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A METHOD FOR DRAG MINIMIZATION IN STOKES FLOW

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ABSTRACT

An inverse procedure for minimizing the hydrodynamic drag of a body is proposed. The distribution of optimum body deformation vector is determined in each iterative step by solving an integral equation.

The method is applied to the axisymmetric Stokes flow for obtaining the optimum shape with the volume specified. Other various restrictions for length, beam, and moments are also taken into account. Numerical results and discussions on the optimum shapes including two-dimensional cases are made in the paper.

The present method can be applied to other flows like Oseen flow, cavity flow, and boundary layer flow, if the flow and the drag are determinable by specifying the body form. The concept of this method will even be applied to the problem of ship viscous resistance optimization.

NOTATION

a	Reference length	B	Body breadth
C	Body contour in (x,r) plane	C_i	Constants $(i=1,2,3,,,)$
$D,\bar{D}$	Region outside, inside of body	E	Dissipation energy integral, or elliptic integral of 1st kind
H	Function of W and W^*		
i,j,k	Suffices (1,2,3) in 3-d coordinates or (x,r) in axisymmetric coordinates		
k	Variable of elliptic integral	K	Elliptic integral of 2nd kind
l_i	Direction cosine of a plane	L	Body length
n	Normal to body surface	p	Pressure
P	Reference point	P_i	Kernel function for pressure
Q	Reference point	r	Radial axis normal to x
R	Resistance or distance $\overline{PQ}$	s	Tangential along contour C
S	Body surface	t_i	Stress in fluid or on body
$T_{k,j}$	Kernel function for stress	u	Velocity $=(u_1,u_2,u_3)$
u_i	Velocity component	$U_{k,j}$	Kernel for velocity
V	Volume of body	$V_{i,j}$	Kernel for velocity in (x,r)
W,W^*	functions in kernels	x	longitudinal axis
Y_j	Kernel for vorticity in ϑ direction	$Z_{k,j}$	Kernel for vorticity in 3-d coordinates
α	Angle of n to x	γ_{ij}	Velocity gradient tensor
$\Gamma_{k,j}$	Kernel for γ	δ	Prefix for variance
δ_{ij}	Kronecker's delta	ε	Infinitesimal sphere at Q
μ	Viscosity	ϑ	Angle in peripheral direction
φ	Angle in elliptic integral	ζ	Vorticity

1. INTRODUCTION

In the field of ship wave resistance, a variety of inverse methods for the inprovement of ship hull form based on linear theory have so far been established and is widely used. In the viscous resistance problem, on the other hand, ordinary orthodox methods have still mainly been used, in which trial and error approachs are made for improving ship stern hull form.

Nevertheless, there have been a few inverse approach in this field. Nagamatsu et al. [1] made a minimization of the viscous resistance using a

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two-dimensional boundary layer equation and a direct search method. Bessho [2] made an optimization of the frame line configuration on the basis of the minimum crossflow energy in a cross section of the hull. Hess [3] also established a minimization of the viscous resistance for two-dimensional and axisymmetric bodies by using approximate solutions of turbulent boundary layer equation.

Recently in ship structure field, Bessho [4],[5] has developed a new theory on a plane stress analysis. He expressed the problem by an integral equation like the boundary element method, and established a new treatment on a case of small deformation of boundary. He applied the theories to the problem for obtaining an optimum boundary shape of mimimum stress concentration. Bessho's method can easily be applicable to the flow problems like Oseen flow, cavity flow [6], and boundary layer flow [7]. The present work is an application of his theory to the three-dimensional Stokes flow.

An inverse approach for mimimum drag in Stokes flow was discussed by Tuck [8] on the basis of a slender body approximation. Pironneau [9] showed a general expression for the optimum condition with a prescribed volume, which gives a constant vorticity on the optimum body hull. Bourot [10] obtained numerically an axisymmetric optimum form based on Pironneau's scheme. Sano and Sakai [11] also showed a two-dimensional solution by using FEM and gold division method. The optimum shapes become elongated forms with finite edge angle.

The present method gives a different approach to this problem including the cases with various additional restrictions as well. The authors [12] have already treated the case of two-dimensional Stokes flow. A three-dimensional analysis is presented here. And numerical examples are shown for axisymmetric flow.

2. BASIC INTEGRAL EQUATIONS FOR STOKES FLOW

Before we preceed on the inverse problem, an ordinary method for obtaining the Stokes flow field with a prescribed body shape is first stated here. We take the coordinate system in three dimensions as shown in Fig.1, and velocities and lengths are normalized by the uniform velocity U (=1) and the reference length a (=1). The pressure p and the stresses are non-dimensionalized by μU/a, and the resistance by μUa.

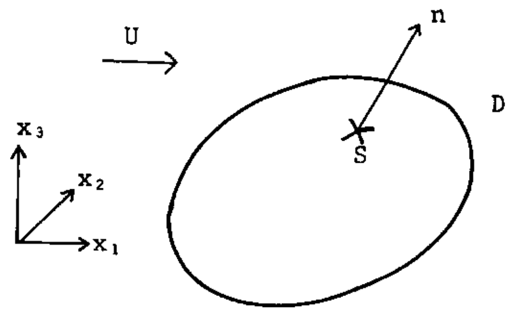

Fig.1 Coordinate system in three-dimensional domain.

The basic equations in three-dimensional Stokes flow are expressed in the form,

$$\partial^2 u_i/\partial x_j^2 = \partial p/\partial x_i, \quad \partial u_j/\partial x_j = 0 \qquad (1)$$

in which u is a perturbation velocity component. The boundary conditions are prescribed as

$$u_1 = -U, \; u_2 = u_3 = 0 \quad \text{on S.} \qquad (2)$$

As is well known, the pressure field is harmonic and the velocity can be broken into potential and viscous parts. The stress t_i on a plane with the direction cosine l_j is represented as the form,

$$t_i = t_{ij} l_j \qquad (3)$$

where the stress tensor t_{ij} is related to the velocity gradient tensor and the pressure.

$$t_{ij} = \gamma_{ij} - p\delta_{ij}, \quad \gamma_{ij} = \gamma_{ji} = \partial u_i/\partial x_j + \partial u_j/\partial x_i \qquad (4)$$

A reverse theorem for Stokes flow can be derived as in the following. Let us consider two flow fields with the velocity vectors $u = (u_1, u_2, u_3)$ and $u' = (u_1', u_2', u_3')$. We here introduce a bilinear form E,

$$E(u,u') = (1/2)\iiint \gamma_{ij}\gamma'_{ij} dv \qquad (5)$$

in which the form E corresponds to the dissipation energy in the flow if $u = \dot{u}'$. Integrating eq.(5) by part gives the reverse theorem.

$$E(u,u') = -\iint u_j t'_j dS = -\iint u'_j t_j dS \tag{6}$$

In the case u=u', applying the boundary condition eq.(2), we obtain the form,

$$E(u,u') = E(u) = U \iint t_1 dS = UR \tag{7}$$

which shows that the dissipation energy directly corresponds to the resistance of the body.

We proceed on the expressions for the flow field. Assuming the flow u' in eq.(7) is created by a singularity at a point Q and using the expressions for the kernel functions in APPENDIX, we can derive the following expression assuming the surface integral consists of the body surface and a small sphere surrounding the singular point Q.

$$u_k(Q) = -\iint [u_j(P)T_{k,j}(P,Q) - t_j(P)U_{k,j}(P,Q)]dS(P) \tag{8}$$

We also assume the flow ceases inside the body $\overline{D}$.

$$\iint u_j(P)T_{k,j}(P,Q)dS(P) = 0 \tag{9}$$

Combining eqs. (8) and (9) we finally obtain the expression for the velocity field,

$$u_k(Q) = \iint t_j(P)U_{k,j}(P,Q)dS(P) \tag{10}$$

where summation is taken for the suffix j. The application of the boundary condition (2) to eq.(10) leads to the integral equation for the stress distribution t_i on the body surface, which can be solved by use of the kernel functions in APPENDIX. Then the pressure p and the vorticity ζ can be obtained by differenciating eq.(10).

$$p(Q) = \iint t_j(P)P_j(P,Q)dS(P) \tag{11}$$

$$\zeta_k(Q) = \iint t_j(P)Z_{k,j}(P,Q)dS(P) \tag{12}$$

where

$$\zeta_i = \partial u_{i+2}/\partial x_{i+1} - \partial u_{i+1}/\partial x_{i+2} \quad \text{for } i = (1,2,3) \tag{13}$$

The conversion of the flow quantities into an axisymmetric coordinate system (x, r, ϑ) as in Fig. 2 can be made. Assume the coordinate transformation as in the form,

$$x_2 = r\cos\vartheta,\ x_3 = r\sin\vartheta,\ x_1 = x \tag{14}$$

then we obtain the transformations for the velocity $(u_x, u_r, 0)$, the stress $(t_x, t_r, 0)$, and the vorticity $(0,0,\zeta)$.

$$u_x = u_1,\ u_r = u_2\cos\vartheta + u_3\sin\vartheta$$
$$t_x = t_1,\ t_r = t_2\cos\vartheta + t_3\sin\vartheta \tag{15}$$
$$\zeta = \zeta_3\cos\vartheta - \zeta_2\sin\vartheta$$

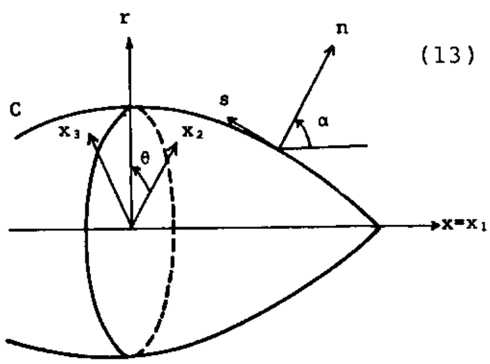

Fig.2 Axisymmetric coordinate system.

Applying these quantities to eqs.(10) and (12), and integrating the kernels along the peripheral direction (ϑ), we can obtain the expressions for the velocities and the vorticity in axisymmetric flow,

$$u_i(Q) = \int_C t_j(P)V_{i,j}(P,Q)ds(P), \text{ for } i = x,r \tag{16}$$

$$\zeta(Q) = \int_C t_j(P)Y_j(P,Q)ds(P) \tag{17}$$

where sum is taken for j=x and r, and the integration is made along the contour line C on the plane (x,r) as shown in Fig. 2. The kernel functions $V_{i,j}$ and Y_j can be expressed by the elliptic integrals of the first and second kinds as stated in APPENDIX. The integral equation for the flow is derived by applying the boundary condition eq.(2) transformed into the axisymmetric coordinates to eq.(16).

We can also show a conjugate relationship between the pressure and the vorticity on the body surface. Transforming the stresses t_x and t_r in the directions of x and r into those t_n and t_s along the normal and tangential to the line C, and using the continuity equation and the boundary condition, we can derive the simple form.

$$t_n = -p, \quad t_s = \zeta \tag{18}$$

Finally we can determine the Stokes flow field by using eq.(12) or eq.(16). The case of the two-dimensional flow can be treated in the same way [12], although the kernels are different and the flow is subject to the so-called Stokes paradox.

3. BOUNDARY DEFORMATION AND DRAG MINIMIZATION

If the variation of resistance due to a deformation of body shape should be obtained, we could inversely determine an optimum deformation so as to minimize the resistance. In particular should the deformation be quite small and be linearly related to the resistance variation, the solution of the problem would be quite simple. The present procedure is a kind of linear analysis and was first developed by Bessho [4],[5] on a plane stress problem. And it is here applied to the three-dimensional Stokes flow.

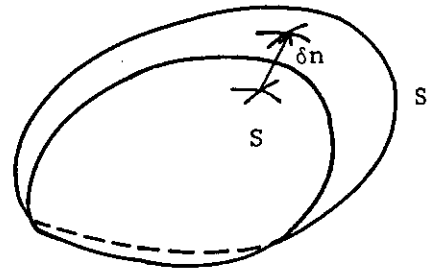

Fig.3 Deformation of body.

As shown in Fig.3, we assume the body surface S associated with the velocity u and the resistance R deforms by an amount of δn to the surface S' with u' and R'. Denoting the variance of the velocity and the stress components as δu_i and δt_i, we define them as follows,

$$\left.\begin{aligned} \delta u_i &= [u'_i]_S - [u_i]_S \\ \delta t_i &= [t'_i]_S - [t_i]_S \end{aligned}\right\} \tag{19}$$

where the deformation δn is assumed to be small and the flow u' around S' be analytically continued to the region around S. We then expand the velocity u' around S and substitute it into eq.(19) to obtain the form,

$$\delta u_i = -(\partial u'_i/\partial n)\delta n = -(\partial u_i/\partial n)\delta n + O(\delta n^2) \tag{20}$$

where the velocity gradient along the normal can be expressed by the vorticity by use of the boundary condition eq.(2) and the continuity equation.

$$[\partial u_i/\partial n]_S = \zeta_{i+1}\, l_{i+2} - \zeta_{i+2}\, l_{i+1} \tag{21}$$

In the above the suffix i takes cyclic value of (1,2,3). In the axisymmetric case, eq.(21) can be converted by the angle α in Fig.2.

$$\delta u_x = \zeta\delta n \sin(\alpha), \quad \delta u_r = -\zeta\delta n \cos(\alpha) \tag{22}$$

The resistance variance δR is defined from eq.(7) as follows,

$$\delta R = E_{D'}(u') - E_D(u) = [E_D(u') - E_D(u)] + E_{\delta D}(u') \tag{23}$$

where the variance of the domain δD = D'-D. The first term in the righthand side of eq.(23) is transformed by use of the reverse theorem eq.(6), and eqs.(3),(4),(20), and (21) into the form.

$$\begin{aligned} E_D(u') - E_D(u) &= -\iint(t_j\delta u_j + u_j\delta t_j)dS \\ &= -2\iint\delta n t_j(\partial u_j/\partial n)dS = 2\iint\zeta_j^{\,2}\delta n dS \end{aligned} \tag{24}$$

The second term in eq.(23) can be expanded to the first order of δn.

$$E_{\delta D}(u') = (1/2) \iint (\gamma_{ij})^2 \delta n dS = - \iint \zeta_j{}^2 \delta n dS \qquad (25)$$

Finally we obtain the form for the resistance variance,

$$\delta R = \iint \zeta_j{}^2 \, n dS \qquad (26)$$

which is completely the same to Pironneau's result [9].

We next consider the problem of the resistance minimization. The optimum condition is expressed in eq.(26) with a volume V constant, i.e.,

$$\delta V = \iint \delta n dS = 0 \qquad (27)$$

The solution for this is clearly the form,

$$\zeta_j{}^2 = \text{const.} \qquad (28)$$

which means the absolute value of the vorticity should be constant on the optimized surface. Therefore an axisymmetric body ($\zeta_x = \zeta_r = 0$) takes smaller resistance than an optimized three-dimensional body. In the former case, taking only the vorticity ζ in the peripheral direction ϑ , the optimization conditions are expressed as follows.

$$\delta R = \int_C r \zeta^2 \delta n ds = 0 \qquad (29)$$

$$\delta V = 2\pi \int_C r \delta n ds = 0 \qquad (30)$$

Then the solution becomes

$$\zeta^2 = C_0 \ (= \text{const.}) \text{ on } C. \qquad (31)$$

The constant vorticity condition for the elementary optimum form also holds in the two-dimensional case [12].

The solution technique for eqs.(30) and (31) should then take an iterative scheme, in which an improved value for δn is solved by use of the variances δu and $\delta\zeta$ from their initial values. Replacing ζ to $\zeta+\delta\zeta$ in eq.(31) and retaining up to the first order term, we obtain the form.

$$\delta\zeta - C_0/(2\zeta) = - \zeta/2 \qquad (32)$$

The relationship between δn and $\delta\zeta$ can be derived by taking variance of eqs. (16) and (17) in connection with the variance δt_i of the stress t_i on the body surface.

$$\delta u_x = \int (\delta t_x V_{x,x} + \delta t_r V_{x,r})ds = \zeta \delta n \sin(\alpha) \qquad (33)$$

$$\delta u_r = \int (\delta t_x V_{r,x} + \delta t_r V_{r,r})ds = -\zeta \delta n \cos(\alpha) \qquad (34)$$

The variance of eq.(17) associated with eq.(32) gives the form.

$$\delta\zeta = \int (\delta t_x Y_x + \delta t_r Y_r)ds = C_0/(2\zeta) - \zeta/2 \qquad (35)$$

The set of eqs.(30),33),(34) and (35) represents the simultaneous integral equations for the unknowns of δn, δt_x, δt_r on the body surface. By solving the equations numerically we obtain the improved value of the deformation δn. The kernels are already available at the stage of the flow calculation so that no more calculation for the kernels is needed, although the size of the equations increases by about one and a half amount of those in the flow calculation. After solving δn and deforming the body, the flow at the new stage is obtainable by solving eq.(16) on the surface, then by using eq.(17) to obtain the vorticity. Eq.(18) can also be used to check the value of the vorticity on the surface.

Finally we consider about additional restriction conditions. Eq.(31) for constant vorticity on the body surface would not hold with more than one restriction condition, so that it should be necessary to increase freedom for the vorticity distribution. Since any longitudinal asymmetry would not contribute to the drag minimization as can be seen in eq.(29), we consider hereafter only the symmetric body in fore and after parts and then assume the form.

$$\zeta^2 = C_0 + C_1 x^2 + C_2 x^4 + ,,,, \qquad (36)$$

The unknowns C_i should be taken by the number of the restriction conditions. Substituting eq.(36) into eq.(29), we can see that the additional unknowns C_1 and C_2 correspond to the additional specification for the higher order moments of the displacement distribution, as follows.

$$\int r\delta n x^2 ds = 0, \quad \int r\delta n x^4 ds = 0 \tag{37}$$

The body length L can be specified as the form,

$$[\ \delta n\ ]_{r=0} = 0 \tag{38}$$

and the breadth B as

$$[\ \delta n\ ]_{x=0} = 0. \tag{39}$$

Various kinds of any other restriction can be added in a similar way. For the iterative procedure eq.(36) can be transferred into the linear form.

$$\delta\zeta = -\ [\ C_0 + C_1x^2 + C_2x^4\ ]/(2\zeta) - \zeta/2 \tag{40}$$

All the preceding discussions in this chapter also hold in two-dimensional case [12].

4. NUMERICAL PROCEDURE AND CALCULATION RESULTS

Computations are made for axisymmetric and longitudinally symmetric body. The after part of the body contour is devided into N (=15) panels in (x,r) plane, and uniform distribution of the stress t_x and t_r on a panel is assumed. The numerical iterative procedure is stated in the following.

(i) Give an initial body shape like sphere, ellipsoid or a polynomial.

(ii) Calculate the panel configulation. Let the control point be at the panel center.

(iii) Calculate the kernel functions $V_{i,j}(P,Q)$ and $Y_j(P,Q)$, which is expressed by use of the elliptic integrals in APPENDIX. On the self inducing panel (P=Q), a singularity treatment should be made. The order of the singularity is logarithmic or the first pole so that the singular part can be integrated analytically and the rest by using Gaussian integration scheme.

(iv) Solve the integral equation (16), and then determine the flow quantities like the pressure, vorticity and resistance.

(v) Solve the integral equations (33) to (35) with any additional restriction condition to obtain the optimum body deformation δn.

(vi) Judge the convergency according to the absolute value of δn.

(vii) If convergence is not attained, deform the body and return to the step (ii).

Preliminary calculations for the present method were carried out by using a 8-bit home computer, PC-8801 (NEC) with the operating system CP/M and the FORTRAN compiler, which takes about 2 hours for one case including four iterations. A relatively large computer ACOS-700 (NEC) at the University of Osaka Prefecture was also used for the series calculations consuming about one minute for the one case.

The results are discussed in the following. Fig.4 represents the elementary optimum shape with the volume specified as a constant (= $4\pi/3$), which completely coincides with Bourot's result [10]. The edge angle is nearly 60 degree which is the exact solution as stated in Pironneau's paper [9]. The resistance decreases to the value 17.98 from 6π (= 18.83) in the case of sphere. Fig.5 shows the distribution of the vorticity which is nearly constant for the optimum shape and is $-(3/2)\sin(\alpha)$ for the sphere. Fig.6 stands for the pressure distribution on the surface.

Instead of specifying constant volume, the case of the length L specified as a constant (=2) is solved as shown in Fig.7. The shape is the same to the volume-specified case, since the vorticity distribution is assumed to be the form in eq.(31). The case when the breadth B is prescribed also produces the same shape.

Fig.8 represents the case when two conditions are specified, like the length and the breadth. When the breadth is prescribed to be small, the optimum shape give rise to an inflexion near the edge like a 'swan neck.' Fig.9 illustrates the results for the volume and the length specified. The case of the prismatic coefficient Cp and the length specified is also shown in the figure. These shapes shoud be similar to those in Fig.8 since the two conditions are specified by using eq.(40). Figs.10 and 11 show the case of three restriction conditions, i.e., the length, beam, and Cp. When the condition is prescribed to be apart from the original elementary optimum shape, unnatural results appear. The half edge angle keeps the value of 60 degree for all cases.

An example of the authors' results for the two-dimensional Stoke flow [12] is shown in Fig.12. The optimum shape is quite similar to the present case. The half edge angle in this case becomes 51.3 degree which is proved by the author in a similar way to the Pironneau's three-dimensional case. Fig.13 shows a comparison of the elementary optimum shapes normalized by the maximum half beams in the different kinds of flow. It can be seen that the solutions of the two- and three-dimensional Stokes flows are quite close to each other, and that they are also close to the solution in which an empirical turbulent boundary layer formula (Hess [3]) is adopted for the resistance prediction and the optimization is made for the outer potential flow (Bessho [7]). Fig.14 represents a comparison of the normalized Cp curves. The present result becomes quite close to parent ship forms, like the aft part of Taylor series (0.6 Cp) and the fore part of Series 60 (0.6 C_B). Although the comparison itself seems meaningless, it is quite interesting to see the fairly good coincidence among the theories based on different flows and the actual ship forms.

The present method gives a technique to solve the inverse problem for obtaining optimum shapes under various resitrictions once the method for predicting the flow and resistance will be established. It is therefore hopeful to apply and modify the present method for the improvement of actual ship forms in the future.

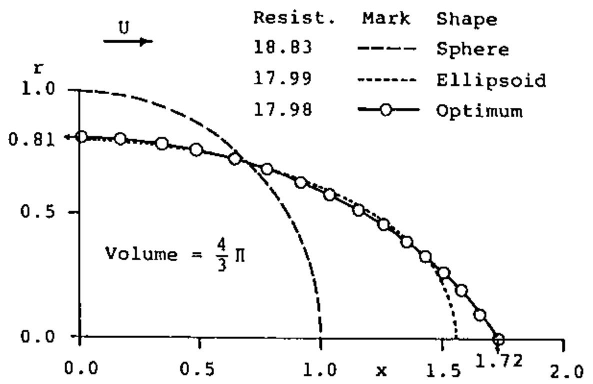

Fig.4 Optimum profile in axisymmetric Stokes flow.

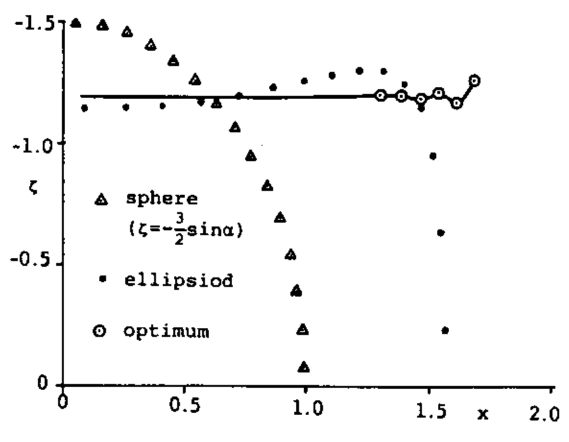

Fig.5 Vorticity distribution.

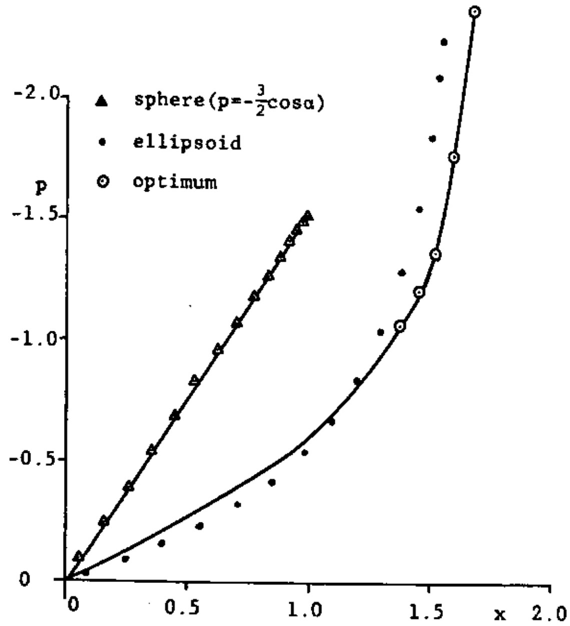

Fig.6 Pressure distribution.

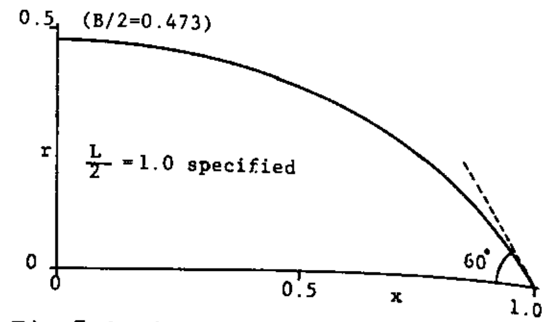

Fig.7 Optimum profile with L specified.

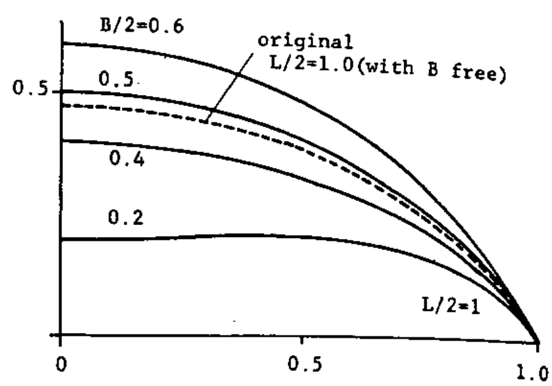

Fig.8 Optimum profiles with length and breadth specified.

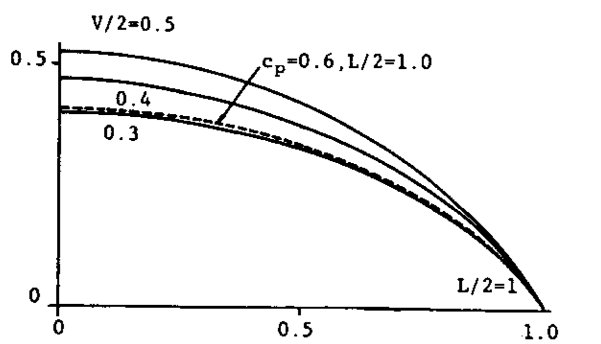

Fig.9 Optimum profiles with volume and length specified.

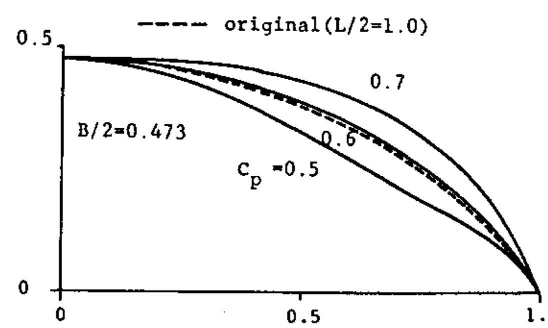

Fig.10 Optimum profiles with length, breadth and C_p specified (B/2= 0.473).

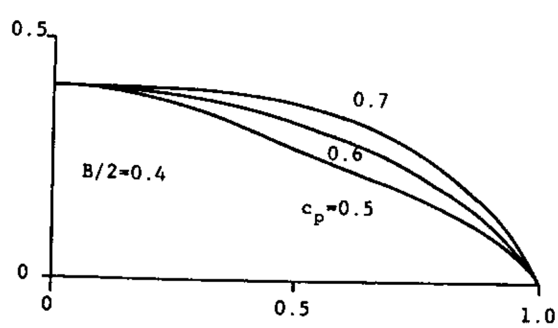

Fig.11 Optimum profiles with length, breadth and C_p specified (B/2= 0.4).

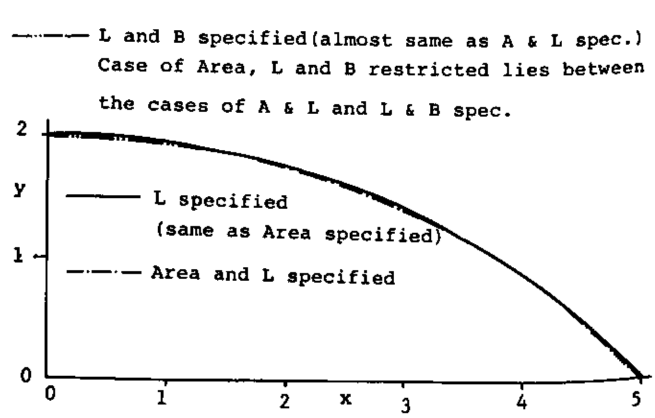

Fig.12 Optimum profiles in two-dimensional Stokes flow (B/2= abt.2, C_B= abt.0.7).

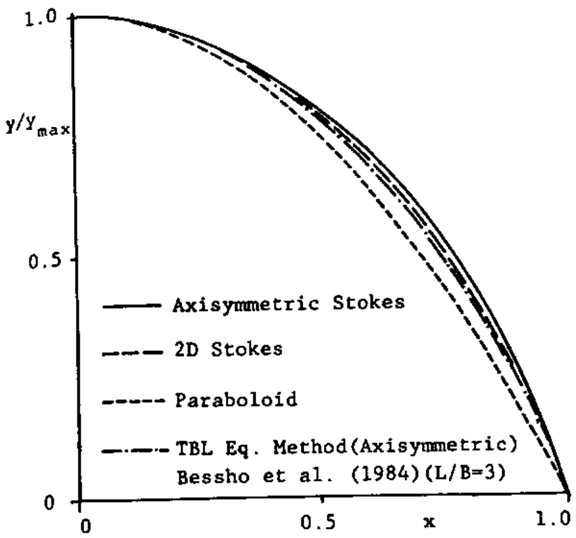

Fig.13 Comparison of normalized optimum shapes.

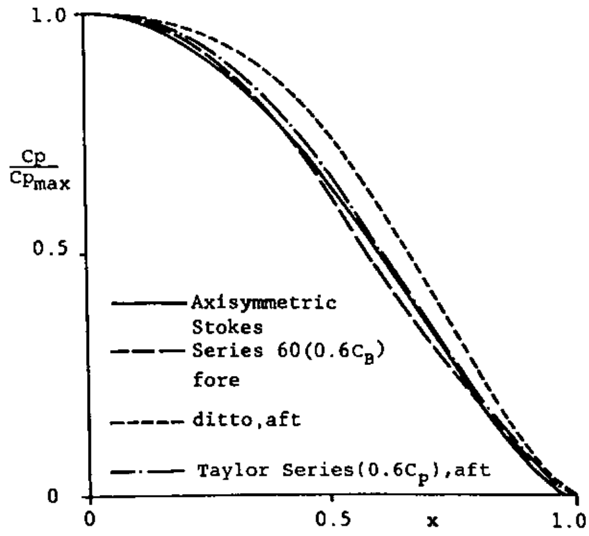

Fig.14 Comparison of normalized Cp curves.

5. CONCLUSIONS

An optimization technique which was proposed by Bessho [4],[5] in a plane stress analysis is applied to the three-dimensional Stokes flow. The analysis and computations for drag minimization are made for axisymmetric flow. Conclusions will be as follows.

(i) The present inverse method consists of an iterative scheme in which the flow is first determined by prescribing the body shape, and secondly the body is deformed slightly so as to minimize the resistance. These two steps require to solve integral equations.

(ii) The elementary optimum shape with a volume specified coincides with Bourot's result by multipole expansion technique. The shape becomes an elongated body with about 0.47 beam-length ratio and 60 degree half edge angle and with a constant value of the surface vorticity.

(iii) Various kinds of restriction condition can be included in the present procedure. As the number of restriction increases, unusual shapes may appear when the shape becomes far from the original elementary optimum shape.

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APPENDIX. KERNEL FUNCTIONS

Let the coordinates of the points P and Q be x_i and x'_i(i=1,2,3), and the distance between them be R. Then the singularity for the pressure is expressed in the form.

$$P_j(P,Q) = (1/4\pi)(\partial/\partial x_j)(1/R) \quad \text{(A1)}$$

The corresponding form for the velocity singularity becomes,

$$U_{k,j}(P,Q) = (1/8\pi)(\partial^2 R/\partial x_k \partial x_j) - \delta_{kj}/(4\pi R) \quad \text{(A2)}$$

and the singularity $T_{k,j}(P,Q)$ for the stress t_k in the direction of k in the plane with the normal defined by $l_i = (x_i - x'_i)/R$ can be derived from eqs. (3) and (4),

$$T_{k,j}(P,Q) = l_i(\Gamma_{k,ji} - P_k \delta_{ij}) \quad \text{(A3)}$$

where summation is taken for the suffix i. For the velocity gradient tensor the singularity becomes the form.

$$\Gamma_{k,ji} = \partial U_{k,j}/\partial x_i + \partial U_{k,i}/\partial x_j \quad \text{(A4)}$$

Substituting eqs. (A1) and (A2) we obtain

$$4\pi\ T_{k,j}(P,Q) = 3(x_k - x'_k)(x_j - x'_j)/R^4. \quad \text{(A5)}$$

The surface integration around a small sphere ε at the point Q gives the form.

$$\int_\varepsilon T_{k,j}(P,Q)dS(Q) = \delta_{kj} \quad \text{(A6)}$$

The vorticity singularity is derived from the velocity.

$$Z_{k,j}(P,Q) = \partial U_{k+1,j}/\partial x_{k+2} - \partial U_{k+2.j}/\partial x_{k+1} \quad \text{(A7)}$$

In case of axisymmetric coordinates, we integrate the singularity expression along the peripheral coordinate ϑ from 0 to 2π. Let the superscript ' concern with the point Q ($\vartheta' = 0$).

$$V_{x,x}(P,Q) = r \int U_{1,1}(P,Q)d\vartheta = r(\partial^2 W^*/\partial x^2/2 - W) \quad \text{(A8)}$$

$$V_{x,r}(P,Q) = r \int(U_{1,2}\cos\vartheta + U_{1,3}\sin\vartheta)d\vartheta = r\partial^2 W^*/\partial x\partial r/2 \quad \text{(A9)}$$

$$V_{r,x}(P,Q) = r \int(U_{2,1}\cos \vartheta' + U_{3,1}\sin \vartheta')d\vartheta = -r\partial^2 W^*/\partial x\partial r'/2 \quad \text{(A10)}$$

$$V_{r,r}(P,Q) = r \int(U_{2,2}\cos\vartheta\cos\vartheta' + U_{2,3}\cos\vartheta'\sin\vartheta + U_{3,2}\sin\vartheta'\cos\vartheta + U_{3,3}\sin\vartheta'\sin\vartheta)d\vartheta = -r(\partial^2 W^*/\partial r\partial r'/2 + H) \quad \text{(A11)}$$

$$Y_x(P,Q) = r\ W/ r',\quad Y_r(P,Q) = r\ H/ x \quad \text{(A12)}$$

$$W^* = \int(R/4\pi)d\vartheta\ ,\quad W = \int 1/(4\pi R)d\vartheta \quad \text{(A13)}$$

$$H = \int\cos\vartheta\ /(4\pi R)d\vartheta = rW/r' - \partial W^*/\partial r/r' \quad \text{(A14)}$$

In the above, the subsequent functions W and W* can be expressed by the first and second kinds of the elliptic functions E(k) and K(k) in the following.

$$r_2{}^2 = (x-x')^2 + (r+r')^2, \quad k^2 = 4rr'/r_2{}^2 \tag{A15}$$

$$W^* = (r_2/\pi) \int(1-k^2\sin^2\varphi)^{1/2}d\varphi = r_2E(k)/\pi \tag{A16}$$

$$W = 1/(\pi r_2) \int(1-k^2\sin^2\varphi)^{-1/2}d\varphi = K(k)/(\pi r_2) \tag{A17}$$

As the point P approaches to the point Q, the term k tends to unity and a logarithmic singularity appears in K(k).