On the Wave Resistance Theory of a Submerged Body

By

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Introduction

In the theory of wave-making resistance in an ideal fluid in which the viscosity of water is ignored, the first requirement is to obtain such velocity potential as to satisfy the boundary condition.

There are two boundary conditions in this case. One of them is called the boundary condition on the surface of the body for which wave-making resistance is considered. The other is called the boundary condition on the free surface.

However, the one is not necessarily independent of the other. When waves are created on the surface of water, their effect, if differ in size, is exerted on the body. If velocity potential is obtained by the second approximation in which the said effect is taken into account, the resultant second approximate value of the waves is different more or less than that of the first approximation. Such mutual dependency goes on when higher approximation is attempted to be obtained for the velocity potential and the waves.

In the existing theory of wave-making resistance the velocity potential is obtained in the first approximation by employing the condition on the surface of the body for which the disturbance caused by the waves is ignored.

The author of this paper was not satisfied with such treatment of the problems. Over a period of several years he has attempted to find a solution by employing the condition on the surface of the body for which the effect of the disturbance caused by the waves was taken into consideration. For this purpose he selected submerged bodies for which mathematical analysis could be made with comparative ease. He examined his solution thus obtained comparing it with that of other researchers.

In Chapter 1 of this paper three of his papers [1]·[2]·[3] on the subject are put together. In § 1 a general equation is obtained for a submerged spheroid and numerical examples are given [1]. In § 2 as a special case of the study made in the preceding section an equation is induced for a submerged sphere [1]. In § 3, with respect to a prolate spheroid with a major-minor-axis ratio of approximately 4, comparisons are made between the numerical examples of the theorectical calculation and the values measured in a model basin experiment [2]. In §4 a general solution is obtaind for a submerged body of an arbitrary form by the method of successive approximations [3].

In §5 in Chapter 2 attention is paid to the fact that, in the existing theory, the treatment of the boundary condition on the free surface is based on the assumption of the so-called infinitesimal wave amplitude, i.e., the condition is linearized. The author examines what change would be found in the form of the velocity potential when the boundary condition on the free surface is treated in a non-linearized form, i.e., by taking the effect of finite wave height into account [4]. In this case, in order to avoid extreme difficulty to be encountered in the frontal attack of the problem of the three-dimensional wave motion, a submerged cylinder is chosen as the subject of study. A two-dimensional wave motion is examined in the case where a submerged cylinder having infinite length and placed horizontally advances at a uniform speed in the direction which forms a right angle to the cylinder's axis.

Needless to say that, in the above discussions, the boundary condition on the cylinder's surface is treated exactly.

In §6 the author discusses the ratio at which the velocity potential, wave-making resistance, lift etc. change when the condition on the body's surface is alone exactly treated, with the condition on the free surface being left linearized.

That change of the ratio is compared with the change of that which is found with the velocity potential, wave-making resistance, lift etc. when the condition on the free surface is treated exactly by cosidering also the effect of finite wave height.

Chapter 1 Higher Approximation to the Wave-Making Resistance of a Submerged Body (Three-Dimensional Problem)

§1 Expressions for the wave-making resistance of a submerged spheroid

1.1 As the example of the comparison made between the first approximate values and the accurate ones of the forces acting on a submerged body in a uniform stream, we have HAVELOCK's work in which he treated the case of a circular cylinder [A–1]. Here the author discusses the case of a sphere, as a limit of an ellipsoid or a prolate spheroid.

At first, let us expand the velocity potential in harmonics. Consider an ellipsoid, submerged in a uniform stream V with the constant depth

f, axes of which are parallel to those of the coordinate system. The axes have a half length a, b and c, respectively.

Take the origin at the centre of the ellipsoid, with Oz vertically upwards and Ox horizontal in the opposite direction of the stream.

The velocity potential of the motion with the source strength $V\sigma$ over the surface of the body S is given as

$$\phi(x, y, z) = Vx + \phi_1 + \phi_2 \tag{1}$$

and

$$\phi_1(x, y, z) = V\iint_S \frac{\sigma(x', y', z')}{D} dS \tag{2}$$

where D is the distance from the point (x, y, z) to (x', y', z'), or

$$\phi_1 = \frac{V}{2\pi}\int_{-\pi}^{\pi}\int_0^{\infty} f(k, t)\exp(-kz + ikw)\,dkdt, \quad z \geq 0 \tag{2·A}$$

and

$$\phi_2 = -\int_{-\pi}^{\pi}\int_0^{\infty} f(k, t)h(k, k_0, t)\exp(-2fk + kz + ikw)\,dkdt, \qquad z \leq 2f \tag{3}$$

where
$$w = x\cos t + y\sin t$$

and

$$f(k, t) = \iint_S \sigma\exp(kz - ikw)\,dS$$

and
$$h(k, k_0, t) = (k\cos^2 t + k_0 + \mu i\cos t)/(k\cos^2 t - k_0 + \mu i\cos t) \tag{4}$$

where μ is the usual infinitesimal factor.

Let us now consider an ellipsoid, expand σ into the ellipsoidal harmonic G_n^m of n-th degree and m-th order on the surface S as [A-3]

$$\sigma(x, y, z) = \frac{\bar{p}}{2\pi abc}\sum_{n,m}\frac{A_n^m}{\kappa^2\{\Pi_{n,m}(\theta)\}^2}G_n^m(x, y, z) \tag{5}$$

where $\bar{p}$ is a distance from the origin to the tangential plane at the point (x, y, z), κ is each of $\left\{1 \begin{matrix} a & ab \\ b & bc \\ c & ca \end{matrix} abc\right\}$, $\{\Pi_{n,m}(\theta)\}$ is the product of characteristic values of n-th degree and m-th order, and A_n^m is a constant independent of the coordinates.

And we have by the integral theorem [A-3]

$$\phi_1 = \frac{V}{2\pi abc}\sum_{n,m}\frac{A_n^m}{\kappa^2\{\Pi_{n,m}(\theta)\}^2}\iint_S \frac{G_n^m\bar{p}}{D}dS = V\sum_{n,m}A_n^m\mathfrak{G}_n^m \tag{2·B}$$

where $\mathfrak{G}_n^m$ is an exterior harmonic corresponding to G_n^m, an interior one.

By the more general integral theorem [A–3] we have

$$\iint_S G_n^m \exp(kz-ikw)\,\bar{p}dS$$
$$=4\pi abc\kappa^2\{\prod_{n,m}(\theta)\}^2 \frac{2^n n!\,\Gamma(n+3/2)}{(2n+1)!\,(1/2D)^{n+\frac{1}{2}}}$$
$$\times I_{n+\frac{1}{2}}(D) H_n^m\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \exp(kz-ikw) \qquad (6)$$

where D^2 stands for an operator $a^2\partial^2/\partial x^2 + b^2\partial^2/\partial y^2 + c^2\partial^2/\partial z^2$. $I_{n+\frac{1}{2}}$ is a Bessel function of an imaginary argument, H_n^m is a homogeneous harmonic. After the operation x, y, z should be put to zero.

Making use of the above expression, we have from (4)

$$f(k,t)=\sum_{n,m} A_n^m \frac{2^{n+1}n!}{(2n+1)!}\frac{\Gamma(n+3/2)}{(1/2D)^{n+\frac{1}{2}}}$$
$$\times I_{n+\frac{1}{2}}(D) H_n^m\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \exp(kz-ikw) \qquad (4\cdot\mathrm{A})$$

Next, the expansion of ϕ_2 is given as

$$\phi_2 = -V \sum_{n,m} I_n^m G_n^m(x,y,z) \qquad (7)$$

where

$$I_n^m = \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^{\infty} f(k,t)h(k,k_0,t)\exp(-fk)\,C_n^m dkdt \qquad (8)$$

if $\exp(kz+ikw)$ can be expanded as

$$\exp(kz+ikw) = \sum_{n,m} C_n^m G_n^m(x,y,z) \qquad (9)$$

where

$$C_n^m = \iint_S G_n^m \exp(kz+ikw)\,\bar{p}dS \Big/ \iint_S \{G_n^m\}^2 \bar{p}dS \qquad (10)$$

Since [A–3]

$$\iint_S \{G_n^m\}^2 \bar{p}dS = 4\pi abc\kappa^2\{\prod_{n,m}(\theta)\}^2 \frac{2^n n!}{(2n+1)!}$$
$$\times H_n^m\left(a\frac{\partial}{\partial x}, b\frac{\partial}{\partial y}, c\frac{\partial}{\partial z}\right) H_n^m\left(\frac{x}{a}, \frac{y}{b}, \frac{z}{c}\right)$$
$$=4\pi abc\kappa^2\{\prod_{n,m}(\theta)\}^2 \frac{1}{(2n+1)\varepsilon_m}\frac{(n+m)!}{(n-m)!} \qquad (11)$$

where ε_m is 1 for $m=0$ and 2 for other cases, we have from (6)

$$C_n^m = \frac{2^n n!}{(2n+1)!}\varepsilon_m \frac{(n-m)!}{(n+m)!}\frac{\Gamma(n+3/2)}{(1/2D)^{n+\frac{1}{2}}} I_{n+\frac{1}{2}}(D) H_n^m\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \exp(kz+ikw) \tag{10·A}$$

When we take the x-axis as the axis of harmonics, we have [A–3]

$$H_n^m\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \frac{(-)^m 2^{m-1}(2n)!}{2^n n!(n-m)!}\left(\frac{\partial}{\partial x}\right)^{n-m}\left\{\left(\frac{\partial}{\partial \eta}\right)^m + \left(\frac{\partial}{\partial \zeta}\right)^m\right\}$$

where $\eta = z+iy$ and $\zeta = z-iy$, so explicitly after the operation

$$f(k, t) = \sum_{n,m} A_n^m \frac{2(-i)^{n+m}k^n}{(2n+1)(n-m)!} S_n^m(t)\psi_n(\vartheta) \tag{4·B}$$

$$C_n^m = \frac{\varepsilon_m i^{n+m}}{(n+m)!} k^n S_n^m(t)\psi_n(\vartheta) \tag{10·B}$$

where

$$2S_n^m(t) = \cos^{n-m} t\{(1+\sin t)^m + (1-\sin t)^m\}$$

and $\psi_n(\vartheta) = \Gamma(n+3/2) I_{n+\frac{1}{2}}(\vartheta)/(\vartheta/2)^{n+\frac{1}{2}}$, $\vartheta^2 = k^2(c^2 - a^2\cos^2 t - b^2 \sin^2 t)$

Further, $\psi_n(\theta)$ equals $\Gamma(n+3/2) I_{n+\frac{1}{2}}(-i\vartheta)/(-i\vartheta/2)^{n+\frac{1}{2}}$, tending to 1 as $\vartheta \to 0$. In the case of a submerged sphere ϑ becomes zero, we have $\psi_n = 1$.

1.2 Let us confine ourselves to the problem of a horizontal prolate spheroid, and represent the harmonics explicitly.

The integral expression of the interior harmonic is generally [A–5]

$$G_n^m(x, y, z) = \frac{2^n n!(n+m)!}{2\pi(2n)!} i^m \int_{-\pi}^{\pi} P_n\left(\frac{B}{A}\right) A^n \cos mu\,du \tag{12}$$

where

$$A^2 = a^2 - c^2\cos^2 u - b^2 \sin^2 u \quad \text{and} \quad B = x + iz\cos u - iy \sin u$$

Here we renew the notation for the prolate spheroid, of which $a > b = c$, and write $a^2 - b^2 = c^2$. Introduce also the spheroidal polar coordinates η, θ and φ, that is,

$$x = c\cosh\eta\cos\theta, \quad y = -c\sinh\eta\sin\theta\sin\varphi, \quad z = c\sinh\eta\sin\theta\cos\varphi$$

or write also $\cosh\eta = \nu$, $\cos\theta = \mu$

From this expression we have directly [A–3]

$$G_n^m(x, y, z) = \frac{2^n n!(n-m)!}{(2n)!} c^n P_n^m(\nu) P_n^m(\mu)\cos m\varphi \tag{12·A}$$

Next, for the exterior harmonic we have had an integral representation from (2·A) and (4·B), that is,

$$\mathfrak{G}_n^m(x, y, z) = \frac{2^{n+\frac{1}{2}}\Gamma(n+3/2)(-i)^{n+m}}{\pi(2n+1)(n+m)!\,c^{n+1}} \int_{-\pi}^{\pi} S_n^m(t)dt$$
$$\times \int_0^\infty \exp\left(-\frac{k}{c}+i\frac{k}{c}w\right) J_{n+\frac{1}{2}}(k\cos t)\frac{k^n dk}{(k\cos t)^{n+\frac{1}{2}}} \quad (13)$$

Since the integral with respect to k is of a LIPSCHITZ-HANKEL type [A-4] and it is expressed by the Legendre function of the second kind, we have

$$\mathfrak{G}_n^m(x, y, z) = \frac{(2n)!\,(-i)^m}{\pi 2^n n!\,(n-m)!\,c^{n+1}} \int_L Q_n\left(\frac{x+iz\cos u - iy\sin u}{c}\right)\cos mu du \quad (13\cdot\mathrm{A})$$

where L is the path from $\pi - i\infty$, $\pi + i\infty$, $i\infty$ to $-i\infty$.

This path can be deformed in the plane of the argument of Q_n without crossing its cut in the manner that u varies $-\pi$ to π.

Thus we have [A-3]

$$\mathfrak{G}_n^m(x, y, z) = \frac{(-)^m(2n)!}{2^{n-1}n!\,(n+m)!\,c^{n+1}} Q_n^m(\nu)P_n^m(\mu)\cos m\varphi \quad (13\cdot\mathrm{B})$$

These functions are reduced to homogeneous harmonics, when the spheroid becomes a sphere, that is,

$$G_n^m(x, y, z) = r^n P_n^m(\mu)\cos m\varphi \quad (12\cdot\mathrm{B})$$

$$\mathfrak{G}_n^m(x, y, z) = \frac{2}{2n+1}\frac{1}{r^{n+1}} P_n^m(\mu)\cos m\varphi \quad (13\cdot\mathrm{C})$$

where $x = r\cos\theta$, $y = -r\sin\theta\sin\varphi$ and $z = r\sin\theta\cos\varphi$

1.3 Now, our problem is to obtain the coefficients A_n^m.

In the case of $x = c\mu\nu = G_1^0(x, y, z)$, using the expansions (2·B) and (7), we have infinite equations to express the boundary condition that the normal velocity at the surface of the spheroid must vanish

$$A_1^0 = B_1^0(-1+I_1^0),$$
$$A_n^m = B_n^m I_n^m \quad (14)$$

where

$$B_n^m = \frac{(-)^m}{2}(n+m)!\,(n-m)!\left\{\frac{n!\,(2c)^n}{(2n)!}\right\}\left|\frac{dP_n^m}{d\nu}\Big/\frac{dQ_n^m}{d\nu}\right|_{\nu=\nu_0}, \quad \cosh\nu_0 = a/c \quad (15)$$

This tends to $-a^{2n+1}n(2n+1)/2(n+1)$ in the case of the sphere with a radius a.

Putting (4·B) into (8), we have

$$I_n^m = \sum_{\nu,\mu} A_\nu^\mu M\left(\begin{smallmatrix} m \\ n \end{smallmatrix}\middle|\begin{smallmatrix} \mu \\ \nu \end{smallmatrix}\right) \tag{8·A}$$

where

$$M\left(\begin{smallmatrix} m \\ n \end{smallmatrix}\middle|\begin{smallmatrix} \mu \\ \nu \end{smallmatrix}\right) = \frac{\varepsilon_m i^{n+m-\nu-\mu}}{(n+m)!(2\nu+1)(\nu-\mu)!} \frac{1}{\pi}\int_{-\pi}^{\pi} S_n^m S_\nu^\mu dt \int_0^\infty h(k, k_0, t) e^{-2fk} \psi_n \psi_\nu k^{n+\nu} dk \tag{16}$$

Let us introduce matrices

$A = \{A_1^0 A_1^1 A_2^0 A_2^1 \cdots\}$, $C = \{-B_1^0\ 0\ 0\ 0\ \cdots\}$. These two are column matrices, B is a diagonal matrix of the sequence $\{B_1^0 B_1^1 B_2^0 \cdots\}$, and $S = \{M\left(\begin{smallmatrix} m \\ n \end{smallmatrix}\middle|\begin{smallmatrix} \mu \\ \nu \end{smallmatrix}\right)\}$, aranged in columns by n, m and in rows by

ν, μ, according to the numerical order of n at first and next of m.

And we have a matrix equation in place of (14)

$$A = C + BSA, \quad \text{or} \quad (E - BS)A = C \tag{14·A}$$

where E is the unit matrix.

Consequently we have a solution

$$A = (E - BS)^{-1} C \tag{17}$$

Expanding this reciprocal matrix in Neumann series, we are able to have a successive approximate solution. In fact we have the so-called first approximate solution as the first term of this expansion, that is, $A_1^0 = -B_1^0$.

For our present purpose, it is sufficient to take as follows:

$$\left.\begin{aligned} A_1^0 &= -B_1^0 - (B_1^0)^2 M\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\middle|\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) - (B_1^0)^2 \{B_1^0 M^2\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\middle|\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) + B_1^1 M\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\middle|\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) M\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\middle|\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\} \\ A_1^1 &= -B_1^0 B_1^1 M\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\middle|\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) - B_1^0 B_1^1 \{B_1^0 M\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\middle|\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) M\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\middle|\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) + B_1^1 M\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\middle|\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) M\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\middle|\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\} \\ A_2^0 &= -B_1^0 B_2^0 M\left(\begin{smallmatrix} 0 \\ 2 \end{smallmatrix}\middle|\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right),\ A_2^1 = -B_1^0 B_2^1 M\left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\middle|\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right),\ A_2^2 = -B_1^0 B_2^2 M\left(\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\middle|\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \end{aligned}\right\} \tag{17·A}$$

§2 Numerical evaluation for a submerged sphere

2.1 We confine ourselves again to the problem of a sphere, and let us evaluate the necessary integral.

Introduce another integral, for the integral $M\left(\begin{smallmatrix} m \\ n \end{smallmatrix}\middle|\begin{smallmatrix} \mu \\ \nu \end{smallmatrix}\right)$ is inconvenient,

$$K(m, n) = \frac{i^m}{2\pi} \gamma^{n+1} \int_{-\pi}^{\pi} h(k, 1, t) \cos mt dt \int_0^\infty e^{-\gamma k} k^n dk \tag{18}$$

$$L(m, \mu; n) = \frac{1}{2} \{(-)^m K(m+\mu, n) + K(m-\mu, n)\} \tag{19}$$

where $\gamma = 2k_0 f$

$K(m, n)$ has the recurrence formula

$$-K(m+2,n)+2K(m,n)-K(m-2,n)$$
$$=4\gamma K(m,n-1)+4i^m\{\gamma\delta_m(n-1)!+\vartheta_m n!\} \qquad (20)$$

where δ_m is 1 for $m=0$ and 0 for $m\neq 0$, and ϑ_m is 1/2 for $m=0$, 1/4 for $m=\pm 2$ and 0 for other cases.

From this formula all $K(m,n)$ are reduced to $K(1,n)$ and $K(0,n)$.

At first integrating with respect to t by the contour integration, they are

$$K(0,n)=n!-2\gamma^{n+1}e^{-\gamma}\int_0^1 e^{\gamma u}(1-u)^n\frac{du}{\sqrt{u}} \qquad (18\cdot\mathrm{A})$$

$$K(1,n)=2\gamma^{n+\frac{1}{2}}e^{-\gamma}\int_0^\infty e^{-t}\left(1+\frac{t}{\gamma}\right)^{n+\frac{1}{2}}\frac{dt}{\sqrt{t}}=2\sqrt{\pi}\gamma^{n+\frac{1}{2}}e^{-\gamma/2}W_{n/2,\,n/2}(\gamma) \qquad (18\cdot\mathrm{B})$$

where $W_{n/2,\,n/2}$ is the Whittaker function [A–5]. This is the case which is represented by making use of the modified Bessel function K_n.

The second term of the former integral is given by polynomials of γ and the following function $E(\gamma)$ [A–6]·[A–7], when we expand $(1-u)^n$ and integrate by parts.

$$E(\gamma)=\frac{1}{2}e^{-\gamma}\int_0^1 e^{\gamma t}\frac{dt}{\sqrt{t}}=\frac{e^{-\gamma}}{\sqrt{\gamma}}\int_0^{\sqrt{\gamma}}e^{t}dt \qquad (21)$$

Thus we have

$$\left.\begin{aligned}
&K(0,1)=1+2\gamma-2\gamma(1+2\gamma)E(\gamma)\\
&K(0,2)=2+3\gamma+2\gamma^2-2\gamma\left(\frac{3}{2}+2\gamma+2\gamma^2\right)E(\gamma)\\
&K(0,3)=6+\frac{15}{2}\gamma+4\gamma^2+2\gamma^3-2\gamma\left(\frac{15}{4}+\frac{9}{2}\gamma+3\gamma^2+2\gamma^3\right)E(\gamma)
\end{aligned}\right\} \qquad (18\cdot\mathrm{C})$$

$$\left.\begin{aligned}
&L(1,1;2)=\gamma K(0,1)+1+\gamma,\ L(1,1;3)=\gamma K(0,2)+3+2\gamma\\
&L(1,1;4)=\gamma K(0,3)+12+6\gamma
\end{aligned}\right\} \qquad (19\cdot\mathrm{A})$$

$$\left.\begin{aligned}
&L(1,0;2)=K(1,2)=\gamma^3e^{-\gamma/2}\left\{K_0\left(\frac{\gamma}{2}\right)+\left(1+\frac{1}{\gamma}\right)K_1\left(\frac{\gamma}{2}\right)\right\}\\
&L(1,0;3)=K(1,3)=\gamma^4e^{-\gamma/2}\left\{\left(1+\frac{1}{2\gamma}\right)K_0\left(\frac{\gamma}{2}\right)+\left(1+\frac{1}{2\gamma}+\frac{2}{\gamma^2}\right)K_1\left(\frac{\gamma}{2}\right)\right\}
\end{aligned}\right\} \qquad (19\cdot\mathrm{B})$$

2.2 The forces acting on a body can be calculated directly by the pressure integration, but this method needs complicated operations. In order to avoid them we shall use the equivalent formula, that is,

$$
\left.\begin{aligned}
X&=-2\rho V^2 i\int_{-\pi}^{\pi}\int_0^{\infty} h(k,k_0,t)\,|f(k,t)|^2 e^{-2fk}k\cos t\,dk\,dt\\
Z&=-2\rho V^2\int_{-\pi}^{\pi}\int_0^{\infty} h(k,k_0,t)\,|f(k,t)|^2 e^{-2fk}k\,dk\,dt\\
M_y&=4\pi\rho V^2\iint_S \sigma z\,dS\\
&\quad -2\rho V^2 i\int_{-\pi}^{\pi}\int_0^{\infty} h(k,k_0,t)\left\{f(k,t)\frac{\partial}{\partial p}\overline{f(k,t)}\right\}e^{-2fk}k\,dk\,dt
\end{aligned}\right\}\qquad(22)
$$

where $p=k\cos t$, and also

$$
\iint_S \sigma z\,dS=\frac{1}{2}\left[\frac{\partial}{\partial p}\{f(k,t)+\overline{f(k,t)}\}\right]_{k=t=0}
$$

In the case of a sphere, since we have from (4.B) and (17.A)

$$
\begin{aligned}
|f(k,t)|^2=\frac{a^4}{4}\Big[&\{1-\beta^3 L(1,1;2)\}(ka)^2\cos t-\frac{4}{9}\beta^4 L(1,1;3)(ka)^3\cos^2 t\\
&+\frac{\beta^6}{4}\{L^2(1,0;2)(ka)^2+3L^2(1,1;2)(ka)^2\cos^2 t\\
&-2K^2(1,2)(ka)^2\cos^2 t\}\Big]
\end{aligned}\qquad(23)
$$

where $\beta=a/2f$, we have from the above formulæ making use of K and L function

$$
\begin{aligned}
C_w=-\frac{X}{\pi\rho a^2V^2}=C_{w_0}\Big[&\{1-\beta^3L(1,1;2)\}-\frac{4}{9}\beta^5L(1,1;3)K(1,3)/K(1,2)\\
&+\frac{\beta^6}{4}\left\{3L^2(1,1;2)-2K^2(1,2)+\frac{1}{\gamma}K(1,2)K(1,3)\right\}\Big]
\end{aligned}\qquad(24)
$$

$$
\begin{aligned}
C_z=\frac{Z}{\pi\rho a^2V^2}=C_{z_0}\Big[&\{1-\beta^3L(1,1;2)\}-\frac{4}{9}\beta^5L(1,1;4)\\
&+\frac{\beta^6}{4}\{3L^2(1,1;2)-2K^2(1,2)+K^2(1,2)K(0,3)/L(1,1;3)\}\Big]
\end{aligned}\qquad(25)
$$

where C_{w_0} and C_{z_0} are the first approximate values and given as [A-8]

$$
C_{w_0}=\beta^4\gamma K(1,2)\,,\qquad C_{z_0}=\beta^4L(1,1;3)\qquad(26)
$$

Besides, the moment is clearly zero in our case, but the first term is not zero, so the second term is not zero either. The second term must be equal to the first term with the opposite sign. In fact it will be found to be so by the same calculation as the above.

The numerical values are given in Tables 1, 2 and Figs. 1, 2. As we may expect, general features are very analogous with those of the

TABLE 1. WAVE RESISTANCE OF A SPHERE

$\gamma=\frac{2}{F^2}$	$F=\frac{V}{\sqrt{gf}}$	C_{w_0}/β^4	$\beta=a/2f=1/4$					$\beta=1/3$				
			C_w/β^4	$r\times100$	$r_1\times100$	$r_2\times100$	$r_3\times100$	C_w/β^4	$r\times100$	$r_1\times100$	$r_2\times100$	$r_3\times100$
18	0.3333	1.393×10^{-3}	1.455×10^{-3}	4.50	2.05	2.42	0.032	1.608×10^{-3}	15.23	4.85	10.20	0.177
12	0.4082	0.1387	0.1460	5.07	2.37	2.65	0.042	0.1624	17.05	5.62	11.19	0.238
8	0.5000	1.891	2.008	6.18	3.28	2.82	0.081	2.275	20.11	7.77	11.89	0.451
7	0.5345	3.265	3.48	6.38	3.63	2.65	0.097	3.93	20.35	8.61	11.19	0.545
6	0.5774	5.264	5.60	6.43	3.97	2.33	0.134	6.30	19.86	9.40	9.82	0.637
5	0.6326	7.748	8.21	5.98	4.12	1.74	0.113	9.16	18.27	9.78	7.85	0.637
4	0.7071	10.007	10.47	4.74	3.73	0.94	0.068	11.32	13.18	8.85	3.95	0.383
3	0.8165	10.544	10.75	1.93	1.91	0.06	−0.042	11.01	4.54	4.52	0.26	−0.235
2.4	0.9129	9.332	9.37	0.45	0.84	−0.31	−0.075	9.35	0.28	2.02	−1.32	−0.421
2	1.0000	7.794	7.23	−0.86	−0.32	−0.46	−0.074	7.55	−3.13	−0.76	−1.95	−0.416
1.6	1.1180	5.788	5.67	−2.04	−1.49	−0.51	−0.040	5.45	−5.90	−3.52	−2.16	−0.222
1.2	1.2910	3.602	3.40	−2.93	−2.46	−0.48	0.018	3.23	−7.79	−5.85	−2.04	0.103
1	1.4142	2.570	2.49	−3.17	−2.77	−0.45	0.045	2.36	−8.19	−6.56	−1.89	0.252
0.8	1.5811	1.655	1.600	−3.28	−2.94	−0.40	0.065	1.518	−8.29	−6.96	−1.70	0.367
0.6	1.8257	0.9140	0.855	−3.16	−2.88	−0.36	0.071	0.842	−7.92	−6.81	−1.51	0.399
0.4	2.2361	0.3872	0.376	−2.78	−2.52	−0.32	0.063	0.359	−7.20	−6.21	−1.34	0.354
0.2	3.1623	0.0891	0.0870	−2.40	−2.16	−0.29	0.043	0.0838	−6.09	−5.12	−1.21	0.242

Notice: $C_{w_0}=\beta^4K(1,2)=-X/\pi\rho a^2V^2$ $\quad C_w=C_{w_0}(1+r)$, $\quad r=r_1+r_2+r_3$

TABLE 2. VERTICAL FORCE ACTING ON A SPHERE

$\gamma=\frac{2}{F^2}$	$F=\frac{V}{\sqrt{gf}}$	C_{z_0}/β^4	$\beta=a/2f=1/4$					$\beta=1/3$				
			C_z/β^4	$r\times100$	$r_1\times100$	$r_2\times100$	$r_3\times100$	C_z/β^4	$r\times100$	$r_1\times100$	$r_2\times100$	$r_3\times100$
18	0.3333	4.34	4.46	2.93	2.05	0.87	0.011	4.71	8.56	4.85	3.67	0.059
12	0.4082	5.08	5.26	3.44	2.37	1.05	0.014	5.60	10.14	5.62	4.44	0.079
8	0.5000	8.02	8.41	4.96	3.28	1.65	0.027	9.21	14.88	7.77	6.96	0.150
7	0.5345	8.60	9.06	5.30	3.63	1.64	0.032	9.95	15.70	8.61	6.91	0.181
6	0.5774	8.75	9.23	5.45	3.97	1.45	0.037	10.10	15.71	9.40	6.10	0.210
5	0.6326	7.80	8.20	5.13	4.12	0.98	0.032	8.91	14.10	9.78	4.14	0.180
4	0.7071	5.13	5.34	4.01	3.73	0.24	0.777	5.65	10.08	8.85	1.03	0.202
3	0.8165	0.44	0.45	2.08	1.91	−0.61	−0.240	0.47	6.34	4.52	−2.55	4.370
2.4	0.9129	−2.64	−2.63	−0.38	0.84	−0.97	−0.162	−2.55	−3.43	2.02	−4.11	−1.342
2	1.0000	−4.47	−4.40	−1.60	−0.32	−1.11	−0.112	−4.19	−6.34	−0.76	−4.70	−0.873
1.6	1.1180	−5.71	−5.50	−2.75	−1.49	−1.15	−0.050	−5.20	−8.99	−0.35	−4.84	−0.629
1.2	1.2910	−6.18	−5.96	−3.58	−2.46	−1.07	−3.58	−5.52	−10.68	−0.59	−4.53	−0.299
1	1.4142	−6.08	−5.85	−3.80	−2.77	−1.01	−0.021	−5.42	−10.93	−6.56	−4.25	−0.116
0.8	1.5811	−5.75	−5.53	−3.87	−2.94	−0.92	−0.005	−5.12	−10.88	−6.96	−3.89	−0.028
0.6	1.8257	−5.23	−5.04	−3.70	−2.88	−0.83	0.008	−4.70	−10.26	−6.81	−3.50	0.047
0.4	2.2361	−4.56	−4.41	−3.23	−2.52	−0.73	0.013	−4.13	−9.21	−6.21	−3.07	0.074
0.2	3.1623	−3.80	−3.70	−2.77	−2.16	−0.62	0.011	−3.51	−7.69	−5.12	−2.63	0.0625

Notice: $C_{z_0}=-\beta^4L(1,1;3)=Z/\pi\rho a^2V^2$ $\quad C_z=C_{z_0}(1+r)$, $\quad r=r_1+r_2+r_3$

circular cylinder. The known first approximate values are much closer quantitatively to the accurate ones than in the case of the cylinder.

With respect to the vertical force, the ratio of the second approximate value to that of the first one tends to be constant at the both limits of the speed, i.e., at zero and infinite. Meanwhile, the ratio between the first and the second approximations in wave-making resistance reaches the maximum just before the hump ($V/\sqrt{gf}=0.55$), the mininum at about $V/\sqrt{gf}=0.3$ and increases as the speed decreases. This relation is given by the approximate expansion

$$\frac{C_w}{C_{w_0}}\approx 1+\beta^3\left(1+\frac{9}{4}F^2\right)+\frac{8\beta^5}{3F^2}\left(1+\frac{13}{4}F^2\right)+\frac{3}{4}\beta^6\left(1+\frac{9}{2}F^2\right) \qquad (27)$$

for the sufficiently small $F=V/\sqrt{gf}$, Froude number.

Finally, it is interesting to see the source distribution around the sphere in this stage of approximation. Fig. 3 gives a glimpe of this relation.

§3 Numerical evaluation for a submerged spheroid and its comparison with model basin experiments

The present author made some numerical evaluations for a submerged spheroid and conducted resistance experiment with a spheroid so that the result of the evaluation could be compared with the experimental result.

The calculation in the second approximation was found to be far more difficult than that in the first approximation. The tedious work involved seemed worthless at the thought of the errors that might possibly be made during the long process of calculation and the dubious value of the conclusion drawn from the calculation. However, as the integral procedures progressed, it was found out that all that the author intended to express could be expressed by the generalized forms of P_n and Q_n, the functions employed by HAVELOCK and others. It is expected that the closer examination of these functions would contribute to the development of the theory of wave-making resistance also in other respects than that taken up in this discussion of the author.

It is usual with the calculation of wave-making resistance that it is carried out over the very wide scope of variables. In the calculation in this study, however, the speed range is so limited that the result of the calculation only shows a general tendency. This is for the reason that the amount of the calculations involved in the author's case is several times as much as that in an ordinary case.

3.1 The notation and the equations employed in the preceding section will also be used in this section.

The wave-making resistance R_w, vertical lift Z and the nose-down moment M_y, all working on a submerged spheroid are given in the second approximation by the following expressions:

$$C_w=\frac{R_w}{4\pi\rho V^2c^2}=\frac{4\alpha}{9c^6}(B_1^0)^2\left\{1+2B_1^0M\binom{0|0}{1|1}\right\}{}_2N_1^{(1,1)}-\frac{8\alpha}{15c^6}(B_1^0)^2B_2^1M\binom{1|0}{2|1}{}_3N_1^{(1,2)} \tag{28}$$

$$C_z=\frac{Z}{4\pi\rho V^2c^2}=\frac{4}{9c^6}(B_1^0)^2\left\{1+2B_1^0M\binom{0|0}{1|1}\right\}{}_3N_2^{(1,1)}-\frac{8}{15c^6}(B_1^0)^2B_2^1M\binom{1|0}{2|1}{}_4N_2^{(1,2)} \tag{29}$$

$$C_m=\frac{M_y}{4\pi\rho V^2c^2}=\frac{2}{3c^3}B_1^0(B_1^1-B_1^0)M\binom{1|0}{1|1}\left\{1+2B_1^0M\binom{0|0}{1|1}+B_1^1M\binom{1|1}{1|1}\right\}$$
$$-\frac{4}{45c^6}(B_1^0)^2{}_4N_3^{(1,2)}\left\{1+2B_1^0M\binom{0|0}{1|1}\right\} \tag{30}$$

where

$$\alpha=\kappa_0c=gc/V^2,\qquad \gamma=2\kappa_0f,\qquad c=\sqrt{a^2-b^2}$$

V denotes the velocity of a uniform stream, $2a$ the length of the major axis, $2b$ the length, i.e., the diameter of the minor axis, and f the distance from the spheroid's centre to the water surface.

Also,

$$\begin{aligned}{}_nN_m^{(\nu,\mu)}&=\frac{i^mc^{n+1}}{2\pi}\int_{-\pi}^{\pi}dt\int_0^{\infty}h(\kappa,\kappa_0,t)e^{-2\kappa f}\psi_\nu\psi_\mu\kappa^n\cos^m t\,d\kappa\\ h(\kappa,\kappa_0,t)&=\frac{\kappa\cos^2 t+\kappa_0+\mu i\cos t}{\kappa\cos^2 t-\kappa_0+\mu i\cos t}\\ \psi_\nu&=\frac{\Gamma(\nu+3/2)J_{\nu+\frac{1}{2}}(\kappa c\cot t)}{(\kappa c\cos t/2)^{\nu+\frac{1}{2}}}\end{aligned} \tag{31}$$

Further, to write M by N we have

$$\left.\begin{aligned}&M\binom{0|0}{1|1}=-\frac{2}{3c^3}{}_2N_2^{(1,1)}, &&M\binom{0|1}{1|1}=-M\binom{1|0}{1|1}=-\frac{2}{3c^3}{}_2N_2^{(1,1)}\\ &M\binom{1|0}{2|1}=\frac{2}{9c^5}{}_3N_2^{(1,1)}, &&M\binom{1|1}{1|1}=\frac{2}{3c^3}{}_2N_0^{(1,1)}\end{aligned}\right\} \tag{31'}$$

$$B_1^0=\frac{c^3}{2}\left(\frac{dP_1}{d\nu}\Big/\frac{dQ_1}{d\nu}\right),\quad B_1^1=2c^3\left(\frac{dP_1^1}{d\nu}\Big/\frac{dQ_1^1}{d\nu}\right),\quad B_2^1=2c^5\left(\frac{dP_2^1}{d\nu}\Big/\frac{dQ_2^1}{d\nu}\right) \tag{32}$$

All of the above equations are general expressions, the first approximation for each of which is given by HAVELOCK.

As regard to the resistance and lift, his results correspond to (28) and (29) in which the second term and the second factor in the braces of the first term are neglected.

As for the trimming moment, his result corresponds to (30) in which the factors except unity in the braces of the both terms are neglected [A-9]·[A-10].

If we induce the following signs for calculation

$$\left.\begin{aligned} I^{(2)}(\alpha)&=\frac{1}{2\alpha^6}\left[\alpha^2\int(d\alpha)^3-4\alpha\int(d\alpha)^4+4\int(d\alpha)^5\right] \\ I^{(2)}(\alpha)&=\frac{1}{2\alpha^8}\left[\alpha^3\int(d\alpha)^4-8\alpha^2\int(d\alpha)^5+24\alpha\int(d\alpha)^6-24\int(d\alpha)^7\right] \end{aligned}\right\} \quad (33)$$

we may write like

$$\left.\begin{aligned} \psi_1^2(\alpha\kappa\cos\theta)&=72I^{(1)}_{(\alpha)}\frac{\sin(2\alpha\kappa\cos\theta)}{\kappa\cos\theta} \\ \psi_1(\alpha\kappa\cos\theta)\,\psi_2(\alpha\kappa\cos\theta)&=720I^{(1)}_{(\alpha)}\frac{\sin(2\alpha\kappa\cos\theta)}{\kappa\cos\theta} \end{aligned}\right\} \quad (34)$$

If we define the following functions, the necessary integral can be expressed by calculating them by (31) or (32) as required.

$$\left.\begin{aligned} O_n^{(1)}(x,t)&=\lim_{\mu\to+0}\frac{(-i)^n}{4\pi}\int_{-\pi}^{\pi}du\int_0^{\infty}\frac{\cos^{n+2}u\exp(-\kappa t+i\kappa x\cos u)}{\kappa\cos^2 u-1+\mu i\cos u}dx \\ O_n^{(2)}(x,t)&=\lim_{\mu\to+0}\frac{i^n}{4\pi}\int_{-\pi}^{\pi}du\int_0^{\infty}\frac{\cos^{n+2}u\exp(-\kappa t-i\kappa x\cos u)}{\kappa\cos^2 u-1-\mu i\cos u}dx \end{aligned}\right\} \quad (35)$$

$$x,t>0 \qquad n:\ \text{integer}$$

Then from the above we define the functions P and Q so as to obtain

$$\left.\begin{aligned} P_n(x,t)&=\frac{1}{2}\{O_n^{(1)}(x,t)-O_n^{(2)}(x,t)\} \\ Q_n(x,t)&=\frac{1}{2}\{O_n^{(1)}(x,t)+O_n^{(2)}(x,t)\} \end{aligned}\right\} \quad (36)$$

Besides, we write as auxiliary functions

$$\begin{matrix} q_{2n} \\ q_{2n+1} \end{matrix}(x,t)=\frac{(-)^n}{4\pi}\int_{-\pi}^{\pi}du\int_0^{\infty}e^{-\kappa t}\begin{matrix}\cos^{2n+2}u \\ \cos^{2n+3}u\end{matrix}\begin{matrix}\cos \\ \sin\end{matrix}(\kappa x\cos u)dx \quad (37)$$

then we find by carrying out the differential the relation

$$\left.\begin{aligned} &\frac{\partial}{\partial x}P_n=P_{n-1}, \quad \frac{\partial}{\partial t}P_n=P_{n-2}, \quad \frac{\partial^2}{\partial x^2}P_n=\frac{\partial}{\partial t}P_n \\ &\frac{\partial}{\partial x}Q_n=Q_{n-1}+q_{n-1}, \quad \frac{\partial}{\partial t}Q_n=Q_{n-1}+q_{n-2}, \quad \frac{\partial^2}{\partial x^2}Q_n=\frac{\partial}{\partial t}Q_n+\frac{\partial}{\partial x}q_{n-1} \end{aligned}\right\} \quad (38)$$

At first we write

$${}_2N_1^{(1,1)}=288\alpha^3I^{(1)}(\alpha)[P_{-4}(2\alpha,\gamma)]=\frac{18}{\alpha}\left[P_{-1}+P^0_{-1}-\frac{2}{\alpha}P_0+\frac{1}{\alpha^2}(P_1-P_1^0)\right] \quad (39)$$

$$\begin{aligned} {}_4N_3^{(1,2)}&=-\alpha_3N_1^{(1,2)}=2880\alpha^5I^{(2)}(\alpha)[P_{-6}(2\alpha,\gamma)] \\ &=90\left[P_{-2}-\frac{4}{\alpha}\left(P_{-1}+\frac{1}{2}P^0_{-1}\right)+\frac{6}{\alpha^2}P_0-\frac{3}{\alpha^3}(P_1-P_1^0)\right] \end{aligned} \quad (40)$$

In the above equations the variables of P means 2α and γ, and the suffix 0 means $\alpha=0$.

Next, if we put

$$ {}_nG_m^{(\nu,\mu)}=\frac{\alpha^{n+1}}{2\pi}\int_{-\pi}^{\pi}du\int_0^{\infty}e^{-\tau\kappa}\psi_\nu(\alpha\kappa\cos t)\psi_\mu(\alpha\kappa\cos t)\kappa^n\cos^m t\,d\kappa\,dt \tag{41} $$

the remaining integral may be expressed like

$$ \begin{aligned} {}_2N_0^{(1,1)}&=-144\alpha^3I^{(1)}(\alpha)\left[\frac{\partial}{\partial\alpha}\left\{Q_{-4}(2\alpha,\gamma)+\frac{1}{2}q_{-4}(2\alpha,\gamma)\right\}\right]\\ &=-{}_2G_0^{(1,1)}-\frac{2}{\alpha}{}_3G_2^{(1,1)}-\frac{36}{\alpha^2}\left[2\int q_{-2}d\alpha-\frac{2}{\alpha}\iint q_{-2}d\alpha^2-\frac{1}{\alpha}\int q_{-1}d\alpha\right]\\ &\quad-\frac{18}{\alpha}\left[Q_{-2}+Q_{-2}^0-\frac{2}{\alpha}Q_{-1}+\frac{1}{\alpha^2}(Q_0-Q_0^0)\right] \end{aligned} \tag{42} $$

In the above, Q_n^0 means $Q_n(0,\gamma)$.

$$ \begin{aligned} {}_2N_2^{(1,1)}&={}_2G_2^{(1,1)}+\frac{2}{\alpha}{}_3G_4^{(1,2)}-\frac{72}{\alpha^2}\left(\int q_0d\alpha-\frac{1}{\alpha}\iint q_0(d\alpha)^2\right)+\frac{36}{\alpha^3}\int q_1d\alpha\\ &\quad-\frac{18}{\alpha}\left[Q_0+Q_0^0-\frac{2}{\alpha}Q_1+\frac{1}{\alpha^2}(Q_2-Q_2^0)\right] \end{aligned} \tag{43} $$

$$ \begin{aligned} {}_3N_2^{(1,1)}&={}_3G_2^{(1,2)}+\frac{2}{\alpha}{}_4G_4^{(1,2)}+\frac{720}{\alpha^2}\left\{\int q_{-1}d\alpha-\frac{3}{\alpha}\iint q_{-1}(d\alpha)^2+\frac{3}{\alpha^2}\iiint q_{-2}(d\alpha)^3\right\}\\ &\quad-\frac{1080}{\alpha^3}\left\{\int q_0d\alpha-\frac{1}{\alpha}\iint q_0(d\alpha)^2\right\}+\frac{540}{\alpha^4}\int q_1d\alpha\\ &\quad+\frac{90}{\alpha}\left[Q_{-1}-\frac{4}{\alpha}\left(Q_0+\frac{1}{2}Q_0^0\right)+\frac{6}{\alpha^2}Q_1-\frac{3}{\alpha^3}(Q_2-Q_2^0)\right] \end{aligned} \tag{44} $$

$$ \begin{aligned} {}_4N_2^{(1,2)}&={}_4G_2^{(1,2)}-\frac{720}{\alpha}\left\{\int q_{-3}d\alpha-\frac{3}{\alpha}\iint q_{-3}(d\alpha)^2+\frac{3}{\alpha^2}\iiint q_{-3}(d\alpha)^3\right\}\\ &\quad+\frac{1080}{\alpha^2}\left\{q_{-2}d\alpha-\frac{1}{\alpha}\iint q_{-2}(d\alpha)^2\right\}-\frac{540}{\alpha^3}\int q_{-1}d\alpha\\ &\quad-90\left[Q_{-3}-\frac{4}{\alpha}\left(Q_{-2}+\frac{1}{2}Q_2^0\right)+\frac{6}{\alpha^2}Q_{-1}-\frac{3}{\alpha^3}(Q_0-Q_0^0)\right] \end{aligned} \tag{45} $$

Now the problem has been reduced to P and Q. These P and Q are what are generalized of those which HAVELOCK [A–11]·[A–12]·[A–13] defined when he calculated the wave-making resistance and wave profiles of a ship with infinite draught. It must be noted, however, that the function Q meant by the author is quite different from that defined by HAVELOCK. O_0 defined by the author nearly equals Q_0 defined by HAVELOCK.

We cannot dwell upon the characteristics of these functions. Their important characteristics are as follows:

$$nP_n+(n-1)P_{n-2}=2t(P_{n-2}+P_{n-4})+x(P_{n-1}+P_{n-3}) \tag{46}$$

where all variables are (x, t),

$$nQ_n+(n-1)Q_{n-2}=2t(Q_{n-2}+Q_{n-4})+x(Q_{n-1}+Q_{n-3}) \\ +2t(q_{n-2}+q_{n-4})+x(q_{n-1}+q_{n-3}) \tag{47}$$

An equation in a form similar to that expressing Q_n may be obtained also with respect to O_n.

Also

$$\left.\begin{aligned} q_{2n+1}&=-\frac{x}{t}q_{2n+2}, \qquad \frac{\partial}{\partial x}q_n=\frac{\partial}{\partial t}q_{n+1} \\ q_{2n+2}&=-\frac{x}{t}q_{2n+1}+\frac{(-)^n}{2t\sqrt{\pi}}\frac{\Gamma(n+3/2)}{\Gamma(n+2)} \end{aligned}\right\} \tag{48}$$

For integral expressions we may write

$$P_{\nu-1}(x, t)=R(-i)^\nu\int_1^\infty \frac{\exp(-t\tau^2+ix\tau)}{\sqrt{\tau^2-1}}\tau^{-\nu}d\tau \tag{49}$$

$$O_2^{(1)}(x, t)=-\frac{1}{2}\int_{L_1+L_2}\exp(tu^2-xu)\left(\frac{1}{\sqrt{1+u^2}}-1+\frac{u^2}{2}\right)\frac{du}{u^3} \tag{50}$$

Both L_1 and L_2 are integral paths. The former is the one from the origin to a point in infinite distance via the imaginary negative axis, and the latter is the other from the point x/t and to a point in infinite distance above the positive and negative axes. For O_n, O_2 is given here to represent it because there is no uniform expression for O_n. For other functions we may give, for example,

$$O_{-1}^{(1)}=\frac{1}{2}\int_{L_1+L_2}\exp(tu^2-xu)\frac{du}{\sqrt{1+u^2}} \tag{50$'$}$$

Various expressions may be considered for expansions and asymptotic expansions. We shall give here only the asymptotic expansions which are actually employed in our calculation.

$$P_{\nu-1}(x, t)\approx R(-i)^\nu\frac{\sqrt{\pi}\,e^{-t+ix}}{2\sqrt{t-ix/2}}\left[1+\frac{C_1}{2t}+\frac{3C_2}{4t^2}+\frac{15C_3}{8t^3}+\cdots\right] \tag{51}$$

$$\left.\begin{aligned} C_1&=-\frac{1}{2C_0}\left(\frac{1}{4}+\nu+\frac{3}{4C_0}\right), \qquad C_0=1-\frac{ix}{2t} \\ C_2&=\frac{1}{(2C_0)^2}\left\{\frac{3}{32}+\frac{\nu}{4}+\frac{\nu(\nu+1)}{2}+\frac{5}{4C_0}\left(\frac{1}{4}+\nu\right)+\frac{35}{32C_0^2}\right\} \\ C_3&=-\frac{1}{(2C_0)^3}\left[\frac{5}{128}+\frac{3\nu}{32}+\frac{\nu(\nu+1)}{8}+\frac{\nu(\nu+1)(\nu+2)}{6}\right. \\ &\quad\left.+\frac{7}{4C_0}\left\{\frac{3}{32}+\frac{\nu}{4}+\frac{(\nu+1)\nu}{2}\right\}+\frac{63}{32C_0^2}\left(\frac{1}{4}+\nu\right)+\frac{231}{128C_0^3}\right] \end{aligned}\right\} \tag{51$'$}$$

The function Q is calculated by putting $(O-P)$, for O is a monotonous function having the following asymptotic expansion at a point away from the origin. The expression given is just typical one, for there is no uniform expression for O either.

$$O_2^{(1)}(x,t) \approx -\frac{1}{2} \sum_{n=0}^{\infty} (A_n^{(2)}+B_n^{(2)})/t^{n+1} \tag{52}$$

$$\left.\begin{aligned} A_n^{(2)} &= \left(\frac{t}{x}\right)^{2n} \sum_{\mu=1}^{n/2\geq} (-)^{\mu} \frac{(1/2)_{\mu+2}(n+1)_{n-2\mu-1}}{(\mu+2)!(n-2\mu-1)!} \left(\frac{x}{t}\right)^{2\mu} \\ B_n^{(2)} &= \frac{(n+4)_n}{n!}\left(\frac{t}{x}\right)^{2n+4} - \frac{(n+2)_n}{2\,n!}\left(\frac{t}{x}\right)^{2n+2} \\ &\quad - \frac{(t/x)^{2n+4}}{\sqrt{1+x^2/t^2}} \sum_{\mu=0}^{n} \frac{(n+4)_{n-\mu}}{(n-\mu)!} p^{\mu} P_{\mu}(p) \end{aligned}\right\} \tag{52'}$$

where

$$p = \frac{1}{\sqrt{1+(t^2/x^2)}}$$

P_{μ} is Legendre function of μ-th order.

3.2 In the numerical calculations, at first P_n, O_n, q_n etc., were calculated by employing the above-mentioned expansion and then what was obtained was put into the equations as required.

The two cases of f/c were chosen for the calculation for the ellipsoid like $a/c=\sqrt{17/4}$. They were

$$f = \frac{c}{2}, \qquad f = \frac{3}{8}c$$

When f is sufficiently large, or α is small the required value may be obtained by calculating series directly from the series expansion.

However, in the author's calculation an asymptotic expansion was employed. The results are shown in Table 3, and Figs. 4 and 5. We see the general tendency that each curve is steeper than that in the first approximation.

The wave-making resistance was found to increase remarkably between $F=0.35$ and 0.40. The increment was so large that the author checked the calculation but no mistake has been found. The accuracy of the asymptotic expansion is low in a range beyond $F=0.4$. An error in each function is in a small percentage. Consequently there may be errors between 10 and 20 per cent in the final value obtained.

In any case nothing can be said for certainty concerning the result of the calculation before similar calculation is made for many other cases. The tendencies shown by the results of author's calculations may be those

TABLE 3. WAVE RESISTANCE, VERTICAL FORCE AND TRIMMING MOMENT ACTING ON A SUBMERGED SPHEROID

$\gamma=(3/4)\alpha$ $(f/c=3/8)$	$F=1/\sqrt{2\alpha}$	$C_{w_0}\times10^4$	$C_w\times10^4$	$C_{z_0}\times10^3$	$C_z\times10^3$	$C_{m_0}\times10^4$	$C_m\times10^4$
2.0	.4330	15.8	38.8	2.11	2.10	20.6	18.7
2.4	.3953	6.98	18.2	2.16	2.81	10.7	10.0
2.8	.3660	1.725	4.69	1.82	2.43	1.75	1.71
3.2	.3423	.295	.287	1.46	1.74	.067	.073
3.6	.3228	.670	.274	1.25	1.42	−.470	−.469
4.0	.3062	1.057	.965	1.20	1.25	.0419	.0261
4.4	.2919	.893	1.023	1.18	1.24	.442	.458
4.8	.2795	.458	.593	1.15	1.20	.427	.436
5.2	.2685	.130	.1912	1.09	1.15	.199	.202
5.6	.2588	.0247	.0279	1.04	1.09	.0133	.0138
6.0	.2500	.0468	.0361	1.004	1.04	−.0406	−.0406
6.4	.2421	.0739	.0704	.979	1.014	−.0132	−.0126
6.8	.2348	.0645	.0684	.962	.996	.0201	.0211
7.2	.2282	.0334	.0381	.945	.977	.0261	.0270
7.6	.2221	.00946	.0115	.928	.959	.0138	.0142
$\gamma=\alpha$ $(f/c=1/2)$	$F=1/\sqrt{2\alpha}$	$C_{w_0}\times10^4$	$C_w\times10^4$	$C_{z_0}\times10^3$	$C_z\times10^3$	$C_{m_0}\times10^4$	$C_m\times10^4$
2.4	.4564	10.1	21.3	1.04	1.08	10.9	9.51
2.8	.4226	5.87	13.4	1.22	1.67	7.65	7.02
3.2	.3953	2.91	7.165	1.18	1.56	4.46	4.20
3.6	.3727	1.05	2.59	1.021	1.42	1.92	1.84
4.0	.3536	.193	.537	.873	1.15	.487	.472
4.4	.3371	.0576	.0022	.764	.943	−.0699	−.0672
4.8	.3228	.1525	.0527	.701	.817	−.0146	−.141
5.2	.3101	.232	.186	.667	.748	−.0469	−.0436
5.6	.2988	.226	.230	.646	.709	.0520	.0539
6.0	.2887	.160	.184	.627	.680	.0910	.1026
6.4	.2795	.0852	.1065	.609	.654	.0786	.0793
6.8	.2712	.0321	.0438	.590	.630	.0450	.0453
7.2	.2635	.0073	.0110	.570	.604	.01438	.0145

which are found only with a submerged ellipsoid. However, if they are general tendencies to be seen with most submerged bodies, the theoretical values of the wave-making resistance, the vertical force and the moment exerted on them, obtained so far by the first approximation, need to be re-examined from the view point of the boundary condition on the surface of each body concerned.

A term which gives a large correction value is the last one in Eq. (28). The term is there due to the fact that the source distribution is asymmetrical upwards and downwards.

3.3 In order to make comparisons with the above results, towing experiment was conducted in a model basin at the University of Tokyo using a wooden ellipsoid. The particular items of the ellipsoid are:

Major length ($2a$)	: 2.062 m
Diameter (minor length) ($2b$)	: 500 mm, $\sqrt{a^2-b^2}=c$: 1 m
Displacement (V)	: 2.699 m^3
Wetted surface area (S)	: 2.543 m^2

Turbulances were stimulated by pins having the front area of 1 mm^2, each of which was planted 5 mm apart along the location 5 per cent of the over all length of the body from the forward end. It seems better to have had more pins. A sword 118 mm long, 18 mm wide and with an arc section was fixed on the ellipsoid. The sword was suspended by a frame and resistance was measured by means of a magneto-striction dynamometer.

The experiment was conducted at three different depths of immersion. They were:

$$f: \ 1.20\text{ m},\ 1/2\text{ m},\ 3/8\text{ m}.$$

The values shown plotted in Fig. 6 are those obtained by reducing the resistance of the sword which was measured separately. The resistance of the pins planted may be ignored. The effect of the finite width of the model basin is about 3 to 3.5 per cent with respect to friction resistance, and approximately 3 per cent with respect to wave-making resistance. In general the experiment was difficult, the resistance value was large and unsteady. At $f=1.2$ m laminar flows seemed to remain at low speed. No laminar flow seemed to remain at $f=3/8$ m. It seems that the form factor K from the formula of HUGHES may be chosen in the neighbourhood of 0.43 to 0.45 if an average line is assumed to be drawn in the neighbourhood of Reynolds numbers from 2.8 to 3.4×10^6. If the curve of the wave-making resistance calculated by the first approximation is added to the line of $K=0.45$, the calculated value seems to be a little smaller.

At low speed the points at $f=0.5$ m are found generally higher than those at $f=1.2$ m. But it seems possible for us to say that when the Froude number is below 0.24 the curve at $K=0.45$ can be regarded as average. If then the value of the wave-making resistance obtained by the second approximation is added to the above curve we obtain the line on the right side excelled high above in Fig. 6. We see from this that in experiment the last hollow seems rather nonexistent. The curve of the experimental value is close to that of the second approximate value, but the slope on the former is more gradual than that on the latter.

If the first approximate value is taken, as can be seen from Fig. 4, at $f=3/8$ m from $F=0.2$ to 0.24 the experimental points are found between $K=0.50$ and 0.55. Since no wave-making resistance is expected to be found there, we must consider that friction resistance and the form factor, too, are affected by the water surface when the immersion is very shallow.

A similar tendency is found if the curve of wave-making resistance is added to the curve of $K=0.55$. All those tendencies show clearly a great difference between the calculated and measured values. So far no explanation has been found for it.

There have been published few experimental results with regard to submerged bodies. The author waits for the publication of the data of the similar experiment to be made in the future by other researchers.

3.4 The following conclusions may be drawn from the above discussions.

(1) The second approximate value of the wave-making resistance theoretically calculated differs greatly when compared to the first approximate one. (See Fig. 4)

(2) Both the vertical force and trimming moment show more exergerated humps and hollows in the second approximation than in the first. But this tendency is not so big as that found with respect to wave-making resistance. (See Fig. 5)

(3) According to the experimental result, in the speed range in which wave-making resistance is hardly found and which is not a transitionary range, the resistance value increases as follows when compared to the friction resistance value of HUGHES:

at $f=1.2$ m : $K=0.40\sim0.45$

at $f=0.5$ m : $K=0.45\sim0.50$

at $f=3/8$ m : $K=0.50\sim0.55$

(4) The wave-making resistance value measured in the experiment may be said to have no last hollow. The value is generally in agreement with that of the theoretical second approximation at the speed where wave-making resistance starts to be observed and at the speed just before the last hump can be seen. However, there is a very great difference between the calculated and measured values. The reason for this cannot be explained.

§4 General expressions for the wave-making resistance of a submerged body of an arbitrary form

One of the major problems in the theory of stationary wave resistance is how to satisfy the surface condition of a dody. For this purpose various attempts have been made. The present author published a report on the results of several calculations with respect to a submerged sphere (§2) and a prolate spheroid (§3). In this section discussions will be made with respect to a submerged body of an arbitrary form. According to his theory the surface condition of the body can be fully satisfied, if we do not mind the tedious procedures involved. As for the water surface condition, however, it remains linearized. The difference in the theory and experiment discussed in §3 shall be recalled in this respect.

4.1 It is assumed that a submerged body advances in one direction at a uniform speed. A righthand coordinate system is employed in which the z-axis is taken vertically upward, the x-axis in the direction of the body's advance and the origin is fixed at a point within the body which is at the f-position below the surface of water. Then the velocity potential may be, as well known, given by

$$U(x, y, z)=V(x, y, z)-\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{0}^{\infty}\mathfrak{V}(\kappa, \theta)h(\kappa, \kappa_0, \theta)\exp(-2f\kappa+i\kappa\tilde{\omega})\,d\kappa d\theta \tag{53}$$

where V is a function having its singularities inside the body only and $\mathfrak{V}$ is deduced by the following equation:

$$V(x, y, z)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{0}^{\infty}\mathfrak{V}(\kappa, \theta)\exp(-\kappa z+i\kappa\tilde{\omega})\,d\kappa d\theta\,,\quad z>0 \tag{54}$$

and further

$$h(\kappa, \kappa_0, \theta)=(\kappa\cos^2\theta+\kappa_0+\mu i\cos\theta)/(\kappa\cos^2\theta-\kappa_0+\mu i\cos\theta)$$

$$\kappa_0=g/(\text{velocity})^2\,,\qquad \tilde{\omega}=x\cos\theta+y\sin\theta$$

Our problem is to determine V so that it satisfies on the body's surface the condition

$$\partial V/\partial\nu=-\partial x/\partial\nu \tag{55}$$

with ν as the outward normal of the body's surface.

Now we are to solve the problem of this boundary value first by setting a system of ortho-normal functions on the body's surface, over which to expand (55) to deduce a simultaneous equation, and then by successive substitutions step by step until the solution is obtained.

4.2 We shall first consider a system of ortho-normal functions [A-15]

$\{\psi_n\}$ which is harmonics on the body's surface S and the inside V_i which satisfies the normalized orthogonal condition

$$\iint_S \psi_n \frac{\partial}{\partial\nu}\psi_m dS = \iiint_{V_i} \text{grad.}\,\psi_n \cdot \text{grad.}\,\psi_m d\tau = \delta_{n,m}, \begin{cases} 1 \cdots n=m \\ 0 \cdots n \neq m \end{cases} \quad (56)$$

Actually this can be done in the following procedure.

For example, we take homogeneous spherical harmonics and put it like

$$H_0 = x\,,\ H_1 = -z\,,\ H_2 = x^2 - \frac{1}{2}(y^2+z^2)\,,\ H_3 = -3xz\,,\ H_4 = \frac{3}{2}(z^2-y^2) \quad (57)$$

where odd functions in y are omitted on the assumption that the body is symmetrical with respect to the $x-z$ plane.

Then putting

$$h_{n,m} = \iint_S H_n \frac{\partial}{\partial\nu} H_m dS = \iiint_{V_i} \text{grad.}\,H_n\,\text{grad.}\,H_m d\tau = h_{m,n} \quad (58)$$

$$D_0 = h_{0,0}, \cdots\cdots\cdots\cdots D_n = \begin{vmatrix} h_{00}\,h_{01} \cdots\cdots h_{0n} \\ h_{10} \quad\quad \vdots \\ \vdots \quad\quad\quad \vdots \\ h_{n0} \cdots\cdots\cdots h_{nn} \end{vmatrix} \quad (59)$$

and if we define $\Gamma_{n,m}$ as a conjugate element of $h_{n,m}$,

$$\psi_n = \{\Gamma_{n0}H_0 + \Gamma_{n1}H_1 + \cdots + \Gamma_{nn}H_n\} / \sqrt{D_n D_{n-1}} \quad (60)$$

then it can easily be found that the condition of (56) is actually satisfied.

Now if we take the origin at the body's centre of gravity and assume that the body is symmetrical upwards and downwards as well as fore and aft, ψ_n will be like

$$\left.\begin{aligned} &\psi_0 = x/\sqrt{\varDelta}\,,\ \psi_1 = -z/\sqrt{\varDelta}\,,\ \psi_2 = \left(x^2 - \frac{y^2+z^2}{2}\right)\Big/\sqrt{h_{22}} \\ &\psi_3 = -3xz/\sqrt{h_{33}}\,,\ \psi_4 = (h_{22}H_4 - h_{24}H_2)/\sqrt{h_{22}(h_{22}h_{44}-h_{22}^2)}\,, \cdots\cdots \end{aligned}\right\} \quad (61)$$

where $\varDelta$ is the displacement volume.

We shall now put as f_n the function which is equal to $-\partial\psi_n/\partial\nu$ on S and regular in the domain V_e outside the body. That is,

$$\partial f_n/\partial\nu = -\partial\psi_n/\partial\nu \quad (62)$$

and expanding on S like

$$\partial V/\partial\nu = -\sum_n A_n(\partial\psi_n/\partial\nu) \quad (63)$$

from the definition of f_n, we have

$$V=\sum_n A_n f_n \tag{64}$$

in V_e.

On S, if we put

$$B_{nm}=\iint_S f_n \frac{\partial}{\partial\nu}\psi_m \partial S=-\iint_S f_n \frac{\partial}{\partial\nu} f_m dS \tag{65}$$

we may expand like

$$V=\sum_{nm} A_n B_{nm}\psi_m \tag{66}$$

Next, in order to obtain an expansion for $\mathfrak{V}$, we first make Fourier transformation of (54) to get the following equation:—

$$\mathfrak{V}(\kappa,\theta)=\frac{1}{4\pi}\iint_{-\infty}^{\infty}\left(V\frac{\partial}{\partial z}-\frac{\partial V}{\partial z}\right)\exp(\kappa z-i\kappa\tilde{\omega})\,dxdy$$

Considering the regularity of the integrands we may write from Green's theorem

$$\mathfrak{V}(\kappa,\theta)=\frac{1}{4\pi}\iint_S\left(\frac{\partial V}{\partial\nu}-V\frac{\partial}{\partial\nu}\right)\exp(\kappa z-i\kappa\tilde{\omega})\,dS \tag{67}$$

Consequently, substituting (63) and (67) we have

$$\mathfrak{V}(\kappa,\theta)=-\sum_{n,m} A_n(B_{nm}E_m+E_n) \tag{68}$$

in which

$$E_n(\kappa,\theta)=\frac{1}{4\pi}\iint_S \psi_n\frac{\partial}{\partial\nu}\exp(\kappa z-i\kappa\tilde{\omega})\,dS=\frac{1}{4\pi}\iint_S\left(\frac{\partial}{\partial\nu}\psi_n\right)\exp(\kappa z-i\kappa\tilde{\omega})\,dS \tag{69}$$

4.3 Since the condition on the body's surface (55) is given by

$$-\frac{\partial x}{\partial\nu}=\frac{\partial V}{\partial\nu}-\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^{\infty}h\mathfrak{V}\frac{\partial}{\partial\nu}\exp\{\kappa(z-2f)+i\kappa\tilde{\omega}\}\,d\kappa d\theta \tag{55'}$$

if we multiply the righthand and lefthand sides of the above by ψ_n and integrate them over S, we may write

$$C_n=A_n+2\int_{-\pi}^{\pi}\int_0^{\infty}\mathfrak{V}\cdot h\cdot e^{-2\kappa f}\bar{E}_n d\kappa d\theta \tag{70}$$

referring to (63) and (69). In the above

$$C_n=\iint_S\frac{\partial x}{\partial\nu}\psi_n dS \tag{71}$$

$\bar{E}_n$ is the conjugate complex of E_n.

When we substitute (68) into the above, we obtain a system of infinite linear equations. It is not easy to obtain the solution for them, but it is easy to raise the degree of approximation by successive substitutions.

First referring to (61) we write

$$C_0=\sqrt{\Gamma}\,,\qquad C_1=C_2=C_3=C_4=0 \tag{72}$$

and for the first approximation if we put

$$A_0=C_0\,,\qquad A_1=A_2=\cdots A_n=0 \tag{73}$$

we may write for $\mathfrak{B}$ from (68) as the first approximation

$$\mathfrak{B}(\kappa,\theta)=-A_0(E_0+\sum_m B_{0m}E_m) \tag{74}$$

and substituting this into (68) we obtain as the second approximation

$$\left.\begin{aligned}A_0&=C_0[1+M_{0,0}+\sum_m B_{0,m}M_{m,0}]\\ A_n&=C_0[M_{0,n}+\sum_m B_{0,m}M_{m,n}]\,,\cdots n=1,2,3,4\end{aligned}\right\} \tag{75}$$

in which

$$M_{n,m}=\overline{M}_{n,m}=2\int_{-\pi}^{\pi}\int_0^{\infty}hE_n\overline{E}_m e^{-2\kappa f}d\kappa d\theta \tag{76}$$

4.4 In the above case wave-making resistance, vertical lift and nose-down trimming moment are given by the following equations respectively:

$$\left.\begin{aligned}R&=2\rho i\int_{-\pi}^{\pi}\int_0^{\infty}h|\mathfrak{B}|^2e^{-2\kappa f}\kappa\cos\theta d\kappa d\theta\\ Z&=-2\rho\int_{-\pi}^{\pi}\int_0^{\infty}h|\mathfrak{B}|^2e^{-2\kappa f}\kappa d\kappa d\theta\\ M_y&=4\pi\rho R_e\left\{\frac{\partial}{\partial p}\mathfrak{B}\right\}_{p=q=0}-2\rho i\int_{-\pi}^{\pi}\int_0^{\infty}h\mathfrak{B}\frac{\partial}{\partial p}\overline{\mathfrak{B}}e^{-2\kappa f}\kappa d\kappa d\theta\end{aligned}\right\} \tag{77}$$

where

$$p=\kappa\cos\theta\,,\qquad q=\kappa\sin\theta$$

We then put like

$$\left.\begin{aligned}M_{n,m}^{(1)}&=2i\int_{-\pi}^{\pi}\int_0^{\infty}hE_n\overline{E}_m e^{-2\kappa f}\kappa\cos\theta d\kappa d\theta=\overline{M}_{n,m}^{(1)}\\ M_{n,m}^{(2)}&=2\int_{-\pi}^{\pi}\int_0^{\infty}hE_n\overline{E}_m e^{-2\kappa f}\kappa d\kappa d\theta=\overline{M}_{n,m}^{(2)}\\ M_{n,m}^{(3)}&=2i\int_{-\pi}^{\pi}\int_0^{\infty}hE_n\frac{\partial}{\partial p}\overline{E}_m e^{-2\kappa f}\kappa d\kappa d\theta=\overline{M}_{n,m}^{(3)}\end{aligned}\right\} \tag{78}$$

If we substitute the approximate value mentioned in the preceding section we obtain as the first approximation

$$\left.\begin{aligned} R/\rho &= \nabla(1+B_{00})^2 M_{0,0}^{(2)} \\ -Z/\rho &= \nabla(1+B_{00})^2 M_{0,0}^{(2)} \\ -M_y/\rho &= \nabla(1+B_{00})[(1+B_{11})M_{0,1}+(1+B_{00})M_{0,0}^{(3)}] \end{aligned}\right\} \tag{79}$$

In our case, where the body is symmetrical upwards and downwards as well as fore and aft, the second approximation will be as follows:

$$\left.\begin{aligned} R/\rho &= \nabla(1+B_{00})^2[M_{0,0}^{(1)}+2(1+B_{00})M_{0,0}M_{0,0}^{(1)}+2(1+B_{33})M_{0,3}M_{0,3}^{(1)}] \\ -Z/\rho &= \nabla(1+B_{00})^2[M_{0,0}^{(2)}+2(1+B_{00})M_{0,0}M_{0,0}^{(2)}+2(1+B_{33})M_{0,3}M_{0,3}^{(2)}] \\ -M_y/\rho &= \nabla(1+B_{00})(1+B_{11})M_{0,1}+\nabla(1+B_{00})^2[M_{0,0}^{(3)} \\ &\qquad +2(1+B_{00})M_{0,0}M_{0,0}^{(3)}+2(1+B_{33})M_{0,3}M_{0,3}^{(3)}] \end{aligned}\right\} \tag{80}$$

From the definition of E, we may write

$$\left.\begin{aligned} E_0 &= \frac{-ip}{4\pi\sqrt{\nabla}}\iiint_{V_i}\exp(\kappa z-i\kappa\tilde{\omega})\,d\tau, \quad E_1=\frac{-\kappa}{4\pi\sqrt{\nabla}}\iiint_{V_i}\exp(\kappa z-i\kappa\tilde{\omega})\,d\tau \\ \frac{\partial}{\partial p}E_0 &= \frac{-i}{4\pi\sqrt{\nabla}}\iiint_{V_i}\exp(\kappa z-i\kappa\tilde{\omega})\,d\tau \\ &\quad +\frac{-ip}{4\pi\sqrt{\nabla}}\iiint_{V_i}(z\cos\theta-ix)\exp(\kappa z-i\kappa\tilde{\omega})\,d\tau \end{aligned}\right\} \tag{81}$$

The first equation of (79) concides in form with the so-called interpolation formula by HOGNER [A–14].

As can be seen from the definition, in (65) and (66), B_{00} in (79) is the so-called inertia coefficient or a virtual mass coefficient, which HOGNER substituted by the coefficient of an ellipsoid [A–16] having as its axes the length, breadth and the draught corresponding to those of a ship. His approximation seems appropriate because it is difficult to find by actual calculation such a coefficient with a ship of an arbitrary shape. The result does not seem to be much different if we take for simplicity the coefficient of a rotatory ellipsoid having a similar displacement-length ratio. That coefficient can easily be calculated if the equivalent source distribution is known of the body. An example is given in Fig. 7.

It is well known that the above formula for resistance gives an interpolation formula for a displacement boat and a ship of pressure distribution type. Fig. 8 shows an example of the calculation with respect to the formula. (In the calculation we employed the coefficient of the virtual mass of a rotatory ellipsoid with the same displacement-length ratio as that of the ship considered). The result shows that the value is always larger than that obtained by MICHELL's formula if the breadth is finite. WIGLEY has pointed out this fact showing a similar quantity. [A-17]

The third equation in (79) is the same as the one which HAVELOCK [A-10] considered deducible for a body of an arbitrary shape from an expression for a rotatory ellipsoid when he calculated the trimming moment of a submerged body.

4.5 In the preceding sections we have been able to deduce theoretically HOGNER's interpolation formula and HAVELOCK's formula of trimming moment.

In summary, a solution has been found for the wave-making resistance of a submerged body of an arbitrary shape by employing a system of ortho-normal functions on the body's surface and by a method of successive substitutions. It has been found, too, that the value of the first approximation agrees with that obtained by expanding the equations developed by MICHELL and others.

The value of the first approximation calculated by the present author's method is always a bit larger, as was pointed out by WIGLEY, than that obtained by MICHELL's equation.

In low speed this augmentation of the resistance seemes strange, for this fact contradicts the results obtained from the theory of INUI. The author cannot explain this contradiction. However, it is conceivable that in low speed the form, especially bow form, of a ship affects the wave making property decisively, but such fine form characteristics are not taken into consideration in this treatment. This negligeance might have been the cause of the contradiction.

As for the virtual mass coefficient, the result does not seem much different whether we employ that of an ellipsoid having as its axes the length, breadth and draught corresponding to those of a ship as HOGNER employed in his approximation or that of a rotatory ellipsoid having a similar displacement-length ratio.

Chapter 2 Effects of Finite Amplitude on the Wave-Making Resistance (Two-Dimensional Motion)

§5. The wave-making resistance of a submerged circular cylinder

In the theory of wave-making resistance of a submerged body the condition of the water surface has always been linearized and researchers' efforts have been directed toward the improvement on the approximation of the body's surface condition. Judging from the experiment and the result of the calculation the author carried out on a submerged spheroid (§3), a great effect was clearly observed of the finite surface disturbance, especially when the body's immersion was shallow.

Already NISHIYAMA [5] has discussed the effect of finite wave-height upon wave-making resistance. He treated the problem only from the energy of regular waves in the rear of a body. There remains the problem that how velocity potential would change according to whether the water surface condition is linerized or not.

The author has taken up this problem of the water surface condition to study it in the limited terms of a second approximation with respect to a two-dimensional submerged circular cylinder. A similar treatment is possible in regard to a case of an approximation of a higher order or a three-dimensional body, but the author preferred a two-dimensional body simply to avoid analytical difficulties.

5.1 We shall assume that there is a cylinder at an immersion depth c under the surface of infinite upper stream, and a uniform stream and the cylinder's radius are taken as a unit. The origin is taken in the centre of the cylinder, the x-axis in the direction of a stream and the y-axis directly upwards. The elevation of the water surface is expressed by

$$\eta = y - c$$

Like in the perturbation method, a stream function is given like

$$\psi(x, y) = y + \psi_1(x, y) + \psi_2(x, y) \tag{82}$$

where ψ_2 is assumed to be of higher order than ψ_1.

As for the water surface elevation, if it is assumed that ψ is $-c$ at infinite upper stream, since ψ is a constant on the free surface we may write

$$\eta(x) = \psi_1(x, c) + \eta(x)\psi_{1y}(x, c) + \psi_2(x, c)$$

where ψ_2 is considered as the second order term.

Hereafter a constant variable c will be omitted in the equations, and suffixes x and y will denote partial differential. Then we may now express as the first approximation

$$\eta(x) = \psi_1(x) \tag{83}$$

and as the second approximation

$$\eta(x) = \psi_1(x) + \{\psi_1(x)\psi_{1y}(x) + \psi_2(x)\} \tag{84}$$

On the other hand, assuming that pressure is zero on the water surface we may write from BERNOULLI's equation

$$g\eta(x)+\frac{1}{2}\{\psi_x^2(x,y)+\psi_y^2(x,y)\}=\frac{1}{2} \tag{85}$$

in which g is a constant for the gravity in this unit system. Taking up to the second term

$$g\eta(x)=\psi_{1y}(x)+\left\{\psi_{2y}(x)+\psi_1(x)\psi_{1yy}(x)-\frac{1}{2}(\psi_{1x}^2(x)+\psi_{1y}^2(x))\right\} \tag{86}$$

Then putting

$$g=\gamma(1+\delta) \tag{87}$$

where δ is regarded as a correction term for the equation expressing the relation between wave height and its speed. Then combining (83), (84), (86) and (87) together, we can write

$$\gamma\psi_1(x)=\psi_{1y}(x) \tag{88}$$

$$\psi_{2y}(x)-\gamma\psi_2(x)=\delta\psi_1(x)+\chi(x) \tag{89}$$

and

$$\chi(x)=\gamma\psi_1(x)\psi_{1y}(x)-\psi_1(x)\psi_{1yy}(x)+\frac{1}{2}(\psi_{1x}^2(x)-\psi_{1y}^2(x)) \tag{90}$$

Eq. (88) gives the linearized water surface condition hitherto in use and (90) gives the condition for the second approximation ψ_2.

Now corresponding to (82), we denote a complex velocity potential as

$$f(z)=-z+f_1(z)+f_2(z),\quad z=x+iy \tag{91}$$

and assume it possible to have a Fourier's expression as follows:

$$\left.\begin{aligned} f_1(z)&=-i\int_0^\infty e^{i\kappa z}P_1(\kappa)\,d\kappa-i\int_0^\infty e^{-i\kappa z-2c\kappa}Q_1(\kappa)\,d\kappa \\ f_2(z)&=-i\int_0^\infty e^{i\kappa z}P_2(x)\,dx-i\int_0^\infty e^{-i\kappa z-2c\kappa}Q_2(\kappa)\,d\kappa \end{aligned}\right\}\ 2c>y>0 \tag{92}$$

Then for a stream function we may write as follows:

$$\begin{aligned}\psi_1(x,y)&=\frac{1}{2i}\{f_1(z)-\overline{f_1(z)}\}\\ &=-\frac{1}{2}\int_0^\infty\{\overline{P_1(\kappa)}\,e^{-\kappa y}+Q_1(\kappa)\,e^{\kappa y-2\kappa c}\}\,e^{i\kappa x}\,d\kappa\\ &\quad-\frac{1}{2}\int_0^\infty\{P_1(\kappa)\,e^{-\kappa y}+\overline{Q_1(\kappa)}\,e^{\kappa y-2\kappa c}\}\,e^{-i\kappa x}\,d\kappa\end{aligned}$$

If we use for simplicity the sign

$$\mathfrak{F} \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \, e^{i\kappa x}$$

we may write

$$\left.\begin{aligned} \mathfrak{F}\{\psi_1(x, y)\} &= -\frac{1}{2}\{\overline{P_1(\kappa)}\, e^{-\kappa y} + Q_1(\kappa)\, e^{\kappa(y-2c)}\} \\ \mathfrak{F}\{\psi_2(x, y)\} &= -\frac{1}{2}\{\overline{P_2(\kappa)}\, e^{-\kappa y} + Q_2(\kappa)\, e^{\kappa(y-2c)}\} \end{aligned}\right\} \kappa > 0 \qquad (93)$$

This conjugate complex value is taken as a minus value of κ. This holds true throughout.

Using the above functions, (89) may be transcribed like

$$Q_1(\kappa) = \frac{\kappa+\gamma}{\kappa-\gamma} \overline{P_1(\kappa)}$$

If we introduce here an artificial friction coefficient to eliminate uncertainty in the integration and write

$$Q_1(\kappa) = \frac{\kappa+\gamma+\mu i}{\kappa-\gamma-\mu i} \overline{P_1(\kappa)} \qquad (94)$$

This is the water surface condition hitherto employed. When this is substituted into (93) and partial integration is carried out we immediately obtain the following equation:

$$\left.\begin{aligned} \mathfrak{F}\{\psi_1\} &= -\frac{\kappa \overline{P_1(\kappa)}\, e^{-\kappa c}}{\kappa-\gamma-\mu i}, \quad \mathfrak{F}\{\psi_{1x}\} = \frac{i\kappa^2 \overline{P_1(\kappa)}\, e^{-\kappa c}}{\kappa-\gamma-\mu i} \\ \mathfrak{F}\{\psi_{1y}\} &= -\frac{\gamma\kappa \overline{P_1(\kappa)}\, e^{-\kappa c}}{\kappa-\gamma-\mu i}, \quad \mathfrak{F}\{\psi_{1yy}\} = -\frac{\kappa^3 \overline{P_1(\kappa)}\, e^{-\kappa c}}{\kappa-\gamma-\mu i} \end{aligned}\right\} \qquad (95)$$

Next, Eq. (89) is transcribed like

$$Q_2(\kappa) = \frac{\kappa+\gamma+\mu i}{\kappa-\gamma-\mu i} \overline{P_2(\kappa)} + \frac{2\delta\kappa \overline{P_2(\kappa)}}{(\kappa-\gamma-\mu i)^2} - \frac{2X(\kappa)\, e^{\kappa c}}{\kappa-\gamma-\mu i} \qquad (96)$$

where $\qquad \mathfrak{F}\{\chi\} = X(\kappa)$

In the above, if P_1 and Q_2 are assumed regular, $X(\kappa)$, too, is regular at $\kappa=\gamma$, as can be seen later. Consequently δ must be zero. This means that in the second approximation the relation between wave length and and its speed remains unchanged. Then we write

$$Q_2(\kappa) = \frac{\kappa+\gamma+\mu i}{\kappa-\gamma-\mu i} \overline{P_2(\kappa)} - \frac{2X(\kappa)\, e^{\kappa c}}{\kappa-\gamma-\mu i} \qquad (97)$$

Now if in general

$$\mathfrak{F}\{f\}=F(\kappa),\ \mathfrak{F}\{g\}=G(\kappa),\cdots\kappa>0$$
$$F(\kappa)=\overline{F(-\kappa)},\ G(\kappa)=\overline{G(-\kappa)},\cdots\kappa<0$$

since we have

$$F\{f\cdot g\}=\int_0^\kappa F(\lambda)\,G(\kappa-\lambda)\,d\mu+\int_0^\infty\{\overline{F(\lambda)}\,G(\kappa+\lambda)+\overline{G(\lambda)}\,F(\kappa+\lambda)\}\,d\lambda$$

we may write employing (95)

$$X(\kappa)\,e^{\kappa c}=\int_0^\kappa\frac{\left\{\frac{3}{2}\gamma^2-\left(\kappa-\frac{\lambda}{2}\right)(\kappa-\lambda)\right\}}{(\lambda-\gamma-\mu i)(\kappa-\lambda-\gamma-\mu i)}\lambda(\kappa-\lambda)\,\overline{P_1(\lambda)}\,\overline{P_1(\kappa-\lambda)}\,d\lambda$$
$$+\int_0^\infty\frac{\{3\gamma^2-(\kappa^2+\lambda^2+\lambda\kappa)\}}{(\lambda-\gamma+\mu i)(\kappa+\lambda-\gamma-\mu i)}\lambda(\kappa+\lambda)\,P(\lambda)\,\overline{P_1(\kappa+\lambda)}\,e^{-2\lambda c}\,d\lambda \quad (98)$$

5.2 In Eq. (92), if we assume on the other hand that Laurent's expansion is possible for f_1 and f_2 and if we write

$$\left.\begin{aligned}f_1(z)&=\sum_{n=1}^\infty i^{n-1}A_{1,n-1}z^{-n}+\sum_{n=0}^\infty B_{1,n}(-i)^{n+1}z^n\\ f_2(z)&=\sum_{n=1}^\infty i^{n-1}A_{2,n-1}z^{-n}+\sum_{n=0}^\infty B_{2,n}(-i)^{n+1}z^n\end{aligned}\right\}\quad (99)$$

then between these functions and (92) there is the relation expressed as

$$P_1(\kappa)=\sum_{n=0}^\infty A_{1,n}\frac{\kappa^n}{n!},\quad P_2(\kappa)=\sum_{n=0}^\infty A_{2,n}\frac{\kappa^n}{n!} \quad (100)$$

$$B_{1,n}=\frac{1}{n!}\int_0^\infty e^{-i\kappa z-2c\kappa}Q_1(\kappa)\kappa^n\,d\kappa,\quad B_{2,n}=\frac{1}{n!}\int_0^\infty e^{-i\kappa z-2c\kappa}Q_2(\kappa)\kappa^n\,d\kappa \quad (101)$$

Now, the boundary condition on the cylinder's surface, i.e., on $|z|=1$ means that a stream function becomes a constant on this surface. We may divide this condition into two and put

$$\left.\begin{aligned}-y+\psi_1&=\text{constant}\\ \psi_2&=\text{constant}\end{aligned}\right\}\quad (102)$$

When this is done, we may obtain from (99) an expression for the relation between A and B coefficients. And if the result thus obtained is rearranged by employing (100) and (101) we obtain

$$P_1(\kappa)=1-\int_0^\infty\overline{Q_1(\lambda)}K(\lambda\kappa)\,e^{-2c\lambda}\lambda\,d\lambda \quad (103)$$

$$P_2(\kappa)=-\int^\infty\overline{Q_2(\lambda)}K(\lambda\kappa)\,e^{-2c\lambda}\lambda\,d\lambda \quad (104)$$

where

$$K(\lambda\kappa)=\sum_{n=0}^{\infty}\frac{(\kappa\lambda)^n}{n!\,(n+1)!}$$

5.3 To obtain the required velocity potential from (99), (100) and (101) is equal to obtain P_1 and P_2. P_1 may be obtained from (94) and (103), and P_2 from (97) and (103). That is, when (93) is substituted into (103) we have

$$P_1(\kappa)=1-\int_0^{\infty}\frac{\lambda+\gamma-\mu i}{\lambda-\gamma+\mu i}P_1(\lambda)K(\lambda\kappa)\,e^{-2c\lambda}\lambda\,d\lambda \tag{105}$$

and P_1 may be obtained from this. In the next step if we obtain X in (98) from this P_1, substitute what has been obtained into (97) and then substitute (97) again into (104) we have

$$P_2(\kappa)=2\int_0^{\infty}\frac{X(\lambda)K(\lambda\kappa)}{\lambda-\gamma+\mu i}e^{-c\lambda}\lambda\,d\lambda-\int_0^{\infty}\frac{\lambda+\gamma-\mu i}{\lambda-\gamma+\mu i}P_2(\lambda)K(\lambda\kappa)\,e^{-2c\lambda}\lambda\,d\lambda \tag{106}$$

P_2 can be obtained from this.

As for Eq. (106), since it has been given by HAVELOCK [A-1] it is rather a simple procedure to obtain a detailed solution.

However, in view of our object of making a qualitative comparison between the effect of the cylinder's surface condition and that of the water surface condition, no higher term is necessary now. Because, on one hand, the third approximation for the water surface condition begins with a term of the order γ^6. Our need should be sufficiently met if we adopt the following equation:

$$P_1(\kappa)=A_{1,0}+\kappa A_{1,1}+\frac{k^2}{2}A_{1,2},$$

$$q_n=\int_0^{\infty}\frac{\lambda+1-\mu i}{\lambda-1+\mu i}e^{-\alpha\lambda}\lambda^n d\lambda=r_n-is,\ \alpha=2c\gamma,\ s=2\pi\,e^{-\alpha} \tag{107}$$

we have

$$\left.\begin{aligned}&A_{1,0}=1-\gamma^2q_1+\gamma^4q_1^2,\quad A_{1,1}=-\frac{\gamma^3}{2}q_2\\&A_{1,2}=-\frac{\gamma^4}{6}q_3\end{aligned}\right\} \tag{108}$$

Next, for the similar reason like the above, in (98) we may just put $P_1=1$. Then

$$X(\kappa)\,e^{\kappa c}=\kappa\left(\gamma^2+\frac{1}{2}\gamma\kappa-\frac{5}{12}\kappa^2\right)+\gamma(\gamma^2-\kappa^2)\log\left(\frac{\kappa-\gamma-\mu i}{-\gamma-\mu i}\right)$$

$$-\frac{1}{2c}\left\{\kappa(\kappa+\gamma)+\frac{1}{2c}(\kappa+2\gamma)+\frac{1}{2c^2}\right\}-\frac{\gamma}{\kappa}(\kappa^2-\gamma^2)(\kappa+2\gamma)p_0(\alpha)$$

$$-\frac{\gamma}{\kappa}\kappa(\kappa^2-\gamma^2)(\kappa-2\gamma)\overline{p_0(\alpha-2\kappa c)} \tag{109}$$

in which

$$p_n(\alpha)=\int_0^\infty\frac{e^{-\alpha\lambda}\lambda^n\,d\lambda}{\lambda-1+\mu i}=t_n-\pi i\,e^{-\alpha} \tag{110}$$

It is found from the above that it is correct to have $\delta=0$ in deducing (97).

When we now put

$$Y(\kappa)=2\int_0^\infty\frac{\overline{X(\lambda)}K(\lambda\kappa)}{\lambda-\gamma+\mu i}\,e^{-c\lambda}\lambda\,d\lambda=y_0+\frac{\kappa}{2}y_1+\frac{\kappa^2}{12}y_2+\cdots \tag{111}$$

we have

$$y_0=2\int_0^\infty\frac{X(\lambda)\,e^{-c\lambda}}{\lambda-\gamma+\mu i}\lambda\,d\lambda \tag{112}$$

and for the reason mentioned above we should be satisfied with the first approximation as a solution for (106). That is,

$$P_2(\kappa)=A_{2,0}=y_0 \tag{113}$$

Substituting (109) into (112)

$$y_0=2\gamma^4\Big[\,p_2+\frac{1}{2}p_2-\frac{5}{12}p_4-\frac{1}{\alpha}\left\{p_3+p_2+\frac{1}{2}(p_2+2p_1)+\frac{2}{\alpha^2}p_1\right\}$$

$$+\left(\frac{2}{\alpha}+\frac{3}{\alpha^2}+\frac{2}{\alpha^3}\right)\overline{p}_0+v_2-v_1-2v_0-u_2-u_1\Big] \tag{114}$$

In the above, u and v are given respectively by

$$u_n=\int_0^\infty e^{-\alpha\lambda}\lambda^n\log\left(\frac{\lambda-1+\mu i}{-1+\mu i}\right)d\lambda \tag{115}$$

$$v_n=\int_0^\infty d\lambda\int_0^\infty\frac{e^{-\alpha(\lambda+\kappa)}\kappa^n\,d\kappa}{\kappa+\lambda-1+\mu i} \tag{116}$$

and between them is the relation like

$$\left.\begin{aligned}
u_n &= \frac{n}{\alpha} u_{n-1} + \frac{1}{\alpha} p_n \\
v_n &= \frac{n!}{(n+1)\alpha^{n+1}} + \frac{n}{n+1} v_{n-1} \\
&\text{especially} \\
u_0 &= \frac{1}{\alpha} p_0 \\
v_0 &= \frac{1}{\alpha} + p_0 \\
p_0 &= t_0 - \pi i e^{-\alpha}, \quad t_0 = e^{-\alpha} \overline{E}_i(\alpha)
\end{aligned}\right\} \tag{117}$$

$\overline{E_i}$ is a principal value of the so-called logarithmic integral.

5.4 The lift Y acting on the cylinder and resistance R_W are given by the following equation [A-1]:

$$R_W + iY = -2\pi i \rho \int_0^\infty \frac{\kappa+\gamma-\mu i}{\kappa-\gamma-\mu i} P(\kappa)\overline{P(\kappa)} e^{-2\kappa c} \kappa^2 \, d\kappa \tag{118}$$

where

$$P(\kappa) = P_1(\kappa) + P_2(\kappa)$$

Therefore

$$P(\kappa)\overline{P(\kappa)} = A_{10}\overline{A}_{10} + \kappa(A_{10}\overline{A}_{11} + \overline{A}_{10}A_{11}) + (A_{10}\overline{A}_{20} + \overline{A}_{10}A_{20}) \tag{119}$$

From (108) and (114)

$$A_{10}\overline{A}_{10} = 1 - 2\gamma^2 r_1 + \gamma^4(3r_1^2 - s^2)$$

$$A_{10}\overline{A}_{11} + \overline{A}_{10}A_{11} = -\gamma^3 r_2, \quad A_{10}\overline{A}_{20} + \overline{A}_{10}A_{20} = y_0 + \overline{y}_0$$

For simplicity we shall choose the following expressions for coefficients:

$$\left.\begin{aligned}
C_W &= R_W/4\pi^2 \rho a V^2 \\
C_Y &= Y/2\pi \rho a V^2
\end{aligned}\right\} \tag{120}$$

We shall now go back to the original unit system and denote the cylinder's radius as a, the speed of the stream as V and the cylinder's immersion depth as f. Accordingly we write

$$c = f/a, \quad \gamma = ga/V^2, \quad \alpha = 2gf/V^2$$

and further putting

$$\left.\begin{aligned}
C_W &= C_{W0} + \varDelta C_{W1} + \varDelta C_{W2} \\
C_Y &= C_{Y0} + \varDelta C_{Y1} + \varDelta C_{Y2}
\end{aligned}\right\} \tag{121}$$

and with C_{W0} and C_{Y0} as the first approximation, $\varDelta C_{W1}$ and $\varDelta C_{Y1}$ as correction terms for the cylinder's surface condition and ΔC_{W2} and ΔC_{Y2} as correction terms for the water surface condition, we may write

$$C_{W0}=\gamma^3 e^{-\alpha}, \quad C_{Y0}=-\gamma^3 r_2 \tag{122}$$

$$\left.\begin{aligned}\Delta C_{W1}&=\gamma^3 e^{-\alpha}[-2\gamma^2 r_1+\gamma^4(3r_1^2-s^2-r_2)]\\ \Delta C_{Y1}&=\gamma^3 r_2[2\gamma^2 r_1-\gamma^4(3r_1^2-s^2-r_3)]\end{aligned}\right\} \tag{123}$$

$$\left.\begin{aligned}\Delta C_{W2}&=(\gamma^3 e^{-\alpha})\times 4\gamma^4\Big[2v_0+v_1-v_2-u_1-u_2-\Big(\frac{2}{\alpha}+\frac{3}{\alpha^2}+\frac{2}{\alpha^3}\Big)t_0\\ &\quad-\Big(\frac{2}{\alpha^2}+\frac{2}{\alpha^2}\Big)s_1+\Big(1-\frac{1}{\alpha}-\frac{1}{\alpha^2}\Big)t_2+\Big(\frac{1}{2}-\frac{1}{\alpha}\Big)t_3-\frac{5}{12}t_4\Big]\\ \Delta C_{Y2}/C_{Y0}&=\Delta C_{W2}/C_{W0}\end{aligned}\right\} \tag{124}$$

in which the real part only is taken regarding functions v and u.

In the above we have been able to obtain the force acting on the cylinder up to the order of γ^4. Next comes a term of γ^6 order. But with regard to this order the calculation cannot be so simple as it has so far been because in this term is included the one which is related to the third approximation for the water surface condition. Be that as it may, we shall show in Figs. 9 to 12 the result of the calculation carried out with respect to the two cases of $f=2a$ and $f=4a$.

§6. Conclusions

In general we may make the following statement:

(1) At $f=4a$, the effect of the water surface is much smaller than that of the cylinder's surface, but at $f=2a$ the two effects are of the same order. This fact corresponds to that, as can be seen from (124), while $\Delta C_{W2}/C_{W0}$ is of the order of γ^4, $\Delta C_{W1}/C_{W0}$ starts at the term of γ^2. In the quoted paper of NISHIYAMA only this $\Delta C_{W2}/C_{W0}$ is given with its value of $4\pi^2\gamma^4 e^{-\alpha}$, that is in the same way as the order of γ^4.

(2) Judging from the procedure of calculation, in the case of $f=4a$ the above approximation seems to give nearly a correct value. However, in the case of $f=2a$ the calculation result shown is to be interpreted qualitatively only because the terms in higher approximations would represent a fairly large quantity.

(3) In Figs. 9 and 8, Curve No. 1 represents the first approximation, No. 2 the value for which only the cylinder's surface condition has been taken into consideration and No. 3 the value for which the water surface condition, too, has been taken into account. To avoid confusion these curves are all for the case of $f=2a$. With regard to wave-making resistance, as can be seen in the figures, at that shallow immersion it is

characteristic that the curves are steep and rather flat on the top. This tendency seems quite different from that found in the approximate value obtained by NISHIYAMA.

A conclusion may be drawn that at $f=4a$ the effect of the cylinder's surface condition is important because it is greater than the other effect, but at $f=2a$ no correct value may be obtained if the water surface condition is linearized.

References

[1] Masatoshi BESSHO: Wave Resistance of A Sphere, J.S.N.A. Japan, **94**, pp. 35–41 (Feb. 1954), read in 1953.

[2] Masatoshi BESSHO: On a Resistance Experiment of a Submerged Prolate Spheroid, J.S.N.A. Japan, **98**, pp. 15–21 (Feb. 1956), (in Japanese).

[3] Masatoshi BESSHO: On the Wave Resistance Theory of a Submerged Body, J.S.N.A. Japan, **99**, 15–20 (July 1956), (in Japanese).

[4] Masatoshi BESSHO: On the Wave Resistance of a Submerged Circular Cylinder—A Consideration of the Finite Surface Disturbance of the Water, J.S.N.A. Japan, **100**, pp. 1–6 (Feb. 1957), (in Japanese).

[5] Tetsuo NISHIYAMA: On the Some Problems of Finite Amplitude in the Theory of Wave Resistance, J.S.N.A. Japan, **94**, pp. 7–17 (Feb. 1954).

Additional Literature by Foreign Authors:

[A–1] T. H. HAVELOCK: Proc. Roy. Soc. (*London*), **A157**, p. 526 (1936).

[A–2] T. H. HAVELOCK: Proc. Roy. Soc. (*London*), **A138**, p. 339 (1932).

[A–3] E. W. HOBSON: The Theory of Spherical and Ellipsoidal Harmonics, (Cambridge, 1931).

[A–4] G. N. WATSON: Theory of Bessel Functions, (Cambridge, 1922).

[A–5] E. T. WHITTAKER and G. N. WATSON: Modern Analysis, (Cambridge, 1935).

[A–6] E. JAHNKE und F. EMDE: Funktionentafeln, (Berlin, 1933).

[A–7] T. J. BROMWICH: An Introduction to the Theory of Infinite Series, (London, 1926).

[A–8] T. H. HAVELOCK: Proc. Roy. Soc. (*London*), **A95**, p. 354 (1919).

[A–9] T. H. HAVELOCK: Proc. Roy. Soc. (*London*), **A131**, p. 275 (1931).

[A–10] T. H. HAVELOCK: Quart. J. Mech. App. Math., **5**, p. 129 (1952).

[A–11] T. H. HAVELOCK: Proc. Roy. Soc. (*London*), **A103**, p. 571 (1923).

[A–12] T. H. HAVELOCK: Proc. Roy. Soc. (*London*), **A108**, p. 77 (1925).

[A–13] T. H. HAVELOCK: Proc. Roy. Soc. (*London*), **A135**, p. 4 (1932).

[A–14] E. HOGNER: Hydromechanische Probleme d. Schiffsantriebs, s. 109 (Hamburg, 1932).

[A–15] S. BERGMANN and M. SCHIFFER: Kernel Functions and Elliptic Differential Equations in Mathematical Physics, Part A, Chaps. 2 and 3, (New York, 1953).

[A–16] H. LAMB: Hydrodynamics, 6th Ed. (Cambridge, 1932).

[A–17] W. C. S. WIGLEY: T.I.N.A., **68**, p. 131 (1926).

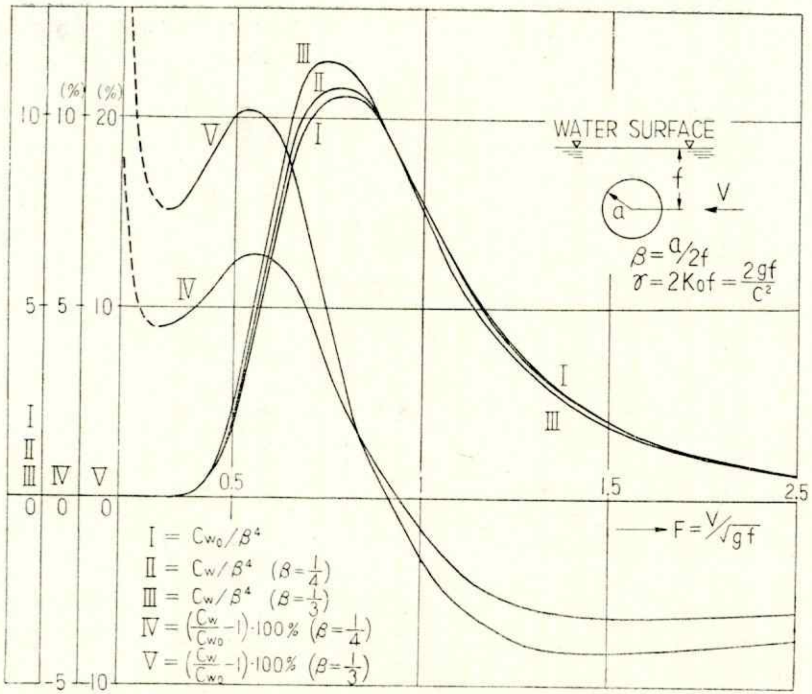

FIG. 1.—WAVE RESISTANCE OF A SUBMERGED SPHERE

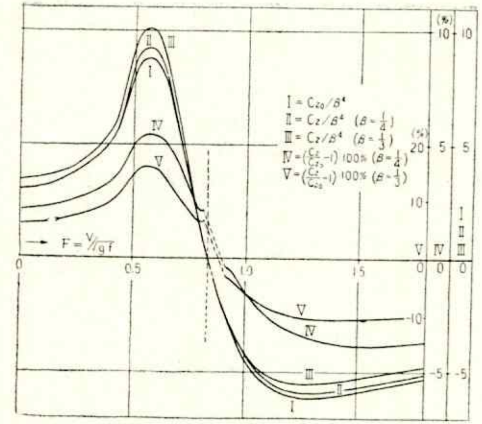

FIG. 2.—VERTICAL FORCE ACTING ON A SUBMERGED SPHERE (STATICAL BUOYANCY EXCLUDED)

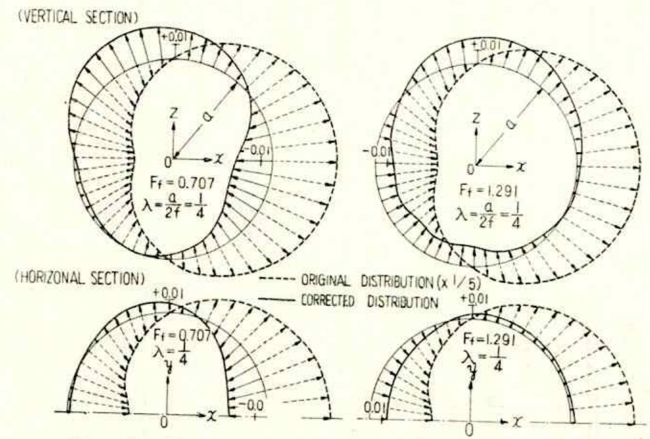

FIG. 3.—SOURCE DISTRIBUTIONS CORRECTED AND UNCORRECTED ON A SUBMERGED SPHERE

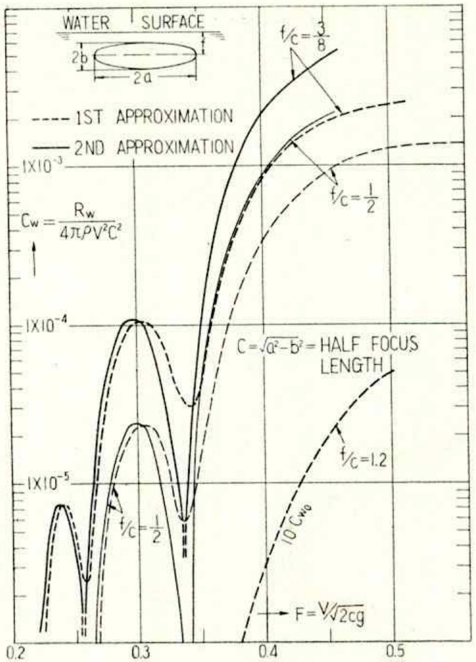

FIG. 4.—WAVE RESISTANCE FOR A SUBMERGED PROLATE SPHEROID

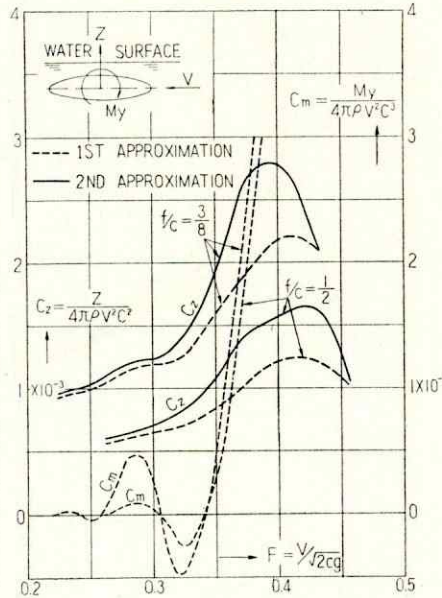

FIG. 5.—VERTICAL FORCE AND MOMENT FOR A SUBMERGED PROLATE SPHEROID

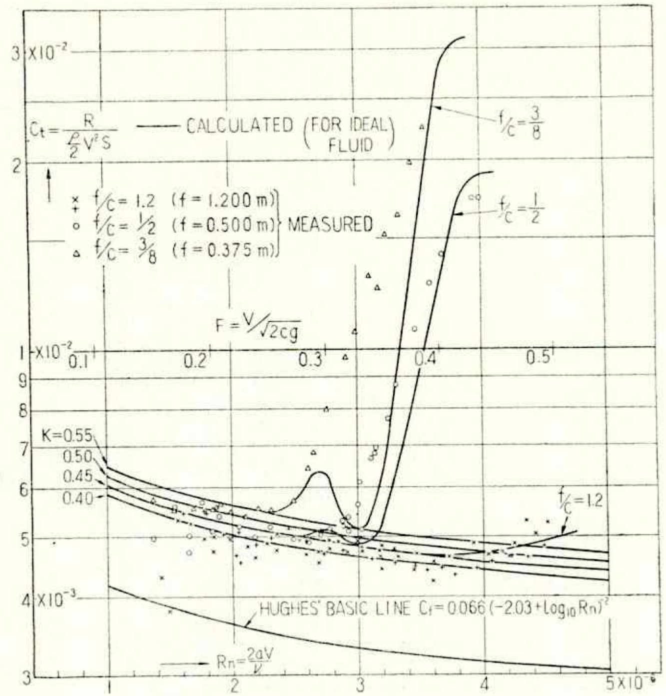

FIG. 6.—EXPERIMENTAL RESULT ON A SUBMERGED PROLATE SPHEROID

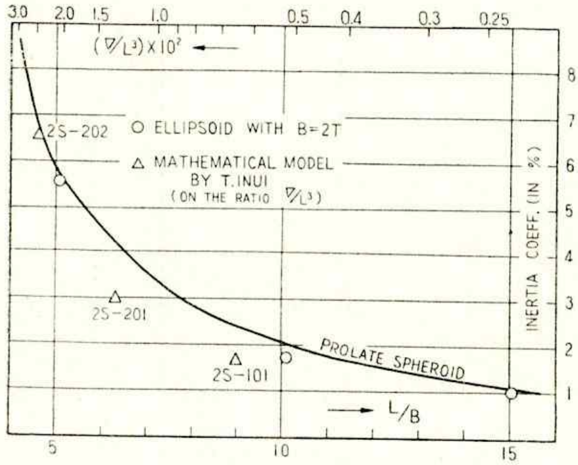

FIG. 7.—INERTIA COEFFICIENT

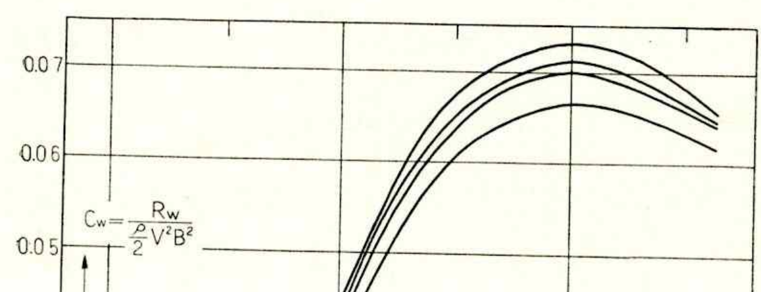

FIG. 8.—WAVE RESISTANCE FOR WALL-SIDED MODELS WITH PARABOLIC WATERLINE

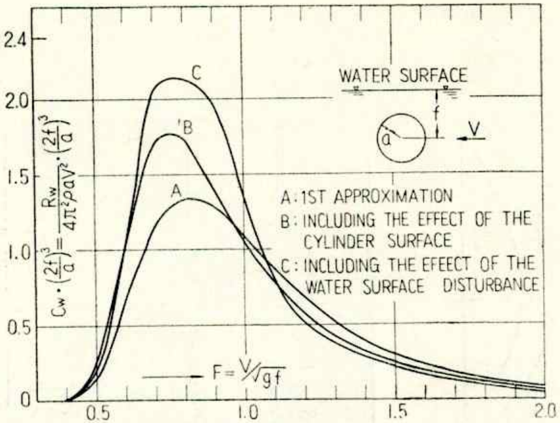

FIG. 9.—WAVE RESISTANCE FOR A SUBMERGED CIRCULAR CYLINDER ($f = 2a$)

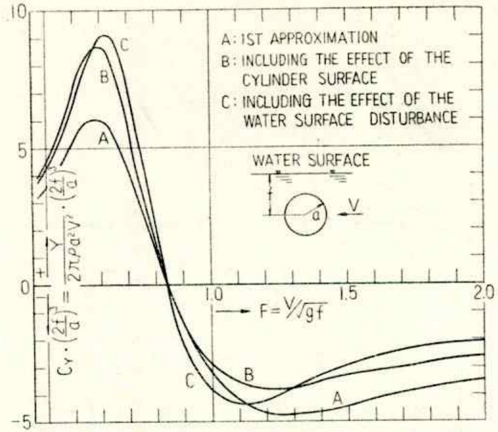

FIG. 10.—VERTICAL FORCE ACTING ON A SUBMERGED CIRCULAR CYLINDER ($f = 2a$)

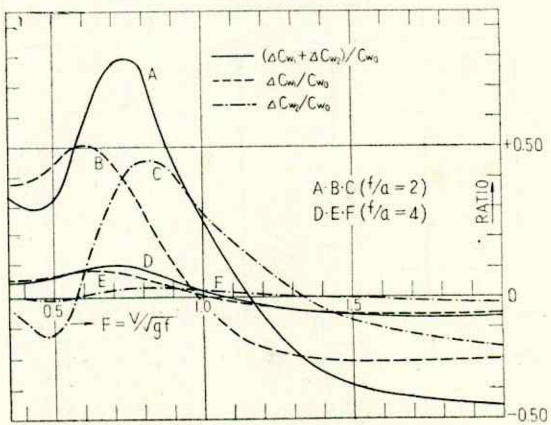

FIG. 11.—COMPARISON IN WAVE RESISTANCE BETWEEN THE PRESENT THEORY AND THE FIRST APPROXIMATION

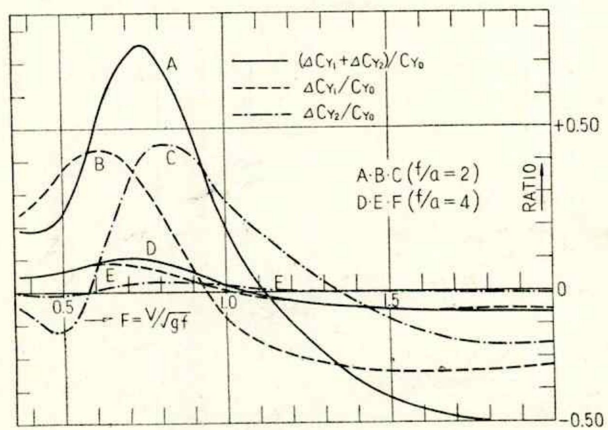

FIG. 312.—COMPARISON IN VERTICAL FORCE BETWEEN THE PRESENT THEORY AND THE FIRST APPROXIMATION