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Two-Dimensional Unsteady Planing Surface

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The two-dimensional unsteady problem of a flat planing surface is analyzed based on airfoil theory. The present analysis treats the effect of the time-varying wetted length on the added mass and damping coefficient. When the reduced frequency becomes very small, the change in wetted length approaches the displacement of the intersection of the planing surface and the undisturbed water surface. In the limiting case of zero frequency, the wetted length change reduces to zero. It is noted that the damping coefficient by the present analysis indicates the negative value in the very small reduced frequency, as Ogilvie and Bessho predicted previously in their theories derived from the different points of view.

Introduction

HYDRODYNAMIC planing boats have distinctive hydrodynamic features quite different from those of displacement ships. The weight of the planing boat, when running at high speed, is supported by hydrodynamic lift instead of hydrostatic buoyancy, and a spray is formed upstream. The fluid past the boat flows off smoothly at the trailing edge, corresponding to Kutta's condition in airfoil theory.

For the steady-state case in planing boats, a number of studies have been carried out experimentally and theoretically. These studies are well described in detail by Savitsky [1],[3] Shen [2], and Shen and Ogilvie [3].

In addition to steady-state characteristics, the hydrodynamic performances of unsteady motions are also very important in actual boat design. The porpoising phenomenon is typical and unique to planing boats. In waves, planing boats oscillate and the violent motions sometimes cause damage to the structure and injuries to the crew. These two unsteady motions appear to be different. However, the porpoising phenomenon belongs to a radiation problem in hydrodynamics, and to calculate motions in waves the radiation problem is a part of the calculations necessary for estimation of motions. Mottard [20, 21] has reported a self-excited planing vibration with only one freedom in motion—the vertical direction—whose frequency is very high. But the reduced frequency ($k = \omega l s/2U$ = circular frequency × wetted length/2 × forward velocity) for that phenomenon is found to be very small because the advance velocity is so high. Therefore the phenomenon discovered by Mottard can be reduced to the radiation problem in which the condition necessary for the unsteady motion is to find the negative damping coefficient.

In the calculation of hydrodynamic forces for actual boats, there exist difficulties because boats are three dimensional. There are a few three-dimensional theories for unsteady motions of ships such as the slender-body theory and thin-ship theory. However, these theories are very complicated as well as inconvenient in practical calculations.

The strip method seems to be only one practical procedure to calculate unsteady motions. Bessho and Komatsu [4–8] have examined the applicability of the strip method to motions of high-speed boats in regular head waves and found that the strip method is very useful for low or moderate Froude numbers. However, for high Froude numbers the calculations by the strip method do not always show good agreement with the experimental data. The result obtained by a forced oscillation test suggests that the discrepancy between the calculated and experimental data must arise from the velocity effect on the hydrodynamic forces, especially the damping coefficient.

Martin [9, 10] has calculated the porpoising instability and motions of high-speed boats in waves by the strip method, the hydrodynamic characteristics of which are based on Bobyleff's flow and Wagner's theory. The calculations agree well with experimental data. Zarnick [11] has also obtained good motion predictions for high-speed boats in waves by the strip method, using a nonlinear mathematical model in which the basis of the hydrodynamic forces are the same as Martin's theory. However, since porpoising and motions in waves are oscillatory phenomena dependent on frequency, it is more natural to calculate hydrodynamics forces acting on a high-speed boat as a function of frequency.

In general it seems that the strip method cannot directly take into account the velocity effects on the hydrodynamic forces. In order to examine the velocity effect, a two-dimensional problem must be calculated where the flow impinges against a planing surface and leaves the trailing edge smoothly.

When a planing surface oscillates in otherwise calm water, the equation to describe the flow field is reduced to an aerodynamic equation for an oscillating wing, when the effect of gravity is considered to be negligible. However, the wetted length of the planing surface, corresponding to a chord length in airfoil theory, changes as the planing surface oscillates. In addition, spray is formed upstream in physical reality.

Sedov [12] calculated a two-dimensional unsteady problem of a planing surface in connection with airfoil theory, but his calculation does not include the effect of wetted length change.

Ogilvie [13] analyzed a heaving two-dimensional planing surface with the wetted length changing. The wetted length is obtained by equating the equation of the planing surface and free-surface equation at the leading edge, where both equations share the same value. The result, however, becomes inconsistent when the heaving frequency approaches zero; that is, the wetted length diverges for zero frequency. Ogilvie and Shen [14] determined, based on Ogilvie's theory, the critical reduced frequency which causes instability to two-dimensional planing surfaces.

Bessho [15] has derived a theory by making use of the difference of the solution for two slightly different wetted lengths, assuming that the wetted length change is small enough for linearization.

An interesting feature of the results of Ogilvie and Shen and Bessho is that the damping coefficients reach the negative values

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for a very small reduced frequency.

In this paper, the two-dimensional problem of a flat planing surface is analyzed based on Bessho's theory [16]. The conditions determining the wetted length are derived from the fact that the planing surface oscillates as a rigid body without deformation. The pressure, water surface elevation, vertical velocity, etc., are expressed in Fourier series.

Property of solution by airfoil theory [16, 17]

The coordinate system is fixed in space as shown in Fig. 1. The uniform flow, the velocity of which is U at infinity, impinges against the flat planing surface. The flat planing surface has a small angle of attack or a trim angle. Generally speaking the flow consists of two parts around the planing surface. One is a spray thrown out forward from the stagnation point on the plate, and the other part is the flow past the plate. The spray produced upstream appears to be very thin and the pressure due to the spray along the planing surface almost vanishes immediately before the stagnation point, when the angle of attack is small. In this paper, therefore, the effect of the spray is neglected, as is usual in linear theory of planing surface problems.

When a two-dimensional planing surface oscillates in otherwise calm water, the water surface including the planing surface must satisfy the kinematic and dynamic conditions. The flow is assumed to be incompressible, inviscid, and irrotational.

Let η denote the water surface elevation, namely

$$F(x,y,t) = y - \eta(x,t) \tag{1}$$

The kinematic condition requires that the total differential of (1) be zero

$$\frac{D}{Dt} F(x,y,t) = 0 \tag{2}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \left(U + \frac{\partial \phi}{\partial x}\right)\frac{\partial}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial y} \tag{3}$$

and ϕ is the velocity potential of the flow field. Thus, neglecting the higher-order terms in (2), the kinematic condition may be written in the form

$$\frac{\partial \phi}{\partial y} = \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\eta \tag{4}$$

The dynamic condition to the water surface is

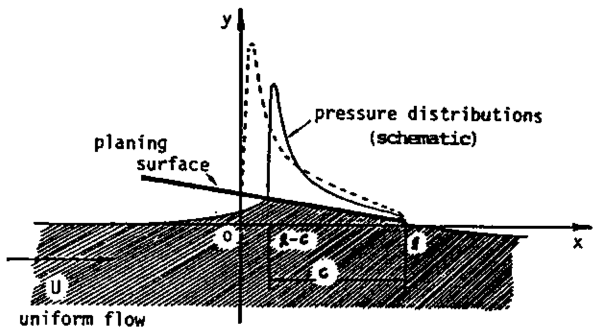

Fig. 1 Coordinate system

$$\frac{\partial \phi}{\partial t} + U\frac{\partial \phi}{\partial x} + g\eta = -\frac{p}{\rho} \quad \text{in } S, \quad y = 0 \tag{5}$$

$$= 0 \quad \text{elsewhere}, \quad y = 0 \tag{6}$$

where S denotes the planing surface, and g, p, and ρ are the acceleration of gravity, pressure, and density of water, respectively. In (5) and (6), the same linearization as used in (2) is applied.

Eliminating η in (5) and (6) with (4) gives

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 \phi + g\frac{\partial \phi}{\partial y} = -\frac{1}{\rho}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)p \quad \text{in } S \tag{7}$$

$$= 0 \quad \text{elsewhere} \tag{8}$$

When the Froude number, $U/\sqrt{gls}$, is very high, the second term of the left hand side in (7) can be neglected. Therefore, the velocity potential may be written in the form

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\phi = -\frac{1}{\rho}p \quad \text{in } S \tag{9}$$

$$= 0 \quad \text{elsewhere} \tag{10}$$

The preceding equations turn out to be the relation between the velocity potential and the acceleration potential, indicating that when the effect of gravity is neglected, the flow field due to an oscillating planing surface can be analyzed by making use of the velocity potential and the acceleration potential.

Rewriting (9) and (10) gives

$$\Phi = -\frac{1}{\rho}p(x,t) \tag{11}$$

Nomenclature

- ρ = fluid mass density
- p = pressure
- t = time
- x,y = fixed rectangular coordinates
- U = velocity of a uniform flow
- η = water surface elevation
- ϕ = velocity potential
- Φ = acceleration potential
- g = acceleration of gravity
- h = heaving motion
- ha = amplitude of heaving motion
- α = pitching motion
- v = vertical velocity of water surface
- l = maximum wetted length of planing surface
- ls = steady-state wetted length of planing surface
- f,f_1,f_2 = arbitrary functions
- L = lift acting on planing surface
- L_0 = lift for steady state
- $H_n^{(2)}$ = Hankel function of second kind
- $H_n^* = \frac{d}{dk}H_n^{(2)}$
- ϵ = amplitude/wetted length ratio ($= ha/ls$)
- ω = circular frequency of motion
- $k' = \omega/U$
- k = reduced frequency $\left(= \frac{\omega ls}{2U} = \frac{k'ls}{2}\right)$
- $C(k)$ = Theodorsen function
- M = added mass
- m = added mass in nondimensional form or integer
- N = damping coefficient
- n = damping coefficient in nondimensional form or integer
- A_n = Fourier coefficient of pressure defined by equation (21)
- B_n = Fourier coefficient of pressure defined by equation (36)
- c = wetted length in Fourier series
- $I_{n,m}$ = integral defined by equation (93)
- $A(x),a(x),b(x)$ = polynomials of x
- $F^{(m)}(mk')$ = terms proportional to $e^{-ik'x}$
- $c^{(1)}$ = amplitude of wetted length change defined by equation (79)
- $a' = c^{(1)}/l \approx c^{(1)}/ls$
- α_0 = steady-state trim angle or angle of attack

$$\Phi = \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\phi(x,y,t) \qquad (12)$$

where Φ denotes the acceleration potential.

The acceleration potential can be expressed in terms of the pressure distribution on the planing surface (see Fig. 1), that is

$$\Phi(x,y,t) = \frac{1}{\pi\rho}\int_{l-c}^{l}\frac{p(\xi,t)y d\xi}{(x-\xi)^2 + y^2} \qquad (13)$$

Integrating (12) yields the velocity potential in the form

$$\phi(x,y,t) = \frac{1}{U}\int_{-\infty}^{x}\Phi\left(\xi,y,t - \frac{x-\xi}{U}\right)d\xi \qquad (14)$$

The vertical velocity of the water surface elevation $\eta(x,\tau)$ may be written as

$$v(x,t) = \left.\frac{\partial\phi}{\partial y}\right|_{y=0} = \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\eta(x,t) \qquad (15)$$

Therefore, (15) has the following expression

$$\eta(x,t) = \frac{1}{U}\int_{-\infty}^{x} v\left(\xi,t - \frac{x-\xi}{U}\right)d\xi \qquad (16)$$

The boundary condition on the wetted surface is given in the form

$$v(x,t) = -\alpha U + \dot{h}(t) + \dot{\alpha}(l-x), \quad l-c<x<l \qquad (17)$$

where α and h denote pitching and heaving motions, respectively, and $l-c<x<l$ means the interval between the leading edge and the trailing edge of the planing surface.

Combining (12) and (17), the boundary condition with respect to the acceleration potential becomes

$$\frac{dv}{dt} = \left.\frac{\partial\Phi}{\partial y}\right|_{y=0} = \ddot{h}(t) + \ddot{\alpha}(l-x) - 2U\dot{\alpha} \qquad (18)$$

On the other hand, differentiating (13) with respect to y gives

$$\left.\frac{\partial\Phi}{\partial y}\right|_{y=0} = \frac{1}{\pi\rho}\int_{l-c}^{l}\frac{p(\xi,t)}{(x-\xi)^2}d\xi$$
$$= -\frac{1}{\pi\rho}\frac{\partial}{\partial x}\int_{l-c}^{l}\frac{p(\xi,t)}{x-\xi}d\xi \qquad (19)$$

Changing the variables in (19) with

$$x = l - \frac{c}{2}(1+\cos\theta) \qquad (20)$$

and

$$dx = \frac{c}{2}\sin\theta d\theta$$

the pressure distribution in (19) can be written in the Fourier series

$$\frac{p}{\rho} = \frac{1}{\sin\theta}\sum_{n=0}^{\infty}A_n\cos n\theta \qquad (21)$$

The Kutta's condition of smooth flow-off from the trailing edge becomes

$$\sum_{n=0}^{\infty}(-1)^n A_n = 0 \qquad (22)$$

Thus, (19) can be written in the form

$$\left.\frac{\partial\Phi}{\partial y}\right|_{y=0} = -\frac{2}{c\pi\sin\theta}\frac{d}{d\theta}\int_0^{\pi}\frac{\sum_{n=0}^{\infty}A_n\cos n\theta' d\theta'}{\cos\theta' - \cos\theta} \qquad (23)$$

Making use of the Glauert integral

$$\int_0^{\pi}\frac{\cos n\theta'}{\cos\theta' - \cos\theta}d\theta' = \pi\frac{\sin n\theta}{\sin\theta} \qquad (24)$$

(23) becomes

$$\left.\frac{\partial\Phi}{\partial y}\right|_{y=0} = \frac{4}{c}(A_2 + 4A_3\cos\theta + \ldots) \qquad (25)$$

Rewriting (18) with (20)

$$\left.\frac{\partial\Phi}{\partial y}\right|_{y=0} = \ddot{h} - 2U\dot{\alpha} + \frac{c}{2}\ddot{\alpha}(1+\cos\theta) \qquad (26)$$

It is found from (25) and (26) that $A_n = 0$ for $n \geq 4$ and A_2 and A_3 can be written in the form

$$A_2 = \frac{c}{4}\left(\ddot{h} - 2U\dot{\alpha} + \frac{c}{2}\ddot{\alpha}\right) \qquad (27)$$

$$A_3 = \frac{c^2}{32}\ddot{\alpha} \qquad (28)$$

On the other hand, the Kutta condition (22) is imposed on A_0, A_1, A_2, and A_3. Therefore the unknown constant reduces to either A_0 or A_1. In the airfoil theory these unknowns are determined in the following procedure.

Substitution of (22), (27), and (28) into (13) and (14) gives the velocity potential, ϕ. From (14) the vertical velocity on the planing surface can be calculated, and comparison of the vertical velocity with (17) yields the solution.

In the case of planing surfaces, however, the wetted length changes with time and is unknown. Therefore, in order to solve the present problem it is necessary to impose another condition.

The water surface elevation on the planing surface may be written in the form

$$\eta(x,t) = h(t) + \alpha(t)(l-x), \quad l-c<x<l \qquad (29)$$

On the other hand, the vertical velocity and water surface elevation contain the following homogeneous solutions of (12) and (15), which convect downstream with time, that is

$$\left.\frac{\partial\phi}{\partial y}\right|_{y=0} = f_1(x-Ut) \qquad (30)$$

$$\eta(x,t) = f_2(x-Ut) \qquad (31)$$

where f_1 and f_2 are arbitrary functions.

On the planing surface, however, the vertical velocity and water surface elevation do not contain such solutions as (30) and (31), as seen from (17) and (29). Therefore those terms in the vertical velocity and water surface elevation must vanish as long as the planing surface is a rigid body. In other words, the acceleration potential satisfying the boundary conditions is determined by (21), (22), (27), and (28) except for one unknown (either A_0 or A_1) and the wetted length varying with time. These unknowns are obtained by imposing the condition on which such terms as (30) and (31) vanish on the plate.

These terms arise from the homogeneous solutions of (12) and (15). Let f be an arbitrary function satisfying the homogeneous equation

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)f = 0 \qquad (32)$$

When phenomena such as velocity distribution and water surface elevation are sinusoidal, f is considered to be sinusoidal and may be written in the form

$$f = f(x-Ut) = C\bar{f}(x)e^{i\omega t} \qquad (33)$$

where C is a constant and ω is frequency; $\bar{f}(x)$ is a complex function of x.

Substituting (33) in (32) gives

$$\frac{\partial\bar{f}}{\partial x} + ik'\bar{f} = 0 \qquad (34)$$

where

$$k' = \frac{\omega}{U}$$

Solving (34) and substituting the solution in (33) yields

$$f = Ce^{-ik'x}e^{i\omega t} \tag{35}$$

Therefore, such terms as (30) and (31) are found to be proportional to $e^{-ik'x}$. The condition on the rigid planing surface can be derived by putting $f = 0$ in (35).

Boundary conditions determining wetted length

An effort is made here to seek the terms corresponding to (35) in the vertical velocity and water surface elevation on the planing surface.

It is assumed that every phenomenon such as the pressure and the change in wetted length results in sinusoidal behavior when the sinusoidal heaving or pitching motion is given on the planing surface in otherwise calm water.

When the planing surface is set in motion, the pressure distribution moves with the wetted length changing. In Fig. 1, the origin is chosen as the maximum point of the stretched wetted length and $x = l - c$ is the point for the instantaneous position. The trailing edge is fixed at $x = l$.

In order to express the pressure distribution (21) in the range $0 < x < l$, (21) must be written in the Fourier series again between the origin and the trailing edge:

$$\frac{p}{\rho} = \frac{1}{\sin\vartheta} \sum_{n=0}^{\infty} B_n \cos n\vartheta \tag{36}$$

$$x = \frac{l}{2}(1 - \cos\vartheta) \tag{37}$$

where

$$B_n = \frac{\epsilon_n}{\pi} \int_0^{\pi} \left(\frac{p}{\rho}\sin\vartheta\right) \cos n\vartheta d\vartheta \tag{38}$$

and

$$\epsilon_0 = 1$$

$$\epsilon_n = 2, \quad n \geq 1$$

Substitution of (21) into (38) with $A_n = 0$ for $n \geq 4$ and using θ as the variable of integration gives

$$B_n = \frac{c\epsilon_n}{\pi l} \int_0^{\pi} \left(\sum_{m=0}^{3} A_m \cos m\theta\right) \cos n\vartheta d\theta \tag{39}$$

where use has been made of the fact that $p = 0$ forward of the instantaneous position of the leading edge to transform the θ limits of the integration.

The coefficient A_n may be written in the form

$$A_n = \sum_{m=0}^{\infty} A_n^{(m)} e^{im\omega t} \tag{40}$$

where $m = 0$ corresponds to the steady state and $m \geq 1$ represents the sinusoidal motions. Usually the components larger than $m = 2$ appear to be minor in hydrodynamic forces.

In the same manner, B_n can be expressed in terms of the frequency components:

$$B_n = \sum_{m=0}^{\infty} B_n^{(m)} e^{im\omega t} \tag{41}$$

Making use of (39) and (41) with (13), it follows that

$$\Phi(x,y,t) = \sum_{m=0}^{\infty} \Phi^{(m)}(x,y) e^{im\omega t} \tag{42}$$

where

$$\Phi^{(m)}(x,y) = \frac{1}{\pi\rho} \int_0^{l} \frac{p^{(m)}(\xi) y}{(x-\xi)^2 + y^2} d\xi \tag{43}$$

and

$$\frac{1}{\rho} p^{(m)}(\xi) = \frac{1}{\sin\vartheta} \sum_{n=0}^{\infty} B_n^{(m)} \cos n\vartheta \tag{44}$$

Integration of (14) with (42) yields

$$\phi^{(m)}(x,y,t) = \frac{1}{U} \int_{-\infty}^{x} \Phi^{(m)}(\xi,y) e^{im\omega(t-(x-\xi)/U)} d\xi$$

$$= \frac{e^{-imk'x + im\omega t}}{U} \int_{-\infty}^{x} \Phi^{(m)}(\xi,y) e^{ik'm\xi} d\xi$$

where $k' = \omega/U$.

Therefore, the spatial part of the velocity potential can be written in the form

$$\phi^{(m)}(x,y) = \frac{e^{-imk'x}}{U} \int_{-\infty}^{x} \Phi^{(m)}(\xi,y) e^{ik'm\xi} d\xi \tag{45}$$

Defining the auxiliary function

$$q^{(m)}(x) = \frac{1}{\pi\rho} \int_0^{l} \frac{p^{(m)}(\xi)}{x - \xi} d\xi \tag{46}$$

and combining (46) with (19) gives

$$\left.\frac{\partial\Phi^{(m)}}{\partial y}\right|_{y=0} = -\frac{\partial}{\partial x} q^{(m)}(x) \tag{47}$$

Using (47) and differentiating (45) with respect to y provides the expression

$$v^{(m)}(x,0) = \left.\frac{\partial\phi}{\partial y}\right|_{y=0} = -\frac{1}{U} q^{(m)}(x) + \frac{imke^{-ik'mx}}{U} \int_{-\infty}^{x} q^{(m)}(\xi) e^{ik'm\xi} d\xi \tag{48}$$

In the same manner, the water surface elevation (16) can be expressed in the form

$$\eta^{(m)}(x) = \frac{e^{-ik'mx}}{U} \int_{-\infty}^{x} v^{(m)}(\xi,0) e^{imk'\xi} d\xi$$

Substituting (48) in the preceding equation gives

$$\eta^{(m)}(x) = -\frac{e^{-imk'x}}{U^2} \int_{-\infty}^{x} q^{(m)}(\xi) e^{imk'\xi} d\xi + \frac{imk'}{U^2} e^{-imk'x} \int_{-\infty}^{x} dx' \int_{-\infty}^{x'} q^{(m)}(\xi) e^{imk'\xi} d\xi$$

Performing the partial integration in the second integral, it follows that

$$\eta^{(m)}(x) = \frac{(imk'x - 1)}{U^2} e^{-imk'x} \int_{-\infty}^{x} q^{(m)}(\xi) e^{imk'\xi} d\xi - \frac{imk'}{U} e^{-imk'x} \int_{-\infty}^{x} \xi q^{(m)}(\xi) e^{imk'\xi} d\xi \tag{49}$$

The conditions on the planing surface concerning the vertical velocity and the water surface elevation are derived from (48) and (49). Before the derivation of the condition it is necessary to evaluate the second term in (48).

Rewriting (44) in the expression

$$\frac{1}{\rho} p^{(m)}(\xi) = \sum_{n=0}^{\infty} B_n^{(m)} p_n^{(m)} \tag{50}$$

where

$$p_n^{(m)} = \frac{\cos n\vartheta}{\sin\vartheta} \tag{51}$$

then (46) can be expressed in the form

$$q^{(m)}(x) = \frac{1}{\pi} \sum_{n=0}^{\infty} B_n^{(m)} q_n^{(m)} \tag{52}$$

where

$$q_n^{(m)}(x) = \int_0^l \frac{p_n^{(m)}(\xi)}{x-\xi}\,d\xi \tag{53}$$

Changing variables in (53) with

$$\xi = \frac{l}{2}(1 - \cos\vartheta)$$

and

$$x = \frac{l}{2}(1 - \cosh u), \quad x < 0$$

$$x = \frac{l}{2}(1 - \cos\vartheta'), \quad 0 < x < l$$

and making use of the Glauert integral together with the integral

$$\int_0^\pi \frac{\cos n\theta\, d\theta}{\cosh u - \cos\theta} = \pi \frac{e^{-nu}}{\sinh u} \tag{54}$$

$q_n^{(m)}$ can be written in the form

$$q_n^{(m)} = \pi \frac{\sin n\vartheta'}{\sin\vartheta'}, \quad 0 < x < l \tag{55}$$

$$q_n^{(m)} = -\pi \frac{e^{-nu}}{\sinh u}, \quad x < 0$$

$$= -\pi\left(\frac{\cosh nu}{\sinh u} - \frac{\sinh nu}{\sinh u}\right) \tag{56}$$

For convenience sake, let $q_n^{(m)}$ divide into two parts

$$q_n^{(m)} = q_{n1}^{(m)} + q_{n2}^{(m)}$$

where

$$q_{n1}^{(m)} = \begin{cases} 0, \quad 0 < x < l & (57) \\ -\pi \dfrac{\cosh nu}{\sinh u}, \quad x < 0 & (58) \end{cases}$$

$$q_{n2}^{(m)} = \begin{cases} \pi \dfrac{\sin n\vartheta}{\sin\vartheta}, 0 < x < l & (59) \\ \pi \dfrac{\sinh nu}{\sinh u}, x < 0 & (60) \end{cases}$$

For $0 < x < l$, $\cos\vartheta = (l - 2x)/l$, then $q_{02}^{(m)} = 0$, $q_{12}^{(m)} = \pi$, $q_{22}^{(m)} = 2\pi(l - 2x)/l$, $q_{32}^{(m)} = \pi[-1 + 4(l - 2x)^2/l^2]$ etc. In the same manner $q_{02}^{(m)} = 0$, $q_{12}^{(m)} = \pi$, $q_{22}^{(m)} = 2\pi(l - 2x)/l$, $q_{32}^{(m)} = \pi[-1 + 4(l - 2x)^2/l^2]$ etc. for $x < 0$, since $\cosh u = (l - 2x)/l$. Therefore q_{n2}^m is found to be a polynomial of x.

Substitution of (57) to (60) in the integral of the first term in (48) provides

$$\int_{-\infty}^x q^{(m)}(\xi)e^{imk'\xi}d\xi = \frac{1}{\pi}\sum_{n=0}^\infty B_n^{(m)} \int_{-\infty}^x (q_{n1}^{(m)} + q_{n2}^{(m)})e^{imk'\xi}d\xi \tag{61}$$

The first term in (61) becomes

$$\int_{-\infty}^x q_{n1}^{(m)}(\xi)e^{imk'\xi}d\xi = \int_{-\infty}^0 q_{n1}^{(m)}(\xi)e^{imk'\xi}d\xi + \int_0^x q_{n1}^{(m)}(\xi)e^{imk'\xi}d\xi \tag{62}$$

The second term in (62) vanishes from (57), and the first term can be evaluated as follows:

$$\int_{-\infty}^0 q_{n1}^{(m)}(\xi)e^{imk'\xi}d\xi = -\frac{l}{2}\int_0^\infty e^{imk'l(1-\cosh u)/2}\cosh nu\,du \tag{63}$$

Making use of the Hankel function of the second kind [18]

$$H_n^{(2)}(k) = -\frac{2e^{n\pi i/2}}{\pi i}\int_0^\infty e^{-ik\cosh\eta}\cosh n\eta\, d\eta$$
$$= J_n(k) - iY_n(k)$$

where J_n and Y_n are the Bessel functions of the first kind and second kind of order n, respectively, (63) can be written in the expression

$$\int_{-\infty}^0 q_{n1}^{(m)}(\xi)e^{imk'\xi}d\xi = -\frac{\pi^2}{4}\, l e^{imk'l/2}\, i^{-(n+1)}H_n^{(2)}\left(\frac{mk'l}{2}\right) \tag{64}$$

On the other hand, since $q_{n2}^{(m)}$ is a polynominal of ξ, the second integral can be evaluated in the form

$$\int_{-\infty}^x q_{n2}^{(m)}(\xi)e^{imk'\xi}d\xi = e^{imk'x}A(x) \tag{65}$$

where $A^{(m)}(x)$ denotes the polynominal of x.

Therefore, (61) can be expressed in the form

$$\int_{-\infty}^x q^{(m)}(\xi)e^{imk'\xi}d\xi = F^{(m)}(mk') + a^{(m)}(x)e^{imk'x} \tag{66}$$

where $a^{(m)}(x)$ is a polynominal of x arising from (65) and

$$F^{(m)}(mk') = -\frac{\pi l}{4}e^{imk'l/2}\sum_{n=0}^\infty i^{-(n+1)}B_n^{(m)}H_n^{(2)}\left(\frac{mk'l}{2}\right) \tag{67}$$

Hence, substituting (66) in (48) yields

$$v^{(m)}(x,0) = -\frac{1}{U}q^{(m)}(x) + \frac{imk'}{U}a^{(m)}(x) + \frac{imk'}{U}F^{(m)}(mk')e^{-imk'} \tag{68}$$

For the water surface elevation, differentiating (66) with respect to mk' as follows:

$$\frac{\partial}{\partial(mk')}\int_{-\infty}^x q^{(m)}(\xi)e^{imk'\xi}d\xi = i\int_{-\infty}^x \xi q^{(m)}(\xi)e^{imk'\xi}d\xi$$
$$= \frac{\partial}{\partial(mk')}F^{(m)}(mk') + b^{(m)}(x)e^{imk'x}$$

where $b^{(m)}(x)$ denotes a polynominal of x, (49) can thus be written in the form

$$\eta^{(m)}(x) = \frac{(imk' - 1)}{U^2}[e^{-imk'x}F^{(m)}(mk') + a^{(m)}(x)] - \frac{mk'}{U^2}[e^{-imk'x}\frac{\partial}{\partial(mk')}F^{(m)}(mk') + b^{(m)}(x)] \tag{69}$$

Therefore, the required condition on the planing surface becomes, from (35), (68), and (69)

$$F^{(m)}(mk') = 0 \tag{70}$$

and

$$\frac{\partial}{\partial(mk')}F^{(m)}(mk') = 0 \tag{71}$$

Finally, rewriting (70) and (71) yields

$$\sum_{n=0}^\infty i^{-(n+1)}B_n^{(m)}H_n^{(2)}\left(\frac{mk'l}{2}\right) = 0 \tag{72}$$

and

$$\sum_{n=0}^\infty i^{-(n+1)}B_n^{(m)}H_n^{*}\left(\frac{mk'l}{2}\right) = 0 \tag{73}$$

where

$$H_n^*\left(\frac{mk'l}{2}\right) = \frac{\partial}{\partial\left(\frac{mk'l}{2}\right)} H_n^{(2)}\left(\frac{mk'l}{2}\right) \quad (74)$$

Solution of a heaving motion

In order to solve the heaving motion, the required conditions (72) and (73) must be calculated.

The heaving motion may be written in the form

$$h = Re\{hae^{i\omega t}\} \quad (75)$$

where ha denotes the amplitude of the heaving motion and is a real number. The heaving amplitude is assumed to be very small.

For convenience sake, the following nondimensional numbers are defined as

$$\epsilon = \frac{ha}{ls} \quad (76)$$

and

$$k = \frac{k'ls}{2} = \frac{\omega ls}{2U} \quad (77)$$

where ϵ is the amplitude-wetted length ratio and k is a reduced frequency; ls denotes the steady-state wetted length of the planing surface.

In general, the wetted length varies with time in the form

$$c = \sum_{m=0}^{\infty} c^{(m)}e^{im\omega t} \quad (78)$$

where $c^{(m)}$ is a complex number. However, since the most important components are the ωt component and steady-state or stationary component, $m = 0$ and 1 are taken in the calculation. Then the foregoing equation reduces to

$$c = c^{(0)} + c^{(1)}e^{i\omega t} \quad (79)$$

where

$$c^{(1)} = c_c^{(1)} + ic_s^{(1)}, \quad c^{(0)} = ls \gg |C^{(1)}|$$

In a manner similar to (79), (40) and (41) may be written in the form

$$A_n = A_n^{(0)} + A_n^{(1)}e^{i\omega t} \quad (80)$$

$$B_n = B_n^{(0)} + B_n^{(1)}e^{i\omega t} \quad (81)$$

where

$$A_n^{(1)} = A_{nc}^{(1)} + iA_{ns}^{(1)} \quad (82)$$

$$B_n^{(1)} = B_{nc}^{(1)} + iB_{ns}^{(1)} \quad (83)$$

and $A_{nc}^{(1)}$, $A_{ns}^{(1)}$, $B_{nc}^{(1)}$, and $B_{ns}^{(1)}$ are real numbers.

The first term in (80) is determined in the following manner. Integrating (21) over the wetted length, only A_0 remains. Since A_0 represents the lift acting on the planing surface, making use of the steady-state solution [19]

$$L_0 = \frac{1}{2}\rho\pi lsU^2\alpha_0 \quad (84)$$

where α_0 denotes the angle of attack, and then substituting (84) into the integral of (21), gives

$$A_0^{(0)} = \alpha_0 U^2 \quad (85)$$

On the other hand, the Kutta condition for the heaving motion can be written

$$A_0 = A_1 - A_2 \quad (86)$$

because $A_3 = 0$ from (28).

The second term, A_2 in (86), reduces from (27) to

$$A_2 = \frac{1}{4}c\ddot{h} \quad (87)$$

Substitution of (75) and (79) in (87) gives

$$A_2 = -\frac{1}{4}ha\omega^2\left[\frac{1}{2}c_c^{(1)} + c_c^{(0)}\cos\omega t + \frac{1}{2}(c_c^{(1)}\cos 2\omega t - c_s^{(1)}\sin 2\omega t)\right] \quad (88)$$

Thus

$$\begin{cases} A_{2c}^{(0)} = -\frac{1}{8}ha\omega^2 c_c^{(1)} \\ A_{2s}^{(0)} = 0 \end{cases} \quad (89)$$

$$\begin{cases} A_{2c}^{(1)} = -\frac{1}{4}ha\omega^2 c_c^{(0)} = -\frac{1}{4}ha\omega^2 ls \\ A_{2s}^{(1)} = 0 \end{cases} \quad (90)$$

Using the Kutta condition, it follows that

$$\begin{cases} A_{1c}^{(0)} = \alpha_0 U^2 - \frac{1}{8}ha\omega^2 c_c^{(1)} \\ A_{1s}^{(0)} = 0 \end{cases} \quad (91)$$

It is necessary to evaluate B_n for the conditions (72) and (73). B_n consists of three integrals, that is

$$B_n = \frac{c}{l}\epsilon_n \sum_{m=0}^{3} A_m I_{n,m} \quad (92)$$

where

$$I_{n,m} = \frac{1}{\pi}\int_0^{\pi} \cos n\vartheta \cos m\theta d\theta \quad (93)$$

$$\cos\vartheta = -1 + \frac{c}{l}(1 + \cos\theta) \quad (94)$$

The integral (93) has the following properties:

$$I_{n,m} = 0, \quad m > n$$

and

$$\begin{aligned} I_{n+1,m} + I_{n-1,m} &= \frac{2}{\pi}\int_0^{\pi} \cos n\vartheta \cos\vartheta \cos m\theta d\theta \\ &= \frac{2}{\pi}\int_0^{\pi} \cos n\vartheta \left[\left(\frac{c}{l} - 1\right) + \frac{c}{l}\cos\theta\right]\cos m\theta d\theta \\ &= 2\left(\frac{c}{l} - 1\right)I_{n,m} + \frac{c}{l}(I_{n,m-1} + I_{n,m+1}) \end{aligned} \quad (95)$$

The integral $I_{n,m}$ can be calculated by making use of the recurrence formula (95).

Also the integral (93) can be expressed in the form

$$I_{n,m} = I_{n,m}^{(0)} + I_{n,m}^{(1)}e^{i\omega t} \quad (96)$$

where

$$I_{n,m}^{(1)} = I_{n,m_c}^{(1)} + iI_{n,m_s}^{(1)} \quad (97)$$

After some algebra, assuming $c^{(0)}/l \approx 1$ and neglecting higher-order terms, it follows that

$$I_{n,n}^{(0)} = \frac{1}{2}na', \quad n = m \quad (98)$$

$$I_{n,m}^{(0)} = 0, \quad n \neq m \quad (99)$$

and

$$I_{n,m}^{(1)} = na' \quad (100)$$

where

$$a' = \frac{1}{l}(c_c^{(1)} + ic_s^{(1)}) \approx \frac{1}{ls}(c_c^{(1)} + ic_s^{(1)})$$

On the other hand, for the heaving motion, from (28)

$$A_3 = 0$$

Therefore, B_n, (92), reduces to

$$B_n = \epsilon_n \left[\frac{c}{l} A_0 I_{n,0} + \frac{c}{l} A_1 I_{n,1} + \frac{c}{l} A_2 I_{n,2}\right] \quad (101)$$

The first-order terms in (101) may be written in the form

$$\left[\frac{c}{l} A_m I_{n,m}\right]^{(1)} \approx A_m^{(0)} I_{n,m}^{(1)} + I_{n,m}^{(0)}(A_m^{(1)} + a' A_m^{(0)}) \quad (102)$$

Thus

$$B_0^{(1)} = A_0^{(1)} + \alpha_0 U^2 a' \quad (103)$$

$$B_1^{(1)} = A_1^{(1)} + 4\alpha_0 U^2 a' \quad (104)$$

and

$$B_2^{(1)} = A_2^{(1)} + 8\alpha_0 U^2 a' \quad (105)$$

For $n \geq 3$, making use of (98), (99), and (100)

$$B_n^{(1)} = 4n\alpha_0 U^2 a', \quad n \geq 3 \quad (106)$$

When the wetted length is fixed constant—that is, if $a' = 0$ is set in (103) to (106)—the condition (72) yields

$$a_0^{(1)} H_0^{(2)}(k) - ia_1^{(1)} H_1^{(2)}(k) - a_2^{(1)} H_2^{(2)}(k) = 0 \quad (107)$$

where $a_n^{(1)}$ is used instead of $A_n^{(1)}$.

The solution of (107) can be written in the form

$$a_0^{(1)} = \left(-1 + i\frac{2}{k} C(k)\right) a_2^{(1)} \quad (108)$$

$$a_1^{(1)} = i\frac{2}{k} C(k) a_2^{(1)} \quad (109)$$

$$a_2^{(1)} = -\frac{1}{4} h a \omega^2 l s \quad (110)$$

where $C(k)$ is the Theodorsen function [17]

$$C(k) = \frac{H_1^{(2)}(k)}{H_1^{(2)}(k) + iH_0^{(2)}(k)} \quad (111)$$

Therefore, when the wetted length is assumed not to change with time, the solution of this problem reduces to the half value of the unsteady airfoil theory [17].

In the case of planing surfaces, however, since the wetted surface varies with motion, it is considered that (108) to (110) are a part of the solutions of (72) to (73), and the other part of the solutions must be added to represent the effects of the wetted length change. Hence, the solutions may be written in the form

$$A_0^{(1)} = a_0^{(1)} + \Delta a_0^{(1)} \quad (112)$$

$$A_1^{(1)} = a_1^{(1)} + \Delta a_1^{(1)} \quad (113)$$

$$A_2^{(1)} = a_2^{(1)} + \Delta a_2^{(1)} \quad (114)$$

where $\Delta a_0{}^{(1)}$, $\Delta a_1{}^{(1)}$, and $\Delta a_2{}^{(1)}$ denote the additional terms produced by the wetted length change.

The Fourier expansion of A_2, (88), shows that the effect of the first-order change in the wetted length does not appear in ωt components. It follows that

$$\Delta a_2^{(1)} = 0 \quad (115)$$

Thus, from the Kutta condition, (86):

$$\Delta a_1^{(1)} - \Delta a_2^{(1)} = 0 \quad (116)$$

Substitution of (108) to (110) and (112) to (116) into (72) and (73) yields

$$(H_0^{(2)} - iH_1^{(2)})\Delta a_0^{(1)} = -\alpha_0 U^2 a' \left[H_0^{(2)} + 4\sum_{n=1}^{\infty} (-i)^n n H_n^{(2)}\right] \quad (117)$$

$$(a_0^{(1)} H_0^* - ia_1^{(1)} H_1^* - a_2^{(1)} H_2^*) + (H_0^* - iH_1^*)\Delta a_0^{(1)} = -\alpha_0 U^2 a' \left[H_0^* + 4\sum_{n=1}^{\infty} (-i)^n n H_n^*\right] \quad (118)$$

The series expansion in (117) can be evaluated as follows using the formulas for the Hankel functions. Let I be the summation of the series, namely

$$I = \sum_{n=1}^{\infty} (-i)^n n H_n^{(2)}$$

Since

$$H_n^{(2)} = \frac{k}{2n}(H_{n-1}^{(2)} + H_{n+1}^{(2)})$$

then

$$I = \frac{1}{2} k \sum_{n=1}^{\infty} (-i)^n (H_{n-1}^{(2)} + H_{n+1}^{(2)})$$

Hence

$$I = -\frac{1}{2} k (H_1^{(2)} + iH_0^{(2)}) \quad (119)$$

Substitution of (119) in (117) provides

$$\Delta a_0 = i\alpha_0 U^2 a' \left(2k - \frac{H_0^{(2)}}{H_1^{(2)} + iH_0^{(2)}}\right) \quad (120)$$

In the same manner, (118) reduces to

$$(a_0^{(1)} H_0^* - ia_1^{(1)} H_1^* - a_2^{(1)} H_2^*) + (H_0^* - iH_1^*)\Delta a_0^{(1)} = -\alpha_0 U^2 a' [-H_1^{(2)} - 2kH_0^{(2)} - 2i(H_0^{(2)} - kH_1^{(2)})] \quad (121)$$

Combining (120) with (121) yields the wetted length change in the form

$$a' = \frac{1}{\alpha_0 U^2} \times \frac{-a_0^{(1)} H_1^{(2)} - ia_1^{(1)} \left(H_0^{(2)} - \frac{H_1^{(2)}}{k}\right) - a_2^{(1)} \left(H_1^{(2)} - \frac{4}{k^2} H_1^{(2)} + \frac{2}{k} H_0^{(2)}\right)}{3H_1^{(2)} + H_0^{(2)} \left[i - \frac{1}{k} C(k)\right]} \quad (122)$$

Substitution of (122) in (120) yields the solution for $\Delta a_0^{(1)}$.

When the reduced frequency becomes very small, the wetted length can be obtained by rewriting (122) in the form

$$a' \approx \frac{\epsilon}{\alpha_0} \cdot \frac{2C(k) - 4}{3 - \left(\frac{Y_0}{Y_1}\right)\frac{1}{k}} \approx \frac{\epsilon}{\alpha_0} \cdot \frac{2C(k) - 4}{3 + \log k} \quad (123)$$

Thus, for very small reduced frequency, say, around $k = 0.01$:

$$|a'| \approx \frac{\epsilon}{\alpha_0} \quad (124)$$

and

$$\lim_{k \to 0} a' = 0 \quad (125)$$

Therefore, the limiting case of the wetted length for zero frequency results in zero, which seems to be different from Ogilvie's analysis [13]. Physically (124) may be interpreted as the change

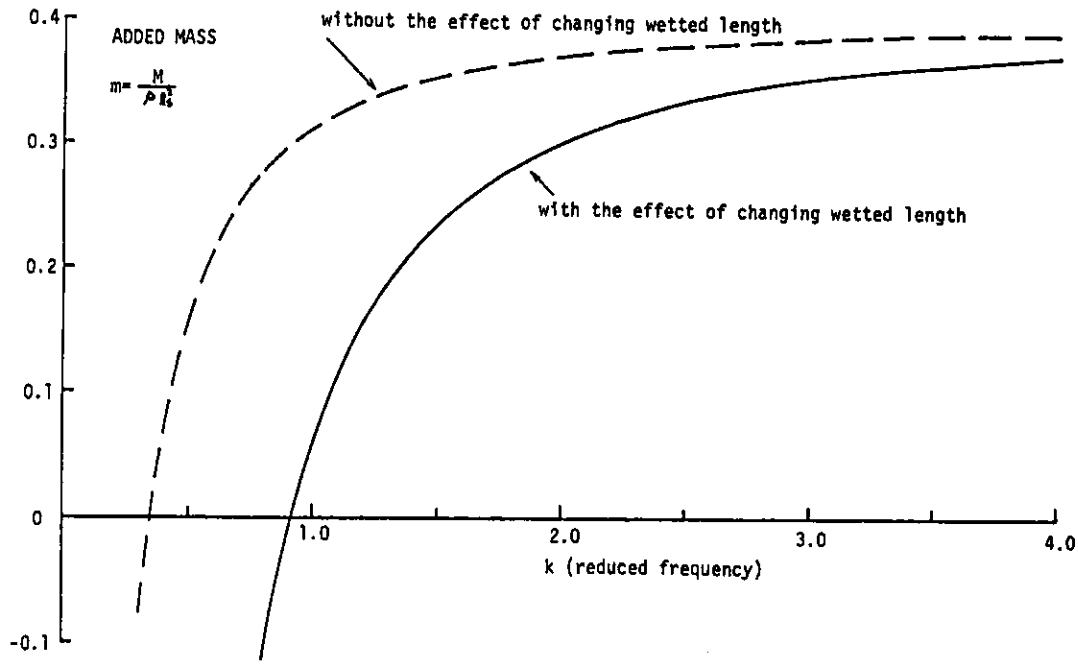

Fig. 2 Added mass

of the intersection of the plate with the calm water surface in a quasi-steady-state case.

The unsteady lift acting on the planing surface can be evaluated by integrating (92) over the wetted surface. Let $\mathcal{L}$ be the lift; then

$$\mathcal{L} = Re\{Le^{i\omega t}\} \tag{126}$$

where

$$L = Lc + iLs \tag{127}$$

Thus

$$L = \int_{l-c}^{l} pdx = \frac{1}{2}\pi\rho l s B_0^{(1)}$$
$$= \frac{1}{2}\pi\rho l s(A_0^{(1)} + aA_0^{(0)}) \tag{128}$$

where $A_0^{(1)}$ and $A_0^{(0)}$ are determined by (112) and (113) with (120) and (122).

On the other hand, using the added mass and damping coefficient, the unsteady lift can be written in the form

$$\mathcal{L} = -M\ddot{h} - N\dot{h}$$
$$= Mha\omega^2\cos\omega t + Nha\omega\sin\omega t \tag{129}$$

where M and N are the added mass and damping coefficient, respectively.

Combination of (128) and (129) yields the expression for the added mass and damping coefficient, and their nondimensional forms can be written in the form

$$m = \frac{M}{\rho l s^2} = \frac{1}{\rho l s^2 ha\omega^2} Lc \tag{130}$$

$$n = \frac{N}{\rho l s^2 \omega} = -\frac{1}{\rho l s^2 ha\omega^2} Ls \tag{131}$$

Finally, rewriting (130) and (131) together with (128) and the solutions obtained in the foregoing, it follows that

$$m - in = \frac{\pi}{8}\frac{1}{\epsilon k^2}\left(\frac{B_0^{(1)}}{U^2}\right) \tag{132}$$

In the nondimensional forms, the added mass and damping coefficient become independent of the trim angle and amplitude of the heaving motion, because $B_0^{(1)}$ is proportional to $\epsilon k^2 U^2$, while the wetted length change is a function of the trim angle. The trim angle determines the steady-state wetted length, which has the effect on the added mass and damping coefficient implicitly in (132).

To examine the effect of Δa_0 in $B_0^{(1)}$ in low frequency, taking the nondimensional form

$$\frac{\pi}{8}\frac{1}{\epsilon k^2}\frac{\Delta a_0^{(1)}}{U^2} = \frac{\pi}{8}\frac{i\alpha_0 a'}{\epsilon k^2}\left(2k - \frac{H_0^{(2)}}{H_1^{(2)} + iH_0^{(2)}}\right) \tag{133}$$

then Δa_0 appears to have a crucial effect on the added mass and damping coefficient.

Figure 2 shows the nondimensional values of the added mass. The values calculated by (132) including the effect of the wetted length change are higher than the result by (108) with no effect of the wetted length change.

The damping coefficients are shown in Fig. 3. The damping coefficient with the effect of the wetted length change has almost the same tendency as the calculation without change of the wetted length in the reduced frequency larger than about 0.4, although the former has higher values. However, it is observed that in the low-frequency range these two calculations have quite different tendencies; that is, the calculation by the analysis with the wetted length effect decreases very quickly when the reduced frequency is smaller than about 0.4, and finally reaches negative value around $k = 0.176$. According to Mottard's [20, 21] experiments, the planing hull which has only heaving motion suffers from the self-excited vibrations. The reduced frequency causing the self-excited vibrations appears to be between around 0.1 and 0.2. The critical reduced frequencies of Mottard's experimental data were estimated by making use of the well-known formula for aspect ratio and lift coefficient. Attention is paid here to the fact that in the case of a planing surface the lift coefficient becomes half of the airfoil case. Figure 4 shows the relation between the critical reduced frequencies and the aspect ratios along with the results of the two-dimensional theories. It is observed that the present analysis agrees very well with the experimental data for large as-

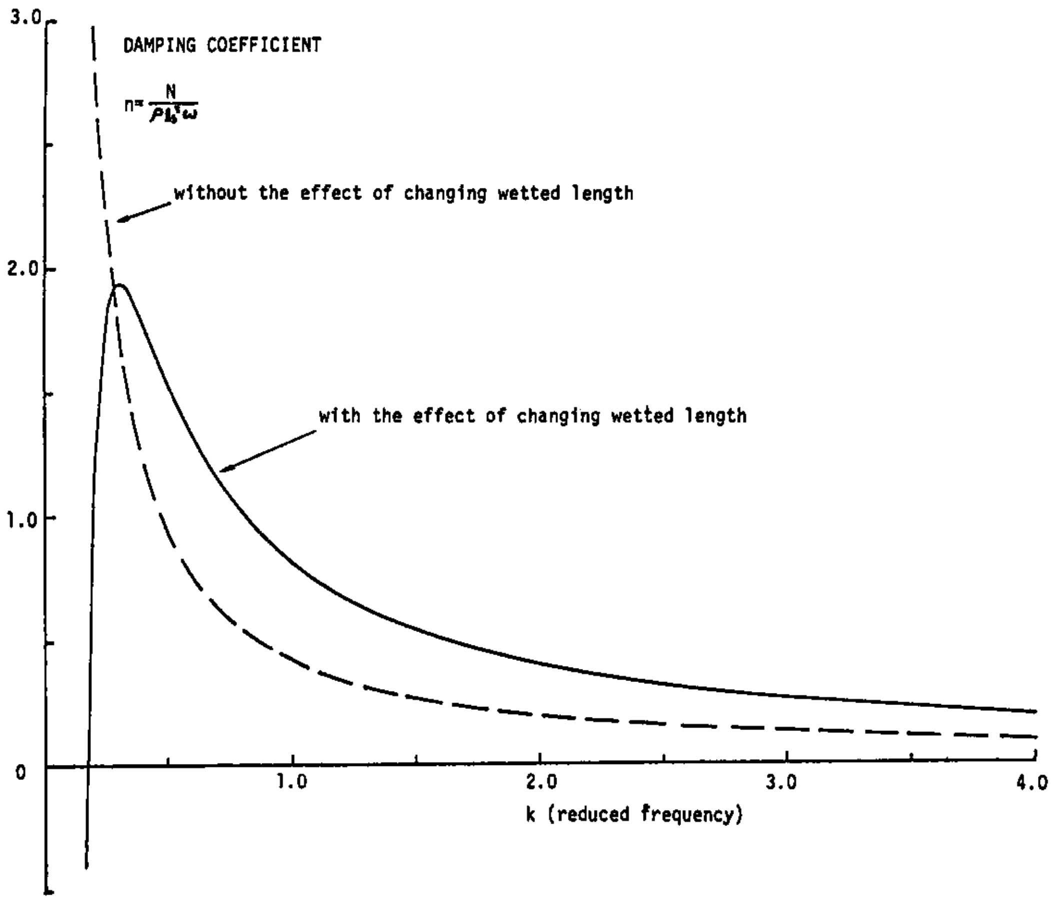

Fig. 3 Damping coefficient

pect ratios. An interesting feature of the experimental data is that the critical reduced frequency becomes small as the aspect ratio decreases. This may suggest that the instability with only heaving motion occurs when the aspect ratio of the planing surface is very large.

It has been found that the present analysis coincides exactly with the result by Bessho's theory [15] derived from the other point of view.

The wetted length change decreases as the reduced frequency increases, as shown in Fig. 5. The behavior of the wetted length around $k = 0$, which is approximately the displacement of the intersection of the planing surface and the undisturbed water level, seems natural physically.

Conclusion

A two-dimensional unsteady problem has been analyzed by making use of acceleration and velocity potentials on the assumption that the effect of gravity is negligible, that is, that the Froude number is very large. The analysis is based on the fact that the planing surface oscillates as a rigid body without deformation while it is set in motion. The calculation has been carried out for a heaving motion.

The analysis includes the effect of the time-varying wetted length on the hydrodynamic forces. The wetted length change appears to be different from that calculated by Ogilvie [13], especially for very small reduced frequency. For very small reduced frequency, the wetted length change becomes the quasi-steady-state displacement of the intersection of the planing surface and the calm water surface elevation. This result seems natural physically. The wetted length change decreases as the reduced frequency increases and is proportional to the amplitude of the heaving motion and inversely proportional to the trim angle.

The general tendency of the added mass and damping coefficient is similar to those by airfoil theory, except for very small reduced frequency, but is different in quantity. It is noted that the damping coefficient becomes negative when the reduced frequency reaches around 0.2. The result is consistent when compared with the experiments of Mottard [20, 21].

The present analysis coincides exactly with Bessho's theory [15] derived from the different concept.

In the future, an experiment appropriate for the present analysis is desired. In order to apply the two-dimensional solution to a three-dimensional problem, there must exist problems to be solved. However, the present analysis may be used to a certain extent with a proper approximation for porpoising and the motions of planing boats in waves.

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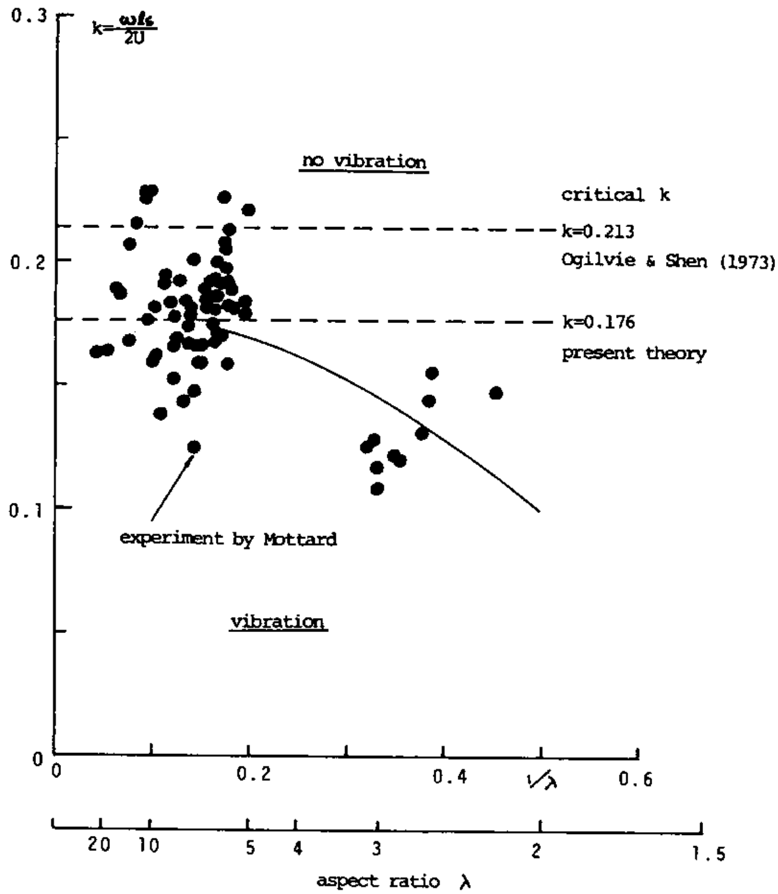

Fig. 4 Critical reduced frequency for planing surfaces

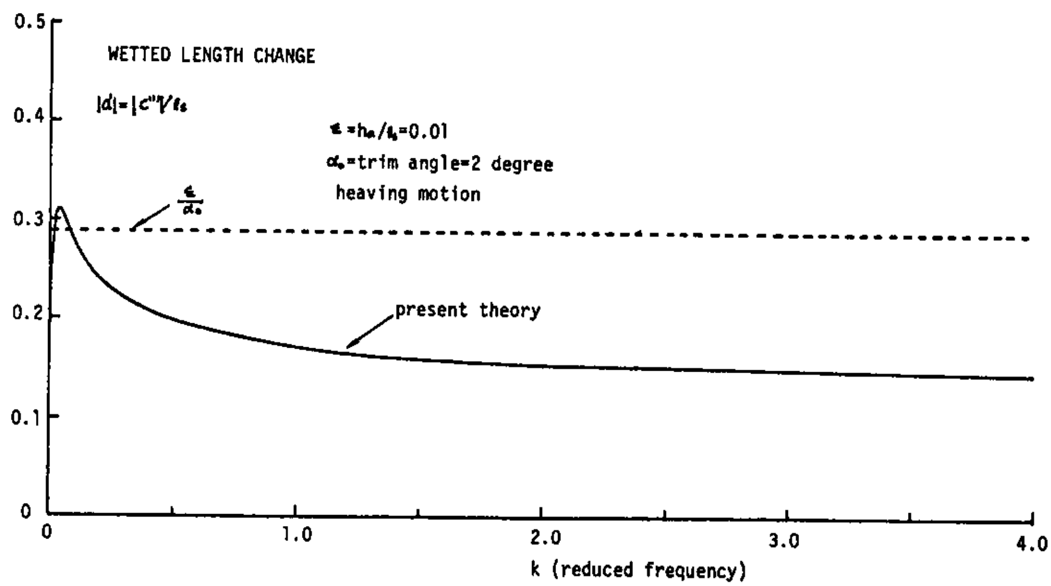

Fig. 5 Wetted length change

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