since B is also a measure of the total lift, this formula shows that the homogeneous solution for the constant surface elevation influences the lift, as we have easily verified by the reciprocity (4.8). [26] It should be noticed that, in this case, the condition $A = 0$ in (A.18) insures the continuity of the planing hull.

Kotchin's function (A.17) is also given in the form

$$H(\theta) = -\iint_S \phi_e x_\nu \, dS - \int_L \phi_e \zeta \, dy + 2L^*(\phi, \tilde{\phi}_d), \qquad (4.27)$$

where $\tilde{\phi}_d$ has the boundary values

$$\tilde{\phi}_{d\nu} = -\phi_{e\nu} \quad \text{on} \quad S$$

and (4.28)

$$g\tilde{\zeta}_d = \tilde{\phi}_{dx} = -\phi_{ex} \quad \text{on} \quad L$$

$\tilde{\phi}_d$ is called the diffraction potential. [23, 26] Here, the second term of (4.27) may be omitted as in (4.25).

For a submerged body, there is no integration along L and H may be written as

$$H(\theta) = -\iint_S (\phi_e + \tilde{\phi}_d) x_\nu \, dS. \qquad (4.29)$$

Finally, for a pressure distribution,

$$H(\theta) = 2\rho \mathfrak{L}^*(p, \tilde{p}_d), \qquad (4.30)$$

where

$$\tilde{\zeta}_d = -\frac{1}{g} \phi_{ex}. \qquad (4.31)$$

V. CONCLUSION

We have presented two variational principles for the boundary value problem associated with the waves of a ship advancing at a constant speed: The first is Flax's principle, which makes use of the stationary character of the drag. This principle is useful only for planing boats or for submerged thin wings. [6,24] The second is based on Gausz's principle, which converts the boundary value problem to a variational problem. This method is shown to be an extension of Riabouchinsky's principle of minimum virtual mass. [3,24]

The latter principle is based on the stationary character of the Lagrangian and has recently been used by Luke, in a more general form, to study water wave dispersion problems. [3,12,13] We also have analogous principles for light and sound wave diffraction and for the radiation of energy due to the heaving, swaying, and rolling oscillations of ships. [25,27,28,29,30]

The variational principles emphasize the dynamical meaning of the boundary value problems and permit us to solve them approximately by the Rayleigh-Ritz-Galerkin procedure. [6,28,29] However, when we try to apply these principles to our problem, there are two difficulties:

The first is that our system is not conservative because of the trailing wave. This may be bypassed by introducing an artificial model, as in Fig. 2, or by introducing a reversed flow for the linearized case.

The second difficulty is for the surface-piercing body, in which case the wave profile is not known, a priori, even in the linearized case. This difficulty may be avoided by introducing homogeneous solutions [27] which appear in the case of a surface pressure distribution. [26]

Finally, although a variational method does not necessarily represent a new method of analysis, it does suggest new methods of approximation. For this reason, it may be useful, especially for engineering purposes.

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APPENDIX A

The Linearized Velocity Potential [2,23]

Let us consider the flow of water around a ship S, taking the coordinate system as in Fig. 1 and the velocity of the stream at upstream infinity to be unity.

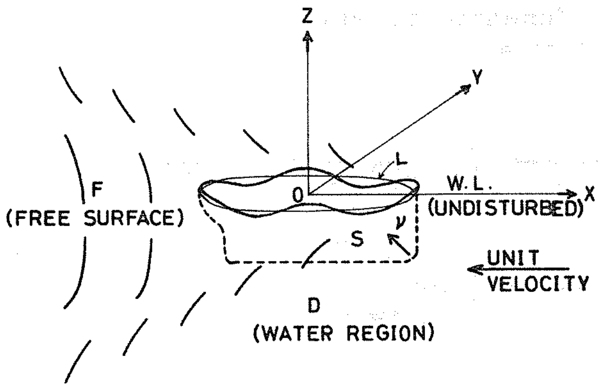

Fig. 1. Coordinate System

The pressure p(x,y) on the water surface is given by

$$\frac{1}{\rho}\, p(x,y) = -\, \phi_x(x,y,0) - g\zeta(x,y), \qquad (A.1)$$

in the linearized theory, where ρ is the water density; g, the gravity constant; ζ, the surface elevation, and ϕ, the perturbation potential ($d\phi = -\,u\,dx - v\,dy - w\,dz$). The suffix stands for differentiation.

The kinematic condition on the water surface is

$$\phi_z(x,y,0) = \zeta_x(x,y). \qquad (A.2)$$

Since the pressure on the free surface is constant, the potential must satisfy the condition

$$\phi_{xx}(x,y,0) + g\,\phi_z(x,y,0) = 0. \qquad (A.3)$$

A solution which has a source singularity at a point Q and satisfies the above water surface condition can be expressed as

$$4\pi S(P,Q) = \frac{1}{r(P,Q)} - \frac{1}{r(P,\overline{Q})} - \frac{g}{\pi}\lim_{\mu\to+0}\int_{-\pi}^{\pi}\int_{0}^{\infty}\frac{e^{k(z+z')+ik(\tilde{\omega}+\tilde{\omega}')}}{k\cos^2\theta - g + \mu i\cos\theta}\,dk\,d\theta\,, \tag{A.4}$$

where $P \equiv (x,y,z)$, $Q \equiv (x',y',z')$, $\overline{Q} \equiv (x',y',-z')$, $r(P,Q) = \overline{PQ}$ and $\tilde{\omega} = x\cos\theta + y\sin\theta$, $\tilde{\omega}' = x'\cos\theta + y'\sin\theta$. Hereafter, we will call this the fundamental singularity. This solution approaches the following values asymptotically:

$$S(P,Q) \xrightarrow[x\gg x']{} \frac{1}{4\pi}\left\{\frac{1}{r(P,Q)} + \frac{1}{r(P,\overline{Q})}\right\}, \tag{A.5}$$

$$S(P,Q) \xrightarrow[x\ll x']{} \frac{g}{\pi}\,\mathrm{Im}\int_{-\pi/2}^{\pi/2} e^{g\sec^2\theta\{(z+z')+i(\tilde{\omega}+\tilde{\omega}')\}}\sec^2\theta\,d\theta. \tag{A.6}$$

By considering the integral

$$\iint [\phi_\nu(Q)S(P,Q) - \phi(Q)S_\nu(P,Q)]\,dS(Q)$$

about a point P in the interior of the fluid, we have the expression

$$\phi(P) = \iint_{S+F}[\phi_\nu S - \phi S_\nu]\,dS.$$

Since ϕ and S satisfy condition (A.3) on F, we have, finally,

$$\phi(P) = \iint_S [\phi_\nu(Q)S(P,Q) - \phi(Q)S_\nu(P,Q)]\,dS(Q)$$
$$- \frac{1}{g}\int_L [\phi(Q)S_{x'}(P,Q) - \phi_{x'}(Q)S(P,Q)]\,dy', \tag{A.7}$$

where L is the curve on which F cuts S.

When the water motion is due to a pressure distribution over the water surface, we have

$$\phi(P) = \frac{1}{\rho g}\iint_S p(Q)S_{x'}(P,Q)\,dx'\,dy', \tag{A.8}$$

where we have used (A.1) and integrated (A.7) by parts. We have also assumed that the potential and surface elevation are continuous over S and F, including L.

Making use of asymptotic characters (A.5) and (A.6) of S, we obtain the asymptotic expansions of ϕ as follows:

$$\phi(p) \xrightarrow[x>>1]{} \frac{1}{2\pi r} A + \frac{x}{2\pi r^3} B, \qquad (A.9)$$

where $r = \overline{PO}$ and

$$A = \iint_S \phi_\nu \, dS + \frac{1}{g}\int_L \phi_x \, dy, \qquad (A.10)$$

$$B = \iint_S [\phi_\nu x - \phi x_\nu] \, dS - \frac{1}{g}\int_L [\phi - x\phi_x] \, dy. \qquad (A.11)$$

The expression for A may also be written as

$$A = \iint_S \phi_\nu \, dS - \frac{1}{g}\iint_F \phi_{xx} \, dx\, dy = \iint_S \phi_\nu \, dS + \iint_F \phi_z \, dx\, dy, \qquad (A.10')$$

by using (A.3). Thus A is the total outward flux from the water domain. This must be zero; otherwise, we would have a large source of the resistance other than from the wave and splash.

We also have the kinematic condition on the surface of the ship,

$$\phi_\nu = -x_\nu \quad \text{on} \quad S. \qquad (A.12)$$

Therefore,

$$\iint_S \phi_\nu \, dS = -\iint_S x_\nu \, dS = 0, \qquad (A.13)$$

where S is the wetted surface of the ship below the undisturbed water surface. From (A.10) and (A.13), we have

$$\frac{1}{g}\int_L \phi_x \, dy = -\int_L \zeta \, dy = 0. \qquad (A.14)$$

But this condition is not adequate in practical cases. One way to avoid this difficulty may be to take the real wetted surface as S. On the other hand, for the consistency of the theory, it may be preferable to take

$$\phi(p) \xrightarrow[x>>1]{} \frac{x}{2\pi r^3} B, \qquad (A.9')$$

instead of (A.9).

Far downstream, we have

$$\phi(p) \xrightarrow[x<<-1]{} \frac{g}{\pi} \operatorname{Im} \int_{-\pi/2}^{\pi/2} \phi_e(P,\theta) H(\theta) \sec^2\theta \, d\theta, \qquad (A.15)$$

where

$$\phi_e(P,\theta) = \exp[g \sec^2\theta (z + i\tilde{\omega})] \qquad (A.16)$$

and

$$H(\theta) = \iint_S [\phi_\nu \phi_e - \phi\phi_{e\nu}] \, dS - \frac{1}{g}\int_L (\phi\phi_{ex} - \phi_x\phi_e) \, dy. \qquad (A.17)$$

For a pressure distribution, we have simply

$$A = 0$$

and

$$B = \frac{1}{\rho g} \iint_S p(x,y) \, dx \, dy, \qquad (A.18)$$

where

$$H(\theta) = \frac{1}{\rho g} \iint_S p(x,y) \phi_{ex} \, dx \, dy. \qquad (A.19)$$

If the flow direction is reversed, the conditions corresponding to (A.1), (A.2), and (A.3) are as follows:

$$\frac{1}{\rho}\tilde{p}(x,y) = \tilde{\phi}_x(x,y,0) - g\tilde{\zeta}(x,y), \qquad (A.20)$$

$$\tilde{\phi}_z(x,y,0) = -\tilde{\zeta}_x(x,y), \qquad (A.21)$$

and

$$\tilde{\phi}_{xx}(x,y,0) + g\tilde{\phi}_z(x,y,0) = 0, \qquad (A.22)$$

so that the fundamental singularity is the same as that for the direct flow, except that the wave follows on the downstream side. This may be expressed as

$$\tilde{S}(P,Q) = S(Q,P), \qquad (A.23)$$

we also have

$$S_{x'}(P,Q) = \tilde{S}_x(Q,P). \qquad (A.24)$$

The boundary conditions for this case are

$$\tilde{\phi}_\nu = - \phi_\nu = x_\nu \quad \text{on} \quad S, \qquad (A.25)$$

and

$$\tilde{\phi}_z = - \tilde{w} = - \phi_z = w = - \zeta_x \quad \text{on} \quad S. \qquad (A.26)$$

APPENDIX B
The Progressing Wave

Let us obtain the solution for a periodic progressing wave, moving at constant unit speed, by the variational method of §3.

We take the form of the complex potential to be

$$\phi + i\psi = - ia \exp(kz - ikx), \qquad (B.1)$$

where the origin is on the undisturbed water level.

The integrals to be evaluated are

$$P = M - T - V, \qquad \frac{1}{\rho}M = - \int_{-\pi/k}^{\pi/k} dx \int_{-\infty}^{0} \phi_x \, dz,$$
$$\frac{1}{\rho}T = \frac{1}{2}\iint (\nabla\phi)^2 \, dx\, dz, \qquad \frac{1}{\rho}V = \frac{g}{2}\int \zeta^2 \, dx, \qquad (B.2)$$

where $2\pi/k$ is the wavelength.

Assuming a surface disturbance of the form,

$$\zeta = b + c \cos kx + d \cos 2kx, \tag{B.3}$$

and integrating the expressions for M, T, and V, we have

$$\frac{1}{\rho} M = \pi ca[1 + k(b + \frac{d}{2}) + \frac{k^2}{8}(c^2 + 4b^2 + 2d^2 + 4bd)],$$

$$\frac{1}{\rho} T = \frac{\pi a^2}{2}[1 + 2kb + k^2(c^2 + 2b^2 + d^2) + k^3c^2(2b + d)], \tag{B.4}$$

$$\frac{1}{\rho} V = \frac{\pi g}{2k}(c^2 + d^2 + 2b^2).$$

Differentiating P with respect to a, b, c and d, and equating the derivatives to zero (by the principle (3.16)), we obtain the following stationary values, neglecting higher order terms:

$$\begin{aligned} c &\doteqdot a(1 + \frac{5}{8}k^2a^2), \\ b &\doteqdot O(k^3), \\ d &\doteqdot \frac{1}{2}ka^2, \end{aligned} \tag{B.5}$$

$$g/k \doteqdot 1 - k^2a^2,$$

$$\begin{aligned} \frac{k}{2\pi} T &\doteqdot \frac{\rho g}{4} c^2(1 + \frac{3}{4}k^2c^2), \\ \frac{k}{2\pi} V &\doteqdot \frac{\rho g}{4} c^2(1 + \frac{1}{4}k^2c^2), \\ \frac{k}{2\pi} P &\doteqdot \frac{k}{2\pi}(T - V) \doteqdot \frac{\rho g}{8} k^2c^4. \end{aligned} \tag{B.6}$$

These expressions agree with other well-known results. [1,2]

APPENDIX C

A Variational Principle for the Stream Function

In the two-dimensional case, we may use a stream function instead of the velocity potential. Let us introduce the stream function as follows:

$$\phi_x(x,z) = \psi_z(x,z), \quad \phi_z(x,z) = -\psi_x(x,z). \tag{C.1}$$

Then, the boundary conditions for ψ become

$$\psi_z(x,0) - g\psi(x,0) = 0 \quad \text{and} \quad \tilde{\psi}_z(x,0) - g\tilde{\psi}(x,0) = 0, \tag{C.2}$$

$$\psi_0(x,z) = -\tilde{\psi}_0(x,z) = -z \quad \text{on} \quad S, \tag{C.3}$$

$$\zeta(x) = -\psi(x,0) \quad \text{and} \quad \tilde{\zeta}(x) = \psi(x,0). \tag{C.4}$$

Introducing a modified Lagrangian integral,

$$L^*(\psi_1,\tilde{\psi}_2) = -\frac{1}{2}\iint_D \nabla\psi_1\nabla\tilde{\psi}_2\, dx\, dy - \frac{g}{2}\int_F \zeta_1\tilde{\zeta}_2\, dx, \tag{C.5}$$

we have, directly, the reciprocity

$$\begin{gathered} L^*(\psi_1,\tilde{\psi}_2) = L^*(\tilde{\psi}_2,\psi_1) \\ \int_S \psi_1\tilde{\psi}_{2\nu}\, dS = \int_S \tilde{\psi}_2\psi_{1\nu}\, dS \end{gathered} \tag{C.6}$$

In particular, from (C.3),

$$L^*(\psi_0\tilde{\psi}_0) = -\frac{1}{2}\int_S \tilde{\psi}_0\psi_{0\nu}\, dS = -\frac{1}{2}\int_S z\psi_{0\nu}\, dS = \frac{1}{2}\int_S \psi_0\psi_{0\nu}\, dS$$

$$= \frac{1}{2}\iint_D (\nabla\psi_0)^2\, dx\, dy - \frac{g}{2}\int_F \zeta_0^2\, dx = L(\psi_0,\psi_0). \tag{C.7}$$

The variational problem with the function

$$I^* = L^*(\psi_0, \tilde{\psi}_0) - L^*(\psi - \psi_0, \tilde{\psi} - \tilde{\psi}_0)$$

$$= \frac{1}{2}\int_S (\psi\tilde{\psi}_\nu - \psi_0\tilde{\psi}_\nu - \tilde{\psi}_0\psi_\nu)\, dS. \qquad (C.8)$$

is equivalent to the boundary value problem for ψ. Here, the boundary values, ψ_0 and $\tilde{\psi}_0$, are given by (C.3). Since a stream function has an arbitrary constant, we should also consider the modified problem with boundary conditions

$$\psi_0 = -\tilde{\psi}_0 = C: \text{ constant on } S, \qquad (C.9)$$

which is the homogeneous problem. [22]

If condition (C.9) holds, the surface elevation at the fore and aft ends is C (instead of zero for the condition (C.3)), but the x-component of the velocity at the same points is $-(1+gc)$, by (C.2). Hence, the water flows in and out the body unless $C = -1/g$. Thus an adequate condition for a surface piercing body is

$$\psi = -z - \frac{1}{g} \quad \text{on} \quad S. \qquad (C.10)$$

Throughout this section, we have treated a class of functions ψ and ζ which are finite and continuous everywhere. As long as the integrals considered exist, the method may be applied with some minor changes to other classes of functions.

The question of the uniqueness of solutions will be left to the future.