E　M-58

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大改訂

3次元弾性の場合

1、応力と歪，基本特異性

2，Bettiの表現等

3. 軸対称の場合

附録　無限長丸棒中の軸対称特異性

1. 応力と歪, 基本特質性

変位のベクトルを $\boldsymbol{u}\,(u_1, u_2, u_3)$ とすると 歪 e_{ij} は.
$(x \equiv x_1,\ y \equiv x_2,\ z \equiv x_3)$

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad (1.1)$$

体積換は $$\gamma = \sum_{i=1}^{3} e_{ii}, \qquad (1.2)$$

フックの法則では 応力 f'_{ij} が次の如く表わされる。

$$\frac{f'_{ij}}{G} \equiv f_{ij} = 2e_{ij} + \frac{2\nu}{1-2\nu}\gamma\delta_{ij}, \qquad (1.3)$$

こゝに $\delta_{ij} = 0$ for $i \neq j$, $\delta_{ii} = 1$,

弾性定数間の関係は.

$$G = \frac{E}{2(1+\nu)} = \mu, \quad \lambda = \frac{2\nu}{1-2\nu}G,$$

Bergman, Schiffer の Text の記号との関係.

$$\alpha = \frac{1}{1-2\nu}, \quad A = -\frac{\alpha}{2(\alpha+1)} = \frac{-1}{4(1-\nu)}, \quad B = 1.$$

$$2A + B = \frac{1-2\nu}{2(1-\nu)},$$

境界力も G で割ってそれを $\boldsymbol{\tau}\,(\tau_1, \tau_2, \tau_3)$ とすると.

$$\tau_i = \sum_{j=1}^{3} f_{ij}\frac{\partial x_j}{\partial n} = 2\sum_{j=1}^{3} e_{ij}\frac{\partial x_j}{\partial n} + \frac{\nu}{1-2\nu}\gamma\frac{\partial x_i}{\partial n}, \qquad (1.4)$$

なお (1.3) より

$$\sigma = \sum_{i=1}^{3} f_{ii} = \frac{2(1+\nu)}{1-2\nu}\gamma, \qquad (1.5)$$

適合条件式は

$$\Delta u_i + \frac{1}{1-2\nu}\frac{\partial \theta}{\partial x_i} = 0, \quad i=1,2,3, \qquad (1.6)$$

今基本特異性として上式を満足する次のものを考える。

$$\left.\begin{aligned} 4\pi U_i^{(k)}(P,Q) &= \frac{\delta_{ik}}{R} - \frac{1}{4(1-\nu)}\frac{\partial^2 R}{\partial x_i \partial x_k}, \\ P &\equiv (x_1, x_2, x_3), \quad Q \equiv (x_1', x_2', x_3'), \quad R = \overline{PQ} \\ U_i^{(k)}(P,Q) &= U_i^{(k)}(Q,P), \quad \Delta R = \frac{2}{R} \end{aligned}\right\} \qquad (1.7)$$

これによる歪は (1) より

$$\left.\begin{aligned} 4\pi E_{ij}^{(k)}(P,Q) &= \frac{1}{2}\left(\frac{\partial}{\partial x_i}\frac{\delta_{jk}}{R} + \frac{\partial}{\partial x_j}\frac{\delta_{ik}}{R}\right) \\ &\quad - \frac{1}{4(1-\nu)}\frac{\partial^3 R}{\partial x_i \partial x_j \partial x_k}, \\ E_{ij}^{(k)}(P,Q) &= -E_{ij}^{(k)}(Q,P), \end{aligned}\right\} \qquad (1.8)$$

(2) より

$$4\pi \Gamma^{(k)}(P,Q) = \frac{1-2\nu}{2(1-\nu)}\frac{\partial}{\partial x_k}\frac{1}{R} = -4\pi\Gamma(Q,P), \qquad (1.9)$$

これによる境界力は

$$\begin{aligned} 4\pi T_i^{(k)}(U^{(k)}; P,Q) &= \delta_{ik}\frac{\partial}{\partial n}\left(\frac{1}{R}\right) + \frac{\partial x_k}{\partial n}\frac{\partial}{\partial x_i}\frac{1}{R} + \frac{\nu}{1-\nu}\frac{\partial x_i}{\partial n}\frac{\partial}{\partial x_k}\frac{1}{R} \\ &\quad - \frac{1}{2(1-\nu)}\frac{\partial^3 R}{\partial n \partial x_i \partial x_k}, \\ &= 8\pi\frac{\partial}{\partial n}U_i^{(k)}(P,Q) + \frac{\partial x_k}{\partial n}\frac{\partial}{\partial x_i}\frac{1}{R} + \frac{\nu}{1-\nu}\frac{\partial x_i}{\partial n}\frac{\partial}{\partial x_k}\frac{1}{R} \\ &\quad - \delta_{ik}\frac{\partial}{\partial n}\frac{1}{R}, \end{aligned} \qquad (1.10)$$

2. Bettiの表現

歪エネルギーは

$$A = \frac{G}{2}\iiint_D \sum_{i,j=1}^{3} f_{ij}\, e_{ij}\, d\tau, \quad d\tau = dx\,dy\,dz,$$

$$= G\iiint_D \left[\sum_{i,j=1}^{3} e_{ij}^2 + \frac{\nu}{1-2\nu}\left(\sum_{i=1}^{3} e_{ii}\right)^2\right] d\tau, \quad \cdots (2.1)$$

であるから 2次形式

$$E(\mathbf{u},\mathbf{v}) = \iiint_D \left[\sum_{i,j=1}^{3} \frac{\partial u_i}{\partial x_j}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) + \frac{2\nu}{1-2\nu}\,\mathrm{div}\,\mathbf{u}\,\mathrm{div}\,\mathbf{v}\right] d\tau \quad (2.2)$$

を導入すると

$$e(\mathbf{u},\mathbf{v}) = e(\mathbf{v},\mathbf{u}) \quad (2.3)$$

部分積分して

$$e(\mathbf{u},\mathbf{v}) = \iint_S \mathbf{u}\,\boldsymbol{\tau}(\mathbf{v})\,dS - \iiint_D \mathbf{u}\left\{\Delta\mathbf{v} + \frac{1}{1-2\nu}\,\mathrm{grad}\,\mathrm{div}\,\mathbf{v}\right\} d\tau, \quad (2.4)$$

となり (1.6) が成立つとすると右辺第2項は消えて (2.3)より Maxwell ~ Betti の相反定理が成立つ

$$\iint_S \left[\mathbf{u}\,\boldsymbol{\tau}(\mathbf{v}) - \mathbf{v}\,\boldsymbol{\tau}(\mathbf{u})\right] dS = 0, \quad (2.5)$$

(Bergman-Schifferの本では法線 D に内向, 今は外向とする)

今 $\boldsymbol{v}$ として (1.7) を考えると 常用手段により

$$\lim_{\varepsilon \to 0} \iint_{\varepsilon} \frac{x_i x_j}{r^4} ds = \frac{4}{3}\pi \delta_{ij}$$

により

$$\boldsymbol{u}(Q) = \iint_S \left[\boldsymbol{\tau}(\boldsymbol{u}(P)) U(P,Q) - \boldsymbol{u}(P) \boldsymbol{T}(U(P,Q))\right] ds_P$$

ここに U はマトリックスで 上式を成分毎に書くと
前節の記号を使って、

$$u_k(Q) = \iint_S \sum_{i=1}^{3} \left[\tau_i(P) U_i^{(k)}(P,Q) - u_i(P) T_i^{(k)}(P,Q)\right] ds(P),$$

$$k=1,2,3 \qquad (2.6)$$

応力等は前節の式を使って計算すればよい。
外部問題（無限遠点を含む領域の）では、
吹出し式表現が便利な事があり、それは

$$u_k(Q) = \iint_S \sum_{i=1}^{3} \tau_i(P) U_i^{(k)}(P,Q) ds(P), \quad (2.7)$$

$$u_k(Q) = - \iint_S \sum_{i=1}^{3} u_i T_i^{(k)}(P,Q) ds(P), \quad (2.8)$$

のようになるが 応力集中問題では これを使用する
(2.8) が便利であるが、この時 右辺の核関数

であるから(2.9)から(2.10)を差引いて、

$$u_k(Q) = \iint_S \sum_{i=1}^{3} (u_i + u_i')\, T_i^{(k)} \, ds_P, \quad \text{in } \bar{D}', \qquad (2.12)$$

を得る。

ここに u_i' は(2.11)によるから $(u_i + u_i')$ は一様引張りも含めた全変位となる。

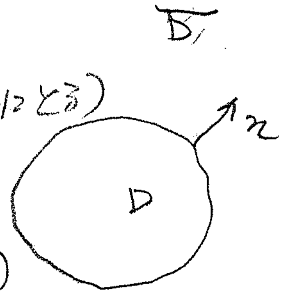
これを次のように書けばこれは S 上で $(u_k + u_k')$ に関するフレドホルムの第2種方程式となる

$$u_k(Q) + u_k'(Q) - \sum_i \iint_S (u_i + u_i')\, T_i^{(k)} \, dS = u_k'(Q), \qquad (2.13)$$

について(1.13)のように応力を計算するのは大変である。

そこで元に戻って今は外部領域なので

(2.6)は符号が変って(法線はそのまま図のようにとる)



$$u_k(Q) = -\iint_S \sum_{i=1}^{3} \left(T_i U_i^{(k)} - u_i T_i^{(k)}\right) ds(p), \quad (2.9)$$

in $\bar{D}$

内部領域で考えると

$$\iint_S \sum_i^{3} \left(T_i U_i^{(k)} - u_i' T_i^{(k)}\right) dsp = \begin{cases} u_k(Q) & \text{in } D \\ 0 & \text{in } \bar{D} \end{cases} \quad (2.10)$$

となるから例えば$\overbrace{\text{一様引張り}}^{\text{一方向の}}$を受ける場合の

応力集中問題では、S上で

$$f_{ij} = 0 \text{ for } i \neq j, \quad f_{11} = 1, \quad f_{22} = f_{33} = 0$$

であるから外部の境界条件は

$$T_1 = -\frac{\partial x_1}{\partial n}, \quad T_2 = T_3 = 0 \quad \cdots \quad (2.10)$$

一方(2.10)については

$$e_{ij} = \frac{1}{2} f_{ij} - \frac{\nu}{1-2\nu} \gamma \delta_{ij} = \frac{1}{2} f_{ij} - \frac{\nu}{2(1+\nu)} \delta_{ij}$$

$$\frac{\partial u_1'}{\partial x_2} = e_{11} = \frac{1}{2(1+\nu)}, \quad e_{22} = -\frac{\nu}{2(1+\nu)} = \frac{\partial u_2}{\partial x_2} = \frac{\partial u_3}{\partial x_3}$$

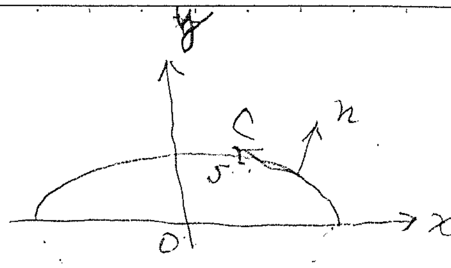
$$e_{ij} = 0 \text{ for } i \neq j$$

$$\therefore \left. u_1' = \frac{x_1}{2(1+\nu)}, \quad u_2' = -\frac{\nu x_2}{2(1+\nu)}, \quad u_3' = -\frac{\nu x_3}{2(1+\nu)} \right\} \quad (2.11)$$

3. 軸対称の場合

x軸のまわりに対称としよう。

乱れ応力分布も



円筒座標を導入し

$$x_1 \equiv x,\quad x_2 \equiv r\cos\theta,\quad x_3 \equiv r\sin\theta,$$

変位は

$$u_1 \equiv u(x,r),\quad u_2 = v(x,r)\cos\theta,\quad u_3 = v(x,r)\sin\theta, \qquad (3.1)$$

$$\left.\begin{aligned}
&e_{11} = \frac{\partial u}{\partial x},\quad e_{22}\Big|_{\theta=0} = \cos^2\theta\frac{\partial v}{\partial r} + \frac{\sin^2\theta}{r}v\Big|_{\theta=0} = \frac{\partial v}{\partial r}\\
&e_{33}\Big|_{\theta=0} = \sin^2\theta\frac{\partial v}{\partial r} + \frac{\cos^2\theta}{r}v\Big|_{\theta=0} = \frac{v}{r},\\
&e_{12}\Big|_{\theta=0} = \frac{1}{2}\left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial x}\right)\cos\theta\Big|_{\theta=0} = \frac{1}{2}\left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial x}\right),\\
&e_{23}\Big|_{\theta=0} = \left(\frac{\partial v}{\partial r} - \frac{v}{r}\right)\sin\theta\cos\theta\Big|_{\theta=0} = 0,
\end{aligned}\right\} \qquad (3.2)$$

以下e_{ij}等の記号はすべて上のように$\theta=0$の値を意味するものとする。

応力は

$$\left.\begin{aligned}
&t_{ij} = 2e_{ij} + \frac{2\nu}{1-2\nu}\gamma\delta_{ij}\\
&t_{11} = 2\frac{\partial u}{\partial x} + \frac{2\nu}{1-2\nu}\gamma,\quad t_{12} = \frac{\partial u}{\partial r} + \frac{\partial v}{\partial x},\\
&t_{22} = 2\frac{\partial v}{\partial r} + \frac{2\nu}{1-2\nu}\gamma,\quad t_{33} = \frac{2v}{r} + \frac{2\nu}{1-2\nu}\gamma\\
&\gamma = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial r} + \frac{v}{r},\\
&\sum_{i=1}^{3} t_{ii} = 2\gamma + \frac{6\nu\gamma}{1-2\nu} = \frac{2(1+\nu)}{1-2\nu}\gamma
\end{aligned}\right\} \qquad (3.3)$$

境界力は

$$\left.\begin{aligned}\tau_1 &= 2\frac{\partial u}{\partial n} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial r}\right)\frac{\partial r}{\partial n} + \frac{2\nu}{1-2\nu}\gamma\frac{\partial x}{\partial n} \\ \tau_2 &= 2\frac{\partial v}{\partial n} - \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial r}\right)\frac{\partial x}{\partial n} + \frac{2\nu}{1-2\nu}\gamma\frac{\partial r}{\partial n}\end{aligned}\right\} \quad (3.4)$$

特に自由境界では

$$\tau_1 = \tau_2 = 0 \qquad , \qquad (3.5)$$

であるから このときは (3.4)から

$$\left.\begin{aligned}&\frac{\partial u}{\partial n}\frac{\partial x}{\partial n} + \frac{\partial v}{\partial n}\frac{\partial r}{\partial n} + \frac{\nu}{1-2\nu}\gamma = 0 \\ &2\left(\frac{\partial u}{\partial n}\frac{\partial r}{\partial n} - \frac{\partial v}{\partial n}\frac{\partial x}{\partial n}\right) + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial r}\right) = 0\end{aligned}\right\} \quad (3.6)$$

なる関係となる。

これを代入すると (3.3)の γ は.

$$\gamma = \frac{\partial u}{\partial s}\frac{\partial x}{\partial s} + \frac{\partial v}{\partial s}\frac{\partial r}{\partial s} + \frac{\partial u}{\partial n}\frac{\partial x}{\partial n} + \frac{\partial v}{\partial n}\frac{\partial r}{\partial n} + \frac{v}{r}$$

$$= \frac{\partial u}{\partial s}\frac{\partial x}{\partial s} + \frac{\partial v}{\partial s}\frac{\partial r}{\partial s} + \frac{v}{r} - \frac{\nu}{1-2\nu}\gamma$$

故 $$\gamma = \frac{1-2\nu}{1-\nu}\left[\frac{\partial u}{\partial s}\frac{\partial x}{\partial s} + \frac{\partial v}{\partial s}\frac{\partial r}{\partial s} + \frac{v}{r}\right], \quad \cdots (3.7)$$

線素に沿う応力を σ_s とすると

$$\sigma_s = 2\left[\frac{\partial u}{\partial x}\left(\frac{\partial x}{\partial s}\right)^2 + \frac{\partial v}{\partial r}\left(\frac{\partial r}{\partial s}\right)^2 + \frac{\partial x}{\partial s}\frac{\partial r}{\partial s}\left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial x}\right)\right] + \frac{2\nu}{1-2\nu}\gamma,$$

$$= 2\left(\frac{\partial u}{\partial s}\frac{\partial x}{\partial s} + \frac{\partial v}{\partial s}\frac{\partial r}{\partial s}\right) + \frac{2\nu}{1-2\nu}\gamma$$

となるから (3.7)を代入すると

$$\sigma_s = \frac{2\gamma}{1-2\nu} - \frac{2v}{r}, \quad \cdots\cdots \quad (3.8)$$

一方 θ方向の応力 $\sigma_\theta \equiv f_{33}$ であるから

$$\sigma_s + \sigma_\theta = \frac{2(1+\nu)}{1-2\nu}\cdot\gamma = \sum_{i=1}^{3} f_{ii}, \quad \cdots\cdots \quad (3.9)$$

これらの式から自由面の応力は u と v の s 方向の微分によって計算出来る事がわかる。

前同様 suffix をえらぶ事とすると基本特異性は

$$
\left.
\begin{aligned}
U_1^1 &= \frac{1}{4\pi}\int_0^{2\pi} d\theta\left[\frac{1}{R} - \frac{1}{4(1-\nu)}\frac{\partial^2 R}{\partial x^2}\right], \\
& \qquad R^2 = (x-x')^2 + r^2 + r'^2 - 2rr'\cos\theta, \\
U_2^1 &= \frac{1}{4\pi}\int d\theta\left[-\frac{1}{4(1-\nu)}\frac{\partial^2 R}{\partial x\partial r}\right], \\
U_1^2 &= \frac{1}{4\pi}\int_0^{2\pi} d\theta\left[-\frac{1}{4(1-\nu)}\frac{\partial^2 R}{\partial r\partial x}\right], \\
U_2^2 &= \;'' \left[\frac{1}{R} - \frac{1}{4(1-\nu)}\frac{\partial^2 R}{\partial r^2}\right],
\end{aligned}
\right\} \quad (3.10)
$$

$$
\left.
\begin{aligned}
T_1^1 &= \;'' \left[\frac{\partial}{\partial n}\frac{1}{R} + \frac{1}{1-\nu}\frac{\partial x}{\partial n}\frac{\partial}{\partial x}\frac{1}{R} - \frac{1}{2(1-\nu)}\frac{\partial^3 R}{\partial n\partial x^2}\right], \\
T_2^1 &= \;'' \left[\frac{\partial x}{\partial n}\frac{\partial}{\partial r}\frac{1}{R} + \frac{\nu}{1-\nu}\frac{\partial r}{\partial n}\frac{\partial}{\partial x}\frac{1}{R} - \frac{1}{2(1-\nu)}\frac{\partial^3 R}{\partial n\partial x\partial r}\right], \\
T_1^2 &= \;'' \left[\frac{\partial r}{\partial n}\frac{\partial}{\partial x}\frac{1}{R} + \frac{\nu}{1-\nu}\frac{\partial x}{\partial n}\frac{\partial}{\partial r}\frac{1}{R} - \frac{1}{2(1-\nu)}\frac{\partial^3 R}{\partial n\partial x\partial r}\right], \\
T_2^2 &= \;'' \left[\frac{\partial}{\partial n}\frac{1}{R} + \frac{1}{1-\nu}\frac{\partial r}{\partial n}\frac{\partial}{\partial r}\frac{1}{R} - \frac{1}{2(1-\nu)}\frac{\partial^3 R}{\partial n\partial r^2}\right],
\end{aligned}
\right\} \quad (3.11)
$$

所で $\dfrac{\partial^3 R}{\partial n\,\partial x^2} = \dfrac{\partial^2}{\partial n\,\partial x}\dfrac{x}{R} = \dfrac{\partial}{\partial n}\left(\dfrac{1}{R} + x\dfrac{\partial}{\partial x}\dfrac{1}{R}\right)$

$$= 2\frac{\partial x}{\partial n}\frac{\partial}{\partial x}\frac{1}{R} + \frac{\partial r}{\partial n}\frac{\partial}{\partial r}\frac{1}{R} + x\frac{\partial^2}{\partial n\,\partial x}\frac{1}{R},$$

（但し $x'=0$ としてある。）

$$\frac{\partial^3 R}{\partial n\,\partial x\,\partial r} = \frac{\partial^2}{\partial n\,\partial r}\frac{x}{R} = \frac{\partial x}{\partial n}\frac{\partial}{\partial r}\frac{1}{R} + x\frac{\partial^2}{\partial n\,\partial r}\frac{1}{R},$$

$$\frac{\partial^3 R}{\partial n\,\partial r^2} = \frac{\partial}{\partial n}\left[\frac{2}{R} - \frac{1}{r}\frac{\partial R}{\partial r} - \frac{\partial^2 R}{\partial x^2}\right]$$

$\because \left(\dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{\partial^2}{\partial x^2}\right)R = \dfrac{2}{R}$ （但し θ で積分したとして）

$$\frac{\partial^3 R}{\partial n\,\partial r^2} = \frac{\partial}{\partial n}\left[\frac{1}{R} - x\frac{\partial}{\partial x}\frac{1}{R} - \frac{1}{r}\frac{\partial R}{\partial r}\right] = \frac{\partial r}{\partial n}\frac{\partial}{\partial r}\frac{1}{R} - x\frac{\partial^2}{\partial n\,\partial x}\frac{1}{R} + \frac{1}{r^2}\frac{\partial r}{\partial n}\frac{\partial R}{\partial r} - \frac{1}{r}\frac{\partial^2 R}{\partial n\,\partial r},$$

$$T_1^1 = 〃\left[\frac{\partial x}{\partial n}\frac{\partial}{\partial x}\frac{1}{R} + \frac{1-2\nu}{2(1-\nu)}\frac{\partial r}{\partial n}\frac{\partial}{\partial r}\frac{1}{R} - \frac{x}{2(1-\nu)}\frac{\partial^2}{\partial n\,\partial x}\frac{1}{R}\right],$$

$$T_2^1 = 〃\left[\frac{\nu}{1-\nu}\frac{\partial r}{\partial n}\frac{\partial}{\partial x}\frac{1}{R} + \frac{1-2\nu}{2(1-\nu)}\frac{\partial x}{\partial n}\frac{\partial}{\partial r}\frac{1}{R} - \frac{x}{2(1-\nu)}\frac{\partial^2}{\partial n\,\partial r}\frac{1}{R}\right]$$

$$T_1^2 = 〃\left[\frac{\partial r}{\partial n}\frac{\partial}{\partial x}\frac{1}{R} - \frac{1-2\nu}{2(1-\nu)}\frac{\partial x}{\partial n}\frac{\partial}{\partial r}\frac{1}{R} - \frac{x}{2(1-\nu)}\frac{\partial^2}{\partial n\,\partial r}\frac{1}{R}\right]. \qquad (3.12)$$

$$T_2^2 = 〃\left[\frac{\partial x}{\partial n}\frac{\partial}{\partial x}\frac{1}{R} + \frac{3-2\nu}{2(1-\nu)}\frac{\partial r}{\partial n}\frac{\partial}{\partial r}\frac{1}{R} + \frac{1}{2(1-\nu)}\left\{x\frac{\partial^2}{\partial n\,\partial x}\frac{1}{R} - \frac{1}{r^2}\frac{\partial r}{\partial n}\frac{\partial R}{\partial r} - \frac{1}{r}\frac{\partial^2 R}{\partial n\,\partial r}\right\}\right]$$

$$= 〃\left[\frac{\partial x}{\partial n}\frac{\partial}{\partial x}\frac{1}{R} + \frac{2-\nu}{1-\nu}\frac{\partial r}{\partial n}\frac{\partial}{\partial r}\frac{1}{R} - \frac{1}{2(1-\nu)}\frac{\partial^3 R}{\partial n\,\partial r^2}\right]$$

今 $r_1^2 = (x-x')^2 + (r-r')^2, \quad r_2^2 = (x-x')^2 + (r+r')^2.$

$$k^2 = \frac{4rr'}{r_2^2} = 1 - k'^2, \quad k'^2 = (r_1/r_2)^2.$$

とおくと
$$R^2 = (x-x')^2 + r^2 + r'^2 - 2rr'\cos\theta$$
$$= r_2^2\left(1 - k^2\cos^2\frac{\theta}{2}\right)$$

故に

$$\left.\begin{aligned} \frac{1}{4\pi}\int_0^{2\pi}\frac{d\theta}{R} &= \frac{1}{\pi r_2}\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} = \frac{K(k)}{\pi r_2} \\ \frac{1}{4\pi}\int_0^{2\pi}R\,d\theta &= \frac{r_2}{\pi}\int_0^{\frac{\pi}{2}}\sqrt{1-k^2\sin^2\varphi}\,d\varphi = \frac{r_2 E(k)}{\pi} \end{aligned}\right\} \quad (3.13)$$

$$\left.\begin{aligned} \dot{K} &= \frac{dK}{d\mu} = k\frac{dK}{dk} = \frac{E}{k'^2} - K, \\ \dot{E} &\equiv \frac{dE}{d\mu} = k\frac{dE}{dk} = E - K. \end{aligned}\right\} \quad \mu = \log k \quad (3.14)$$

$$\frac{\partial\mu}{\partial x} = \frac{1}{k}\frac{\partial k}{\partial x} = -\frac{x}{r_2^2}, \quad \frac{1}{k}\frac{\partial k}{\partial r} = \frac{1}{2r} - \frac{r+r'}{r_2^2} = \frac{\partial\mu}{\partial r} \quad (x'=0 とする)$$

$$\left.\begin{aligned} \frac{\partial K}{\partial x} &= \frac{\partial k}{\partial x}\frac{dK}{dk} = \frac{x}{r_2^2}\left(K - \frac{E}{k'^2}\right), \\ \frac{\partial K}{\partial r} &= \left(\frac{1}{2r} - \frac{r+r'}{r_2^2}\right)\left(\frac{E}{k'^2} - K\right), \\ \frac{\partial E}{\partial x} &= \frac{x}{r_2^2}(K - E), \\ \frac{\partial E}{\partial r} &= \left(\frac{1}{2r} - \frac{r+r'}{r_2^2}\right)(E - K), \end{aligned}\right\} \quad (3.15)$$

$$\frac{\partial}{\partial r}(E-K) = -\frac{k^2}{k'^2}\left(\frac{1}{2r} - \frac{r+r'}{r_2^2}\right)E$$

$$\frac{\partial}{\partial x}\frac{K}{r_2} = -\frac{x}{r_2^3}K + \frac{x}{r_2^3}\left(K - \frac{E}{k'^2}\right) = -\frac{x}{r_2 r_1^2}E ,$$

$$\frac{\partial}{\partial r}\frac{K}{r_2} = -\frac{r+r'}{r_2^3}K + \frac{1}{r_2}\left(\frac{1}{2r} - \frac{r+r'}{r_2^2}\right)\left(\frac{E}{k'^2} - K\right)$$

$$= -\frac{K}{2r r_2} + \frac{r_2}{r_1^2}\left(\frac{1}{2r} - \frac{r+r'}{r_2^2}\right)E ,$$

$$\frac{\partial}{\partial x} r_2 E = \frac{x}{r_2}E + \frac{x}{r_2}(K - E) = \frac{x}{r_2}K .$$

$$\frac{\partial}{\partial r} r_2 E = \frac{r+r'}{r_2}E + r_2\left(\frac{1}{2r} - \frac{r+r'}{r_2^2}\right)(E - K)$$

$$= \frac{r_2}{2r}E - \left(\frac{r_2}{2r} - \frac{r+r'}{r_2}\right)K$$

$$\frac{\partial^2}{\partial x^2} r_2 E = \frac{K}{r_2} - \frac{x^2}{r_1^2 r_2}E , \quad \text{o.K.}$$

$$\frac{\partial^2}{\partial x \partial r} r_2 E = -\frac{x}{2r r_2}K + \frac{x r_2}{r_1^2}\left(\frac{1}{2r} - \frac{r+r'}{r_2^2}\right)E .$$

$$\frac{\partial^2}{\partial r^2} r_2 E = -\frac{r_2}{2r^2}E + \frac{1}{2r}\left[\frac{r_2}{2r}E - \left(\frac{r_2}{2r} - \frac{r+r'}{r_2}\right)K\right]$$

$$- \left(-\frac{r_2^2}{2r^2} + \frac{2(r+r')}{2r} - 1\right)\frac{K}{r_2} - \left(\frac{r_2}{2r} - \frac{r+r'}{r_2}\right)\left[-\frac{K}{2r r_2} + \frac{r_2}{r_1^2}\left(\frac{1}{2r} - \frac{r+r'}{r_2^2}\right)E\right]$$

$$= \left\{\frac{r_2^3}{r_1^2}\left(\frac{1}{2r} - \frac{r+r'}{r_2^2}\right)^2 - \frac{r_2}{4r^2}\right\}E + \left(\frac{r_2}{2r^2} - \frac{r'}{r r_2}\right)K$$

$$= \left[\frac{x^2}{r_1^2 r_2} - \frac{r_2}{2r^2}\right]E + \left[\frac{r_2^2 - 2rr'}{2r^2 r_2}\right]K \quad // \ \text{o.K.}$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial x^2}\right) r_2 E = \frac{2K}{r_2},$$

$$\frac{\partial^2}{\partial x^2}\frac{K}{r_2} = -r_2 E\left(\frac{1}{r_2^2 r_1^2} - \frac{2x^2}{r_1^4 r_2^4}(r_1^2 + r_2^2)\right) - \frac{x}{r_1^2 r_2^2}\frac{x}{r_2}K$$

$$\frac{\partial^2}{\partial x \partial r}\frac{K}{r_2} = +2 r_2 E\left(\frac{x(r-r')}{r_1^4 r_2^2} + \frac{x(r+r')}{r_1^2 r_2^4}\right) - \frac{x}{r_1^2 r_2^2}\left(\frac{r_2}{2r}E - \left(\frac{r_2}{2r} - \frac{r+r'}{r_2}\right)K\right)$$

$$= \frac{x}{r_1^2 r_2}\left(\frac{1}{2r} - \frac{r+r'}{r_2^2}\right)K + \left(\frac{2(r-r')}{r_1^4 r_2} + \frac{2(r+r')}{r_1^2 r_2^3} - \frac{1}{2r r_1^2 r_2}\right)xE$$

$$\frac{\partial^2}{\partial r^2}\frac{K}{r_2} = \frac{K}{2r^2 r_2} - \frac{1}{2r}\left(-\frac{K}{2r r_2} + \frac{r_2}{r_1^2}\left(\frac{1}{2r} - \frac{r+r'}{r_2^2}\right)E\right)$$

$$+ \left\{-\frac{2(r-r')}{r_1^4}\left(\frac{1}{2r} - \frac{r+r'}{r_2^2}\right) + \frac{1}{r_1^2}\left(-\frac{1}{2r^2} - \frac{1}{r_2^2} + \frac{2(r+r')^2}{r_2^4}\right)\right\} r_2 E$$

$$+ \frac{1}{r_1^2}\left(\frac{1}{2r} - \frac{r+r'}{r_2^2}\right)\left(\frac{r_2 E}{2r} - \left(\frac{r_2}{2r} - \frac{r+r'}{r_2}\right)K\right),$$

$$= \left\{\frac{3}{4r^2 r_2} - \frac{r_2}{r_1^2}\left(\frac{1}{2r} - \frac{r+r'}{r_2^2}\right)^2\right\}K$$

$$+ \left\{\frac{2(r^2 - r'^2)}{r_1^4 r_2^2} - \frac{(r-r')}{r r_1^4} - \frac{1}{r_1^2 r_2^2} - \frac{1}{2r^2 r_1^2} + \frac{2(r+r')^2}{r_1^2 r_2^4}\right\} r_2^2 E,$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial x^2}\right)\frac{K}{r_2} = \left[-\frac{x^2}{r_1^2 r_2^3} - \frac{1}{2r^2 r_2} + \frac{3}{4r^2 r_2} - \frac{r_2}{r_1^2}\left(\frac{1}{2r} - \frac{r+r'}{r_2^2}\right)^2\right]K$$

ok.

$$+ \left[\frac{2x^2(r_1^2 + r_2^2)}{r_1^4 r_2^3} - \frac{1}{r_2 r_1^2} + \frac{r_2}{r r_1^2}\left(\frac{1}{2r} - \frac{r+r'}{r_2^2}\right)\right.$$

$$\left. + \frac{2(r^2 - r'^2)}{r_1^4 r_2} + \frac{2(r+r')^2}{r_1^2 r_2^3} - \frac{(r-r')r_2}{r r_1^4} - \frac{1}{r_1^2 r_2} - \frac{r_2}{2r^2 r_1^2}\right]E.$$

$= 0$ o.k.

以上の諸式によって核関数はすべて楕円積分で表わされる事がわかった。

これらの核関数を使えば(2.12)と全く同形式に

$$u_k(Q) = \sum_i \int_c (u_i + u_i') T_i^{(k)}(P,Q)\, r\, ds, \quad \cdots\cdots$$

と置け積分方程式は次のようになる。

$$u_k(Q) + u_k'(Q) - \sum \int_c (u_i + u_i') T_i^{(k)}\, r\, ds = u_k'(Q),$$

$T_i^{(k)}$ なる核の特異性は P=Qで丁度留数分だけジャンプするものであったので積分の時はその点注意する必要がある。

また

$$\left.\begin{aligned} &K(k) \xrightarrow[k\to 1]{} 2\log 2 - \log(k') \\ &E(k) \xrightarrow[k\to 1]{} 1 \\ &K(k) \xrightarrow[k\to 0]{} E(k) \xrightarrow[k\to 0]{} \frac{\pi}{2} \end{aligned}\right\}$$

また次頁以下のように S_j^i なる積分を導入すれば直線区間で変位一定ならば上の積分は直ちに実行出来る。

$$S = \frac{1}{4\pi}\int_0^{2\pi}\frac{d\theta}{R} = \frac{1}{2}\int_0^{\infty} e^{-tx} J_0(tr)J_0(tr')dt.$$

$$\frac{\partial}{\partial r} J_0(tr) = -t\, J_1(tr), \quad \frac{d}{dr}\{rJ_1(tr)\} = rt\, J_0(tr)$$

$$S^* = \frac{1}{4\pi}\int_0^{2\pi} R\, d\theta = \frac{r}{2}\int_0^{\infty} e^{-tx} J_1(tr)J_0(tr')\frac{dt}{t},$$

$$\text{or} \quad = \frac{r'}{2}\int_0^{\infty} e^{-tx} J_1(tr')J_0(tr)\frac{dt}{t}.$$

$$T = \frac{r}{2}\int_0^{\infty} e^{-tx} J_1(tr)J_0(tr')dt. ,$$

これは第3種楕円積分で表わされる他[illegible]

$$\frac{\partial S}{\partial x} = -\frac{1}{r}\frac{\partial T}{\partial r}, \quad \frac{\partial S}{\partial r} = \frac{1}{r}\frac{\partial T}{\partial x},$$

$$\frac{\partial S}{\partial n} = -\frac{1}{r}\frac{\partial T}{\partial s}, \quad \frac{\partial S}{\partial s} = \frac{1}{r}\frac{\partial T}{\partial n},$$

$$T^* = -\frac{rr'}{2}\int_0^{\infty} e^{-tx} J_1(tr)J_1(tr')\frac{dt}{t}$$

$$\left.\begin{aligned}
\frac{\partial T^*}{\partial r} &= -\frac{rr'}{2}\int_0^{\infty} e^{-tx} J_0(tr)J_1(tr')dt = r\frac{\partial S^*}{\partial x} \\
\frac{\partial T^*}{\partial x} &= \frac{rr'}{2}\int_0^{\infty} e^{-tx} J_1(tr)J_1(tr')dt = -r\frac{\partial S^*}{\partial r} \\
\frac{\partial S^*}{\partial n} &= \frac{1}{r}\frac{\partial T^*}{\partial s}, \quad \frac{\partial S^*}{\partial s} = -\frac{1}{r}\frac{\partial T^*}{\partial n},
\end{aligned}\right\}$$

$$\frac{\partial T^*}{\partial r} = r\frac{\partial S^*}{\partial x} = -rT, \quad \frac{\partial T^*}{\partial x} = -r\frac{\partial S^*}{\partial r} = -r^2 S,$$

$$\left.\begin{aligned}
\frac{\partial^2 T^*}{\partial x^2} &= -r^2\frac{\partial S}{\partial x}, \quad \frac{\partial^2 T^*}{\partial r^2} = -T + r^2\frac{\partial S}{\partial x} \\
\frac{\partial^2 T^*}{\partial x \partial r} &= -r\frac{\partial T}{\partial x} = -r^2\frac{\partial S}{\partial r}
\end{aligned}\right\}$$

$$T_1^1 = \left\{1 + \frac{\left(\frac{\partial X}{\partial n}\right)^2}{1-\nu}\right\}\frac{\partial S}{\partial n} + \frac{\frac{\partial Y}{\partial n}\frac{\partial X}{\partial s}}{1-\nu}\frac{\partial S}{\partial s} - \frac{1}{2(1-\nu)}\frac{\partial^2}{\partial X'^2}\frac{\partial S^*}{\partial n}$$

$$T_2^1 = \frac{1}{1-\nu}\frac{\partial X}{\partial n}\frac{\partial Y}{\partial n}\frac{\partial S}{\partial n} + \left(\frac{\partial X}{\partial n}\frac{\partial Y}{\partial s} + \frac{\nu}{1-\nu}\frac{\partial Y}{\partial n}\frac{\partial X}{\partial s}\right)\frac{\partial S}{\partial s} - \frac{1}{2(1-\nu)}\frac{\partial^2}{\partial X'\partial Y'}\frac{\partial S^*}{\partial n}$$

$$T_1^2 = \frac{1}{1-\nu}\frac{\partial X}{\partial n}\frac{\partial Y}{\partial n}\frac{\partial S}{\partial n} + \left(\frac{\partial Y}{\partial n}\frac{\partial Y}{\partial s} + \frac{\nu}{1-\nu}\frac{\partial X}{\partial n}\frac{\partial Y}{\partial s}\right)\frac{\partial S}{\partial s} - \frac{1}{2(1-\nu)}\frac{\partial^2}{\partial X'\partial Y'}\frac{\partial S^*}{\partial n}$$

$$T_2^2 = \left\{1 + \frac{\left(\frac{\partial Y}{\partial n}\right)^2}{1-\nu}\right\}\frac{\partial S}{\partial n} + \frac{\frac{\partial Y}{\partial n}\frac{\partial Y}{\partial s}}{1-\nu}\frac{\partial S}{\partial s} - \frac{1}{2(1-\nu)}\frac{\partial^2}{\partial Y'^2}\frac{\partial S^*}{\partial n}$$

$$S_1^1 = \int ds\, r\, T_1^1 = -\left(1 + \frac{\left(\frac{\partial X}{\partial n}\right)^2}{1-\nu}\right)T - \frac{X_n Y_n}{1-\nu}S - \frac{1}{2(1-\nu)}\frac{\partial^2}{\partial X^2}T^*$$

$$S_2^1 = \int ds\, r\, T_2^1 = -\frac{X_n Y_n}{1-\nu}T + \left(X_n^2 - \frac{\nu}{1-\nu}Y_n^2\right)S - \frac{1}{2(1-\nu)}\frac{\partial^2 T^*}{\partial X\partial Y}$$

$$S_1^2 = \int ds\, r\, T_1^2 = -\frac{X_n Y_n}{1-\nu}T + \left(-Y_n^2 + \frac{\nu}{1-\nu}X_n^2\right)S - \frac{1}{2(1-\nu)}\frac{\partial^2 T^*}{\partial X\partial Y}$$

$$S_2^2 = \int ds\, r\, T_2^2 = -\left(1 + \frac{Y_n^2}{1-\nu}\right)T + \frac{X_n Y_n}{1-\nu}S - \frac{1}{2(1-\nu)}\frac{\partial^2}{\partial Y^2}T^*$$

$$S_1^1 = -\left(1 + \frac{X_n^2}{1-\nu}\right)T - \frac{X_n Y_n}{1-\nu}S + \frac{r^2}{2(1-\nu)}\frac{\partial S}{\partial X}$$

$$S_2^1 = -\frac{X_n Y_n}{1-\nu}T + \left(X_n^2 - \frac{\nu Y_n^2}{1-\nu}\right)S + \frac{r^2}{2(1-\nu)}\frac{\partial S}{\partial Y}$$

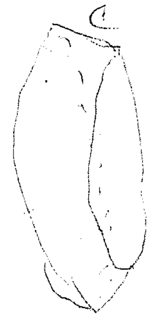
$$S_1^2 = -\frac{X_n Y_n}{1-\nu}T - \left(Y_n^2 - \frac{\nu X_n^2}{1-\nu}\right)S + \frac{r^2}{2(1-\nu)}\frac{\partial S}{\partial Y}$$

$$S_2^2 = \left[\frac{1}{2(1-\nu)} - 1 - \frac{Y_n^2}{1-\nu}\right]T + \frac{X_n Y_n}{1-\nu}S - \frac{r^2}{2(1-\nu)}\frac{\partial S}{\partial X}$$

以上の方法を少し一般化して見よう。

微小円環要素上の積分を考えよう

$$I(Q)=\int_C (u T_1^{(k)} + v T_2^{(k)}) r\,ds$$



この形でのBettiの相反定理から

Q点が外にあるならば

$$I(Q)=\int_C (u T_1^{k} + v T_2^{k}) r\,ds = \int_C (\tau_1 U_1^{k} + \tau_2 U_2^{k}) r\,ds$$

$$+\iint_{A-B} [u T_1^{k} + v T_2^{k} - \tau_1 U_1^{k} - \tau_2 U_2^{k}] r\,dr$$

Aは前の円積面，Bは後。

今 C 上で $u = \text{Const.}$
$v = \nu \dfrac{r}{a}$，（$v = \text{Const}$ とすると $\tau \propto \frac{1}{r}$ で工合が悪い）

とすると $\gamma = \dfrac{2\nu}{a}$，$\tau_1 = \dfrac{4\nu}{1-2\nu}\dfrac{\nu}{a}\dfrac{\partial x}{\partial n}$

$$\tau_2 = 2\frac{\nu}{a}\frac{\partial r}{\partial n} + \frac{4\nu}{1-2\nu}\frac{\nu}{a}\frac{\partial r}{\partial n} = \frac{2}{1-2\nu}\frac{\nu}{a}\frac{\partial r}{\partial n},$$

それ故，これを上式に代入して，

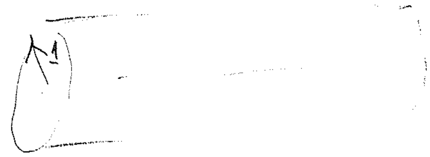
$$I(Q) = \frac{2(\frac{\nu}{a})}{1-2\nu}\int_C \left[2\nu U_1^{k}\frac{\partial x}{\partial n} + U_2^{k}\frac{\partial r}{\partial n}\right] r\,ds$$

$$+\iint_{A-B}\left[u T_1^{k} + \frac{\nu}{a} r T_2^{k} - \frac{4\nu}{1-2\nu}\frac{\nu}{a} U_1^{k}\right] r\,dr$$

附録A 無限長丸棒の中の軸対称な特異性

丸棒の外周（半径1）は自由辺とする。

そうすると そこでは



$$\left.\frac{\partial u}{\partial r} + \frac{\partial v}{\partial x}\right|_{r=1} = 0$$

$$\left.\frac{\partial v}{\partial r} + \frac{\nu}{1-2\nu}\gamma\right|_{r=1} = 0 \qquad (A\cdot1)$$

この第2式は $\gamma = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial r} + \frac{v}{r}$ であるから

$$\left.\frac{1-\nu}{1-2\nu}\gamma\right|_{r=1} = \left.\frac{\partial u}{\partial x} + \frac{v}{r}\right|_{r=1} \qquad (A\cdot1')$$

としてもよい。

そして

$$\frac{1}{R} = \frac{2}{\pi}\int_0^\infty \cos xt\, K_0(t\varpi)\,dt, \qquad (A\cdot2)$$

$$R^2 = x^2 + \varpi^2, \quad \varpi^2 = r^2 + r'^2 - 2rr'\cos\theta$$

であり

$$K_0(t\varpi) = \sum_{m=-\infty}^{\infty} K_m(tr)\, I_m(tr')\cos m\theta, \quad r > r'$$

であるから

$$I = \frac{1}{4\pi}\int_0^{2\pi}\frac{d\theta}{R} = \frac{1}{\pi}\int_0^\infty \cos xt\, K_0(tr)\, I_0(tr')\,dt, \quad \text{for } r > r'$$

$$= \frac{1}{\pi}\int_0^\infty \cos xt\, I_0(tr)\, K_0(tr')\,dt. \quad \text{for } r' > r \qquad (A\cdot3)$$

次で

$$\Delta \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial x^2}, \quad \Delta R = \frac{2}{R},$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - t^2\right) r \begin{Bmatrix} K_1(tr) \\ I_1(tr) \end{Bmatrix} = 2t \begin{Bmatrix} -K_0(tr) \\ I_0(tr) \end{Bmatrix}, \qquad (A\cdot4)$$

$K_0' = -K_1, \quad I_0' = I_1$

$\frac{\partial}{\partial r}(rK_1) = -rtK_0, \quad \frac{\partial}{\partial r}(rI_1) = rtI_0,$

$\frac{\partial^2}{\partial r^2}(rK_1) = -tK_0 + rt^2K_1, \quad \frac{\partial^2}{\partial r^2}(rI_1) = tI_0 + rt^2I_1,$

No. A-2

Date

$$J = \frac{1}{4\pi}\int_0^{2\pi} R\,d\theta = -\frac{r'}{\pi}\int_0^\infty \cos xt\, K_1(tr) I_0(tr')\frac{dt}{t} \quad \text{for } r > r'$$

$$= \frac{r'}{\pi}\int_0^\infty \cos xt\, I_1(tr) K_0(tr')\frac{dt}{t}, \quad \text{for } r' > r \qquad (A\cdot5)$$

この積分は $t=0$ の特異性から積分不可能であるけれど実際はこの微係数のみ必要でそれは積分可能であるから別に不便はないのでこのまゝ書いておく。

これらを使うと (3.10) は

$$U_1^1 = \frac{1}{\pi}\int_0^\infty \cos xt\, I_0(tr')\left[K_0(tr) - \frac{rt}{4(1-\nu)}K_1(tr)\right]dt,$$

$$U_2^1 = \frac{1}{\pi}\int_0^\infty \sin xt\, I_0(tr')\left[-\frac{1}{4(1-\nu)}\frac{\partial}{\partial r}(rK_1(tr))\right]dt, \qquad (A\cdot6)$$

$\text{for } r > r'$

$$U_1^1 = \frac{1}{\pi}\int_0^\infty \cos xt\, K_0(tr')\left[I_0(tr) + \frac{rt}{4(1-\nu)}I_1(tr)\right]dt,$$

$$U_2^1 = \frac{1}{\pi}\int_0^\infty \sin xt\, K_0(tr')\left[+\frac{1}{4(1-\nu)}\frac{d}{dr}(rI_1(tr))\right]dt, \qquad (A\cdot7)$$

$\text{for } r < r'$

$$U_1^2 = U_2^1$$

$$U_2^2 = \frac{1}{\pi}\int_0^\infty \cos xt\, I_0(tr')\left[K_0(tr) + \frac{1}{\underbrace{t}_{4(1-\nu)}}\frac{\partial^2}{\partial r^2}(rK_1(tr))\right]dt, \quad \text{for } r > r'$$

$$= \frac{1}{\pi}\int_0^\infty \cos xt\, K_0(tr')\left[I_0(tr) - \frac{1}{4(1-\nu)t}\frac{\partial^2}{\partial r^2}(rI_1)\right]dt, \quad \text{for } r < r' \qquad (A\cdot8)$$

$r=1$で(A.1)の条件を満たすためには外側の領域に特異性のあるものを考えてまず

$$\left.\begin{aligned} V_1^1 &= \frac{1}{\pi}\int_0^\infty \cos xt\, F_1(t)\left[I_0(tr) + \frac{rt}{4(1-\nu)} I_1(tr)\right] dt, \\ V_2^1 &= \frac{1}{\pi}\int_0^\infty \sin xt\, F_1(t)\left[\frac{1}{4(1-\nu)}\frac{\partial}{\partial r}(r I_1)\right] dt, \end{aligned}\right\} \quad (A.9)$$

とさらに調和分を加えて

$$\left.\begin{aligned} W_1^1 &= \frac{1}{\pi}\int_0^\infty \cos xt\, A_1(t) I_0(tr)\, dt, \\ W_2^1 &= \frac{1}{\pi}\int_0^\infty \sin xt\, A_1(t) I_1(tr)\, dt, \end{aligned}\right\} \quad (A.10)$$

$$\left.\begin{aligned} V_1^2 &= \frac{1}{\pi}\int_0^\infty \sin xt\, F_2(t)\left[\frac{1}{4(1-\nu)}\frac{\partial}{\partial r}(r I_1)\right] dt, \\ V_2^2 &= \frac{1}{\pi}\int_0^\infty \cos xt\, F_2(t)\left[I_0 - \frac{1}{4(1-\nu)t}\frac{\partial^2}{\partial r^2}(r I_1)\right] dt, \end{aligned}\right\} \quad (A.11)$$

$$\left.\begin{aligned} W_1^2 &= -\frac{1}{\pi}\int_0^\infty \sin xt\, A_2(t) I_0(tr)\, dt, \\ W_2^2 &= \frac{1}{\pi}\int_0^\infty \cos xt\, A_2(t) I_1(tr)\, dt, \end{aligned}\right\} \quad (A.12)$$

としてこれらを加えて新しくU_j^iを定義しよう。

$$U_j^i + V_j^i + W_j^i \longrightarrow U_j^i$$

xの前に

$$\Gamma_U^i = \frac{\partial}{\partial x} U_1^i + \frac{\partial}{\partial r} U_2^i + \frac{1}{r} U_2^i ,$$

$$\Gamma_U^1 = \frac{1}{\pi}\int_0^\infty \sin xt\, I_0(tr')\Big[-tK_0 + \frac{rt^2}{4(1-\nu)}K_1 - \frac{1}{4(1-\nu)}\frac{\partial^2}{\partial r^2}(rK_1) - \frac{1}{4(1-\nu)}\frac{1}{r}\frac{\partial}{\partial r}(rK_1)\Big]dt$$

$$= -\frac{1}{\pi}\frac{1-2\nu}{2(1-\nu)}\int_0^\infty \sin xt\, I_0(tr')K_0(tr)t\,dt , \quad \cdots (A\cdot 13)$$

$$\Gamma_V^1 = -\frac{1}{\pi}\frac{1-2\nu}{2(1-\nu)}\int_0^\infty \sin xt\, F_1(t)\, I_0(tr)t\,dt , \quad (A\cdot 14)$$

$$\Gamma_W^1 = \Gamma_W^2 = 0 , \quad \cdots (A\cdot 15)$$

$$\Gamma_U^2 = \frac{1}{\pi}\int_0^\infty \cos xt\, I_0(tr')\Big[-\frac{t}{4(1-\nu)}\frac{\partial}{\partial r}(rK_1) - tK_1 + \frac{1}{4(1-\nu)t}\frac{\partial^3}{\partial r^3}(rK_1) + \frac{K_0}{r} + \frac{1}{4(1-\nu)tr}\frac{\partial^2}{\partial r^2}(rK_1)\Big]dt$$

$$= -\frac{1}{\pi}\frac{1-2\nu}{2(1-\nu)}\int_0^\infty \cos xt\, I_0(tr')K_1(tr)t\,dt , \quad \cdots (A\cdot 16)$$

$$\Gamma_V^2 = \frac{1}{\pi}\int_0^\infty \cos xt\, F_2\Big[\frac{t}{4(1-\nu)}\frac{\partial}{\partial r}(rI_1) + tI_1 - \frac{1}{4(1-\nu)t}\frac{\partial^3}{\partial r^3}(rI_1) + \frac{I_0}{r} - \frac{1}{4(1-\nu)tr}\frac{\partial^2}{\partial r^2}(rI_1)\Big]dt$$

$$= \frac{1}{\pi}\frac{1-2\nu}{2(1-\nu)}\int_0^\infty \cos xt\, F_2(t)\, I_1(tr)t\,dt , \quad \cdots (A\cdot 17)$$

$$\left.\begin{aligned}\Gamma_U^1 + \Gamma_V^1 = \Gamma^1 &= -\frac{1-2\nu}{\pi\, 2(1-\nu)}\int_0^\infty \sin xt\Big[I_0(tr')K_0(tr) + F_1 I_0(tr)\Big]t\,dt , \\ \Gamma_U^2 + \Gamma_V^2 = \Gamma^2 &= -\frac{1-2\nu}{2\pi(1-\nu)}\int_0^\infty \cos xt\Big[I_0(tr')K_1(tr) - F_2 I_1(tr)\Big]t\,dt ,\end{aligned}\right\}$$

(A.18)

$$U^1_1 = \frac{1}{\pi}\int_0^\infty \cos xt\left[I_0(tr')\left\{K_0 - \frac{rt}{4(1-\nu)}K_1\right\} + F_1\left\{I_0 + \frac{rt}{4(1-\nu)}I_1\right\} + A_1 I_0\right]dt,$$

$$U^1_2 = \frac{1}{\pi}\int_0^\infty \sin xt\left[I_0(tr')\left\{+\frac{rt}{4(1-\nu)}K_0\right\} + \frac{F_1}{4(1-\nu)}rt I_0 + A_1 I_1\right]dt, \quad (A\cdot13)$$

$$U^2_1 = \frac{1}{\pi}\int_0^\infty \sin xt\left[I_0(tr')\left\{+\frac{rt}{4(1-\nu)}K_0\right\} + \frac{F_2}{4(1-\nu)}rt I_0 - A_2 I_0\right]dt$$

$$U^2_2 = \frac{1}{\pi}\int_0^\infty \cos xt\left[\frac{I_0(tr')}{4(1-\nu)}\left\{(3-4\nu)K_0 + rtK_1\right\} + \frac{F_2}{4(1-\nu)}\left\{(3-4\nu)I_0 - rtI_1\right\} + A_2 I_1\right]dt \quad (A\cdot14)$$

(A·1)の条件は,

$$I_0(tr')\left[-tK_1 + \frac{rt^2}{2(1-\nu)}K_0\right] + F_1\left[tI_1 + \frac{rt^2}{2(1-\nu)}I_0\right] + 2A_1 t I_1 = 0$$

$$-\frac{t}{2}\left[I_0(tr')K_0(tr) + F_1 I_0\right] = -t\left[I_0(tr')\left\{K_0 - \frac{rt}{4(1-\nu)}K_1\right\} + F_1\left\{I_0 + \frac{rt}{4(1-\nu)}I_1\right\}\right]$$
$$+ \frac{I_0(tr')}{r}\left\{-\frac{1}{4(1-\nu)}\frac{\partial}{\partial r}(rK_1)\right\} + \frac{F_1}{4(1-\nu)r}\frac{\partial}{\partial r}(rI_1)$$
$$+ A_1\left(-I_0 t + \frac{I_1}{r}\right),$$

$$I_0(tr')\left[\left(\frac{1}{2} - \frac{1}{2(1-\nu)}\right)K_0 - \frac{rt}{4(1-\nu)}K_1\right] + F_1\left[\frac{-\nu}{2(1-\nu)}I_0 + \frac{rt}{4(1-\nu)}I_1\right]$$
$$= A_1\left(\frac{I_1}{rt} - I_0\right) = -A_1 I_1'$$

$$I_0(tr')\left[\frac{K_1}{2} - \frac{rtK_0}{4(1-\nu)}\right] + F_1\left[-\frac{I_1}{2} - \frac{rtI_0}{4(1-\nu)}\right] = A_1 I_1, \quad (A\cdot15)$$

$$I_0(tr')\left[\left(\frac{1}{2rt} + \frac{rt}{4(1-\nu)}\right)K_1 - \frac{1-2\nu}{4(1-\nu)}K_0\right] - F_1\left[\left(\frac{1}{2rt} + \frac{rt}{4(1-\nu)}\right)I_1 + \frac{1-2\nu}{4(1-\nu)}I_0\right]$$
$$= A_1 I_0$$

以下同様 未完

No.
Date 54.6.21

3次元弾性の場合

1、 応力と歪 ，基本特異性
2， Bettiの表現等
3． 軸対称の場合

附録 積分

1. 応力と歪，基本特質性

変位のベクトルを $\boldsymbol{u}(u_1, u_2, u_3)$ とすると歪 e_{ij} は，
$(x \equiv x_1,\ y \equiv x_2,\ z \equiv x_3)$

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad \cdots \quad (1.1)$$

体積増は $\gamma = \sum_{i=1}^{3} e_{ii}$, $\cdots$ (1.2)

フックの法則では応力 f'_{ij} が次の如く表わされる。

$$\frac{f'_{ij}}{G} \equiv f_{ij} = 2e_{ij} + \frac{2\nu}{1-2\nu}\gamma\delta_{ij}, \quad (1.3)$$

こゝに $\delta_{ij} = 0$ for $i \neq j$, $\delta_{ii} = 1$,

弾性定数間の関係は，

$$G = \frac{E}{2(1+\nu)} = \mu, \quad \lambda = \frac{2\nu}{1-2\nu}G,$$

Bergman, Schiffer の Text の記号との関係，

$$\alpha = \frac{1}{1-2\nu}, \quad A = -\frac{\alpha}{2(\alpha+1)} = \frac{-1}{4(1-\nu)}, \quad B = 1.$$

$$2A + B = \frac{1-2\nu}{2(1-\nu)},$$

境界力も G で割ってそれを $\boldsymbol{\tau}(\tau_1, \tau_2, \tau_3)$ とすると，

$$\tau_i = \sum_{j=1}^{3} f_{ij}\frac{\partial x_j}{\partial n} = 2\sum_{j=1}^{3} e_{ij}\frac{\partial x_j}{\partial n} + \frac{\nu}{1-2\nu}\gamma\frac{\partial x_i}{\partial n}, \quad (1.4)$$

なお (1.3) より

$$\sigma = \sum_{i=1}^{3} f_{ii} = \frac{2(1+\nu)}{1-2\nu}\gamma, \quad \cdots$$

適合条件式は

$$\Delta u_i + \frac{1}{1-2\nu}\frac{\partial \gamma}{\partial x_i} = 0, \quad i=1,2,3, \qquad (1.6)$$

今基本特異性として上式を満足する次のものを考える。

$$\left.\begin{aligned} U_i^{(k)}(P,Q) &= \frac{\delta_{ik}}{R} - \frac{1}{4(1-\nu)}\frac{\partial^2 R}{\partial x_i \partial x_k}, \\ P &\equiv (x_1, x_2, x_3), \quad Q \equiv (x_1', x_2', x_3'), \quad R = \overline{PQ} \\ U_i^{(k)}(P,Q) &= U_i^{(k)}(Q,P), \quad \Delta R = \frac{2}{R} \end{aligned}\right\} \quad (1.7)$$

これによる歪は(1)より

$$\left.\begin{aligned} E_{ij}^{(k)}(P,Q) &= \frac{1}{2}\left(\frac{\partial}{\partial x_i}\frac{\delta_{jk}}{R} + \frac{\partial}{\partial x_j}\frac{\delta_{ik}}{R}\right) \\ &\quad - \frac{1}{4(1-\nu)}\frac{\partial^3 R}{\partial x_i \partial x_j \partial x_k}, \\ E_{ij}^{(k)}(P,Q) &= -E_{ij}^{(k)}(Q,P), \end{aligned}\right\} \quad (1.8)$$

(2)より $$P^{(k)}(P,Q) = \frac{1-2\nu}{2(1-\nu)}\frac{\partial}{\partial x_k}\frac{1}{R} = -P^{(k)}(Q,P), \qquad (1.9)$$

これによる境界力は

$$\begin{aligned} T_i^{(k)}(U_\cdot^{(k)}; P,Q) &= \delta_{ik}\frac{\partial}{\partial n}\left(\frac{1}{R}\right) + \frac{\partial x_k}{\partial n}\frac{\partial}{\partial x_i}\frac{1}{R} + \frac{\nu}{1-\nu}\frac{\partial x_i}{\partial n}\frac{\partial}{\partial x_k}\frac{1}{R} \\ &\quad - \frac{1}{2(1-\nu)}\frac{\partial^3}{\partial n \partial x_i \partial x_k}\frac{1}{R} \\ &= 2\frac{\partial}{\partial n}U_i^{(k)}(P,Q) + \frac{\partial x_k}{\partial n}\frac{\partial}{\partial x_i}\frac{1}{R} + \frac{\nu}{1-\nu}\frac{\partial x_i}{\partial n}\frac{\partial}{\partial x_k}\frac{1}{R} \\ &\quad - \delta_{ik}\frac{\partial}{\partial n}\frac{1}{R}, \end{aligned} \qquad (1.10)$$

これもまた変位ベクトルと見なしてよいのでこれによる歪を計算すると今度は x_i で微分して.

$$E_{ij}^{(k)}(T^{(k)};Q,P)=\frac{1}{2}\left(\frac{\partial}{\partial x_i}T_j^{(k)}+\frac{\partial}{\partial x_j}T_i^{(k)}\right)$$

$$=2\frac{\partial}{\partial n}E_{ij}^{(k)}(Q,P)+\frac{1}{2}\frac{\partial x_k}{\partial n}\left(\frac{\partial^2}{\partial x_i\partial x_j}+\frac{\partial^2}{\partial x_i\partial x_j}\right)\frac{1}{R}$$

$$+\frac{\partial}{\partial n}\left(\delta_{ik}\frac{\partial}{\partial x_j}+\delta_{jk}\frac{\partial}{\partial x_i}\right)\frac{1}{R}-\frac{\nu}{2(1-\nu)}\left(\frac{\partial x_j}{\partial n}\frac{\partial}{\partial x_i}+\frac{\partial x_i}{\partial n}\frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_k}\frac{1}{R}, \quad (1.11)$$

$$\Gamma^{(k)}(E_{ij}^{(k)}(T);Q,P)=2\frac{\partial}{\partial n}\Gamma^{(k)}(Q,P)-\frac{\nu}{1-\nu}\frac{\partial^2}{\partial n\partial x_k}\frac{1}{R}$$

$$-\frac{\partial^2}{\partial n\partial x_k}\frac{1}{R},$$

$$=2\frac{\partial}{\partial n}\Gamma^{(k)}(Q,P)+\frac{1-2\nu}{1-\nu}\frac{\partial^2}{\partial n\partial x_k}\frac{1}{R}, \quad (1.12)$$

応力は

$$F_{ij}^{k}(T;Q,P)=2E_{ij}(T;Q,P)+\frac{2\nu}{1-2\nu}\Gamma^{(k)}\delta_{ij}$$

$$=4\frac{\partial}{\partial n}E_{ij}^{(k)}(Q,P)+2\frac{\partial x_k}{\partial n}\frac{\partial^2}{\partial x_i\partial x_j}\frac{1}{R}$$

$$+\frac{\partial}{\partial n}\left(\delta_{ik}\frac{\partial}{\partial x_j}+\delta_{jk}\frac{\partial}{\partial x_i}\right)\frac{1}{R}$$

$$-\frac{\nu}{1-\nu}\left(\frac{\partial x_j}{\partial n}\frac{\partial}{\partial x_i}+\frac{\partial x_i}{\partial n}\frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_k}\frac{1}{R}$$

$$+\frac{4\nu\delta_{ij}}{1-2\nu}\frac{\partial}{\partial n}\Gamma^{(k)}(Q,P)+\frac{2\nu}{1-\nu}\delta_{ij}\frac{\partial^2}{\partial n\partial x_k}\frac{1}{R}, \quad (1.13)$$

境界力は

$$T_i^{(k)}(T;Q,P) = \sum_{j=1}^{3} F_{ij}^{(k)}(T;Q,P)\frac{\partial x_j}{\partial n'}$$

$$= 2\frac{\partial}{\partial n} F_{ij}^{(k)}(Q,P) + 2\frac{\partial x_k}{\partial n}\frac{\partial^2}{\partial n'\partial x_i}\frac{1}{R}$$

$$-\delta_{ik}\frac{\partial^2}{\partial n\partial n'}\frac{1}{R} + \frac{\partial x_k}{\partial n'}\frac{\partial^2}{\partial n\partial x_i}\frac{1}{R} + \frac{2\nu}{1-\nu}\frac{\partial x_i}{\partial n'}\frac{\partial^2}{\partial n\partial x_k}\frac{1}{R}$$

$$-\frac{\nu}{1-\nu}\left[\left(\sum_{j=1}^{3}\frac{\partial x_j}{\partial n'}\frac{\partial x_j}{\partial n}\right)\frac{\partial^2}{\partial x_i\partial x_k}\frac{1}{R} - \frac{\partial x_i}{\partial n}\frac{\partial^2}{\partial n'\partial x_k}\frac{1}{R}\right], \quad \cdots (1.14)$$

ただし

$$F_{ij}^{(k)}(P,Q) = -F_{ij}^{(k)}(Q,P) = 2E_{ij} + \frac{2\nu}{1-2\nu}\Gamma^{(k)}\delta_{ij}$$

$$= \left(\delta_{jk}\frac{\partial}{\partial x_i} + \delta_{ik}\frac{\partial}{\partial x_j}\right)\frac{1}{R} - \frac{1}{2(1-\nu)}\frac{\partial^3 R}{\partial x_i\partial x_j\partial x_k}$$

$$+\frac{\nu}{1-\nu}\delta_{ij}\frac{\partial}{\partial x_k}\frac{1}{R}, \quad (1.\ )$$

$$= 2\frac{\partial}{\partial x_j} U_i^{(k)}(P,Q)$$

$$+\left(\frac{\nu}{1-\nu}\delta_{ij}\frac{\partial}{\partial x_k} + \delta_{jk}\frac{\partial}{\partial x_i} - \delta_{ki}\frac{\partial}{\partial x_j}\right)\frac{1}{R}, \quad (1.15)$$

2. Betti の表現

歪エネルギーは

$$A = \frac{G}{2}\iiint_D \sum_{i,j=1}^{3} f_{ij}\, e_{ij}\, d\tau, \quad d\tau = dx\,dy\,dz,$$

$$= G\iiint_D \left[\sum_{i,j=1}^{3} e_{ij}^2 + \frac{\nu}{1-2\nu}\left(\sum_{i=1}^{3} e_{ii}\right)^2\right] d\tau, \quad \cdots \quad (2.1)$$

であるから 2次形式

$$E(\mathbf{u},\mathbf{v}) = \iiint_D \left[\sum_{i,j=1}^{3} \frac{\partial u_i}{\partial x_j}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) + \frac{2\nu}{1-2\nu}\,\mathrm{div}\,\mathbf{u}\,\mathrm{div}\,\mathbf{v}\right] d\tau \quad (2.2)$$

を導入すると

$$e(\mathbf{u},\mathbf{v}) = e(\mathbf{v},\mathbf{u}) \quad (2.3)$$

部分積分にて

$$e(\mathbf{u},\mathbf{v}) = \iint_S \mathbf{u}\,\boldsymbol{\tau}(\mathbf{v})\,dS - \iiint_D \mathbf{u}\left\{\Delta\mathbf{v} + \frac{1}{1-2\nu}\,\mathrm{grad}\,\mathrm{div}\,\mathbf{v}\right\} d\tau, \quad (2.4)$$

となり (1.6) が成立つとすると右辺第2項は消えて (2.3) より Maxwell-Betti の相反定理が成立つ

$$\iint_S \left[\mathbf{u}\,\boldsymbol{\tau}(\mathbf{v}) - \mathbf{v}\,\boldsymbol{\tau}(\mathbf{u})\right] dS = 0, \quad (2.5)$$

(Bergman-Schiffの本では法線Dに内向, 今は外向とする)

今 $\boldsymbol{v}$ として (1.7) を考えると 常用手段により

$$\lim_{\varepsilon\to 0}\iint_{\varepsilon}\frac{x_i x_j}{r^4}ds=\frac{4}{3}\pi\delta_{ij}$$

により

$$\boldsymbol{u}(Q)=\frac{1}{4\pi}\iint_S\left[\boldsymbol{\tau}(\boldsymbol{u}(P))U(P,Q)-\boldsymbol{u}(P)\boldsymbol{T}(U(P,Q))\right]ds_P$$

ここに U はマトリックスで 上式を成分毎に書くと
前節の記号を使って

$$u_k(Q)=\frac{1}{4\pi}\iint_S\sum_{i=1}^{3}\left[\tau_i(P)U_i^{(k)}(P,Q)-u_i(P)T_i^{(k)}(P,Q)\right]ds(P),$$

$$k=1,2,3 \qquad (2.6)$$

応力等は前節の式を使って計算すればよい。
外部問題(無限遠点を含む領域の)では、
吹出し式表現が便利な事があり、それは

$$u_k(Q)=\frac{1}{4\pi}\iint_S\sum_{i=1}^{3}\tau_i(P)U_i^{(k)}(P,Q)\,ds(P), \qquad (2.7)$$

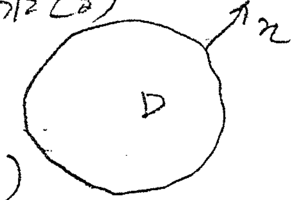
$$u_k(Q)=-\frac{1}{4\pi}\iint_S\sum_{i=1}^{3}u_i T_i^{(k)}(P,Q)\,ds(P), \qquad (2.8)$$

のようになるが 応力集中問題では 2次元問題
(2.8) が便利であるが、この時右辺の核関数

について (1.13) のように応力を計算するのは大変である。

そこで元に戻って今は外部領域なので $\bar{D}$

(2.6) は符号が変って (法線はそのまゝ図のようにとる)



$$u_k(Q) = \frac{1}{4\pi}\iint_S \sum_{i=1}^{3}\left(t_i U_i^{(k)} - u_i T_i^{(k)}\right) dS(P), \quad \text{in } \bar{D} \qquad (2.9)$$

内部領域で考えると

$$\frac{1}{4\pi}\iint_S \sum_{i}^{3}\left(t_i U_i^{k} - u_i' T_i^{(k)}\right) dS_P = \begin{cases} u_k(Q) & \text{in } D \\ 0 & \text{in } \bar{D} \end{cases} \qquad (2.10)$$

となるから例えば x一方向の一様引張りを受ける場合の

応力集中問題では、S上で

$$\tau_{ij} = 0 \ \text{for } i \neq j, \quad \tau_{11} = 1, \quad \tau_{22} = \tau_{33} = 0$$

であるから外部の境界条件は

$$t_1 = -\frac{\partial x_1}{\partial n}, \quad t_2 = t_3 = 0 \ \cdots \qquad (2.10)$$

一方 (2.10) については

$$e_{ij} = \frac{1}{2}\tau_{ij} - \frac{\nu}{1-2\nu}\gamma\delta_{ij} = \frac{1}{2}\tau_{ij} - \frac{\nu}{2(1+\nu)}\delta_{ij}$$

$$\frac{\partial u_1'}{\partial x_2} = e_{11} = \frac{1}{2(1+\nu)}, \quad e_{2,2} = -\frac{\nu}{2(1+\nu)} = \frac{\partial u_2}{\partial x_2} = \frac{\partial u_3}{\partial x_3}$$

$$e_{ij} = 0 \ \text{for } i \neq j$$

$$\therefore\ u_1' = \frac{x_1}{2(1+\nu)}, \quad u_2' = -\frac{\nu x_2}{2(1+\nu)}, \quad u_3' = -\frac{\nu x_3}{2(1+\nu)} \qquad (2.11)$$

$$t_1' = \frac{\partial x_1}{\ }, \quad t_2' = t_3' = 0$$

であるから(2.9)から(2.10)を差引いて、

$$u_k(Q) = \frac{1}{4\pi}\iint_S \sum_{i=1}^{3}(u_i + u_i')\, T_i^{(k)}\, ds_P\,,\quad \text{in } D^i, \qquad (2.12)$$

を得る。

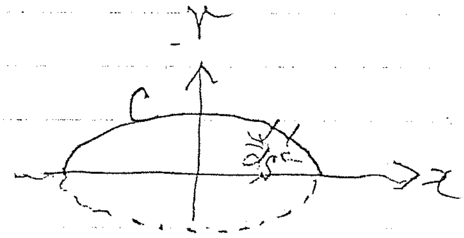
ここに u_i' は(2.11)によるから $(u_i + u_i')$ は一様引張りも含めた全変位となる。

上式は S 上でフレドホルムの第2種方程式となるのでこれを解けばよい。

3. 軸対称の場合

x軸のまわりに回転対称としよう。

$x_2 \equiv y = r\cos\theta$, $\quad x_3 \equiv z = r\sin\theta$ とおくと、



面積要素は $dS = r\,d\theta\,ds$

$$ds = \sqrt{dx^2+dr^2}$$

変位および表面力は

$$\left.\begin{aligned} &u_1 \equiv u(x,r), \quad u_2 = v(x,r)\cos\theta, \quad u_3 = v(x,r)\sin\theta \\ &\tau_1 \equiv \tau(x,r), \quad \tau_2 = \sigma(x,r)\cos\theta, \quad \tau_3 = \sigma(x,r)\sin\theta \end{aligned}\right\} \quad (3.1)$$

歪は

$$\frac{\partial}{\partial y} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}, \quad \frac{\partial}{\partial z} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta},$$

であるから

$$\left.\begin{aligned} &e_{11} = \frac{\partial u_1}{\partial x_1} \equiv \frac{\partial u}{\partial x}, \quad e_{22} = \frac{\partial u_2}{\partial x_2} \equiv \cos^2\theta\frac{\partial v}{\partial r} + \frac{\sin^2\theta}{r}v \\ &e_{33} = \sin^2\theta\frac{\partial v}{\partial r} + \frac{\cos^2\theta}{r}v, \quad e_{12} = \frac{1}{2}\left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial x}\right)\cos\theta, \\ &e_{23} = \frac{1}{2}\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right)\sin\theta\cos\theta \\ &\gamma = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial r} + \frac{v}{r} \end{aligned}\right\} \quad (3.2)$$

C 上の s に沿う歪 e_s は $\quad \left(x_s \equiv \frac{\partial x}{\partial s},\ y_s \equiv \frac{\partial y}{\partial s}\right)$

$$\left.\begin{aligned} e_s &= e_{11}x_s^2 + 2\,e_{12}\big|_{\theta=0}\,x_s y_s + e_{22}\big|_{\theta=0}\,y_s^2 \\ &= \frac{\partial u}{\partial x}x_s^2 + \left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial x}\right)x_s y_s + \frac{\partial v}{\partial r}y_s^2 \\ &= \frac{\partial}{\partial s}\left(u\frac{\partial x}{\partial s} + v\frac{\partial r}{\partial s}\right) \end{aligned}\right\} \quad (3.3)$$

これらの関係を(2.6)に代入してθで積分すると次のようになる。

$$\left.\begin{aligned} u(Q) &= \int_C \Big[\tau(p)\, U^{(1)}(p,Q) + \sigma(p)\, U^{(2)}(p,Q) \\ &\qquad - u(p)\, T^{(1)}(p,Q) - v(p)\, T^{(2)}(p,Q)\Big]\, ds(p), \\ v(Q) &= \int_C \left[\tau V^{(1)} + \sigma V^{(2)} - u S^{(1)} - v S^{(2)}\right] ds_p, \end{aligned}\right\} \quad (3.4)$$

$$P \equiv (x, r), \quad Q \equiv (x', r').$$

ここに

$$U^{(1)}(p,Q) = \frac{r}{4\pi}\int_0^{2\pi} U_1^{(1)}(p,Q)\, d\theta, \quad \cdots$$

以下積分記号の中の P, Q は $\left\{\begin{array}{l} P \equiv (x, r\cos\theta, r\sin\theta), \\ Q \equiv (0, r', 0), \end{array}\right\}$ とする。

そうすると左辺の $Q \equiv (0, r')$ となるが常にこれらの関数は常に $(x - x')$ の形で x に関する項を含むので最後に x を $(x - x')$ に戻しておけばよい。

$$\left.\begin{aligned} U^{(2)} &= \frac{r}{4\pi}\int_0^{2\pi}\left[U_2^{(1)}\cos\theta + U_3^{(1)}\sin\theta\right] d\theta, \\ V^{(1)} &= \frac{r}{4\pi}\int_0^{2\pi} U_1^{(2)}\, d\theta, \\ V^{(2)} &= \frac{r}{4\pi}\int_0^{2\pi}\left(U_2^{(2)}\cos\theta + U_3^{(2)}\sin\theta\right) d\theta, \end{aligned}\right\} \quad (3.5)$$

$$
\left.
\begin{aligned}
T^{(1)}(P,Q) &= \frac{r}{4\pi}\int_0^{2\pi} T_1^{(1)}(P,Q)\,d\theta, \\
T^{(2)} &= \frac{r}{4\pi}\int_0^{2\pi}\left(T_2^{(1)}\cos\theta + T_3^{(1)}\sin\theta\right)d\theta, \\
S^{(1)} &= \frac{r}{4\pi}\int_0^{2\pi} T_1^{(2)}\,d\theta, \\
S^{(2)} &= \frac{r}{4\pi}\int_0^{2\pi}\left(T_2^{(2)}\cos\theta + T_3^{(2)}\sin\theta\right)d\theta,
\end{aligned}
\right\} \quad (3.6)
$$

$4 - 4\nu - 1$

$$U_1^{(1)}(P,Q) = \frac{1}{R} - \frac{1}{4(1-\nu)}\frac{\partial^2}{\partial x^2}R \,, \qquad \frac{\partial^2 R}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{x}{R}\right) = \frac{1}{R} - \frac{x^2}{R^3}$$

$$= \frac{(3-4\nu)}{4(1-\nu)}\frac{1}{R} + \frac{x^2}{4(1-\nu)}\left(\frac{1}{R^3}\right),$$

$$U_2^{(1)}\cos\theta + U_3^{(1)}\sin\theta = +\frac{x}{4(1-\nu)R^3}\left[(y-y')\cos\theta + z\sin\theta\right]$$

$$= \frac{xr}{4(1-\nu)R^3}\left(1 - \frac{r'}{r}\cos\theta\cos\theta'\right),$$

$$U_1^{(2)} = +\frac{x}{4(1-\nu)}\frac{(y-y')}{R^3} = \frac{x}{4(1-\nu)R^3}\left(r\cos\theta - r'\cos\theta'\right),$$

$$U_2^{(2)}\cos\theta + U_3^{(2)}\sin\theta = \frac{\cos\theta}{R} - \frac{1}{4(1-\nu)}\left[\left(\frac{1}{R} - \frac{(y-y')^2}{R^3}\right)\cos\theta + \frac{(y-y')z\sin\theta}{R^3}\right]$$

$(y-y')\cos\theta + z\sin\theta$

$$= \frac{(3-4\nu)}{4(1-\nu)}\cdot\frac{\cos\theta}{R} + \frac{1}{4(1-\nu)R^3}\left[(r\cos\theta - r'\cos\theta')(r - r'\cos\theta\cos\theta')\right]$$

$\theta' \to 0$

$(r\cos\theta - r')(r - r'\cos\theta)$

$\simeq (r^2 + r'^2)\cos\theta - rr'(1+\cos^2\theta)$

$$= \frac{-(3-2\nu)}{2(1-\nu)} \frac{x}{R^3} \frac{\partial x}{\partial n} - \frac{1-2\nu}{2(1-\nu)} \left(\frac{(y-y')}{R^3} \frac{\partial y}{\partial n} + \frac{z}{R^3} \frac{\partial z}{\partial n} \right),$$

$$T_1^{(1)}(P,Q) = \frac{1-2\nu}{2(1-\nu)} \frac{\partial}{\partial n} \frac{1}{R} + \frac{1}{1-\nu} \frac{\partial x}{\partial n} \frac{x}{R^3} = -\frac{(3-2\nu)}{2(1-\nu)} \frac{x \frac{\partial y}{\partial n}}{R^3} - \frac{(1-2\nu)(y-y')}{2(1-\nu) R^3}$$

$$T_2^{(1)} = -\frac{\partial x}{\partial n} \frac{(y-y')}{R^3} - \frac{\nu}{1-\nu} \frac{\partial y}{\partial n} \frac{x}{R^3} + \frac{1}{2(1-\nu)} \frac{\partial}{\partial n} \frac{x(y-y')}{R^3}$$

$$= \frac{\partial}{\partial n} \frac{x(y-y')}{R^3} = \frac{\partial x}{\partial n} \left(\frac{y-y'}{R^3} - \frac{3x^2(y-y')}{R^5} \right)$$

$$+ \frac{\partial y}{\partial n} \left(\frac{x}{R^3} - \frac{3x(y-y')^2}{R^5} \right) - 3 \frac{\partial z}{\partial n} \frac{x(y-y')z}{R^5}$$

$$T_2^{(1)} = \frac{\frac{\partial x}{\partial n}}{2(1-\nu)} \left[-\frac{(1-2\nu)(y-y')}{R^3} - \frac{3x^2(y-y')}{R^5} \right]$$

$$+ \frac{\frac{\partial y}{\partial n}}{2(1-\nu)} \left[(1-2\nu) \frac{x}{R^3} - \frac{3x(y-y')^2}{R^5} \right] - \frac{\partial z / \partial n}{2(1-\nu)} \frac{3x(y-y')z}{R^5}$$

$$T_3^{(1)} = \frac{\frac{\partial x}{\partial n}}{2(1-\nu)} \left[-\frac{(1-2\nu) z}{R^3} - \frac{3xz^2}{R^5} \right] - \frac{\frac{\partial y}{\partial n}}{2(1-\nu)} \frac{3x(y-y')z}{R^5}$$

$$+ \frac{\frac{\partial z}{\partial n}}{2(1-\nu)} \left[(1-2\nu) \frac{x}{R^3} - \frac{3xz^2}{R^5} \right]$$

$$2(1-\nu) \left[T_2^{(1)} \cos\theta + T_3^{(1)} \sin\theta \right] = -\frac{\partial x}{\partial n} \left[(1-2\nu) \frac{1}{R^3} + \frac{3x^2}{R^5} \right] \{ y - y' \cos\theta \}$$

$$+ \frac{\partial r}{\partial n} \left[(1-2\nu) \frac{x}{R^3} - \frac{3x}{R^5} \left\{ (y-y')^2 \cos^2\theta + r^2 \sin^4\theta \right\} \right]$$

$$-2 \frac{\partial r}{\partial n} \left[\frac{3x(y-y') r \cos\theta \sin^2\theta}{R^5} \right]$$

$$(y-y')^2 \cos^2\theta + 2(y-y') r \cos\theta \sin^2\theta + r^2 \sin^4\theta$$

$$= (y-y') \left((y-y') \cos^2\theta + 2y \sin^2\theta \right) + r^2 \sin^4\theta$$

$$y - y' \cos^2\theta$$

$$T_1^{(2)} = -\frac{\partial y}{\partial n}\frac{x}{R^3} - \frac{\nu}{1-\nu}\frac{\partial x}{\partial n}\frac{(y-y')}{R^3} + \frac{1}{2(1-\nu)}\frac{\partial}{\partial n}\frac{x(y-y')}{R^3},$$

$$2(1-\nu)T_1^{(2)} = \frac{\partial x}{\partial n}\left[(1-2\nu)\frac{y-y'}{R^3} - \frac{3x^2(y-y')}{R^5}\right]$$

$$+\frac{\partial y}{\partial n}\left[-(1-2\nu)\frac{x}{R^3} - \frac{3x(y-y')^2}{R^5}\right] - \frac{\partial z}{\partial n}\left[\frac{3x(y-y')z}{R^5}\right]$$

$$T_2^{(2)} = \frac{\partial}{\partial n}\frac{1}{R} - \frac{1+\nu}{1-\nu}\frac{\partial y}{\partial n}\frac{(y-y')}{R^3} - \frac{1}{2(1-\nu)}\frac{\partial}{\partial n}\left(\frac{1}{R} - \frac{(y-y')^2}{R^3}\right)$$

$$2(1-\nu)T_2^{(2)} = -(1-2\nu)\left[\frac{\partial x}{\partial n}\frac{x}{R^3} + \frac{\partial y}{\partial n}\frac{y-y'}{R^3} + \frac{\partial z}{\partial n}\frac{z-z'}{R^3}\right]$$

$$-2(1+\nu)\frac{\partial y}{\partial n}\frac{y-y'}{R^3} - \left[\frac{3x(y-y')^2}{R^5}\frac{\partial x}{\partial n} + \frac{\partial y}{\partial n}\left(\frac{3(y-y')^3}{R^5} - \frac{2(y-y')}{R^3}\right) + 3\frac{\partial z}{\partial n}\frac{(y-y')^2 z}{R^5}\right]$$

$$= -\frac{\partial x}{\partial n}\left[-(1-2\nu)\frac{x}{R^3} + \frac{3x(y-y')^2}{R^5}\right]$$

$$-\frac{\partial y}{\partial n}\left[\frac{3(y-y')^3}{R^5} - \frac{y-y'}{R^3}\right] - \frac{\partial z}{\partial n}\left[-(1-2\nu)\frac{z}{R^3} + \frac{3z(y-y')^2}{R^5}\right],$$

$2-2\nu$

$$T_3^{(2)} = -\frac{\partial y}{\partial n}\frac{z}{R^3} - \frac{\nu}{1-\nu}\frac{\partial z}{\partial n}\frac{y-y'}{R^3} + \frac{1}{2(1-\nu)}\frac{\partial}{\partial n}\frac{z(y-y')}{R^3}$$

$$T_3^{(2)} = \frac{-3}{2(1-\nu)}\frac{\partial x}{\partial n}\frac{xz(y-y')}{R^5} - \frac{\partial y}{\partial n}\left\{\frac{z}{R^3} + \frac{1}{2(1-\nu)}\left(\frac{3z(y-y')^2}{R^5} - \frac{z}{R^3}\right)\right\}$$

$$-\frac{\partial z}{\partial n}\left[\frac{\nu}{1-\nu}\frac{(y-y')}{R^3} - \frac{1}{2(1-\nu)}\left(\frac{y-y'}{R^3} - \frac{3z^2(y-y')}{R^5}\right)\right],$$

$$2(1-\nu)T_3^{(2)} = -3\frac{\partial x}{\partial n}\frac{x(y-y')z}{R^5} - \frac{\partial y}{\partial n}\left[(1-2\nu)\frac{z}{R^3} + \frac{3z(y-y')^2}{R^5}\right]$$

$$-\frac{\partial z}{\partial n}\left[-(1-2\nu)\frac{y-y'}{R^3} + \frac{3z^2(y-y')}{R^5}\right],$$

$$2(1-\nu)\left[T_2^{(2)}\cos\theta + T_3^{(2)}\sin\theta\right]$$

$$= -\frac{\partial x}{\partial n}\left[-(1-2\nu)\frac{x}{R^3}\cos\theta + \frac{3x(y-y')}{R^5}\overset{(r-r'\cos\theta)}{\{(y-y')\cos\theta + z\sin\theta\}}\right]$$

$$-\frac{\partial y}{\partial n}\left[\frac{3(y-y')^2}{R^5}(r-r'\cos\theta) - \frac{(y-y')\cos\theta}{R^3} + (1-2\nu)\frac{z\sin\theta}{R^3}\right]$$

$$-\frac{\partial z}{\partial n}\left[-(1-2\nu)\frac{r\sin\theta\cos\theta + (r\cos\theta - r')\sin\theta}{R^3} + \frac{3z(y-y')}{R^5}(r-r'\cos\theta)\right]$$

$$=$$

等となって附録のように(3.5)(3.6)はすべて完全楕円積分で表わされるがかなり複雑になるので実際上は(3.6)を数値積分する方が得策のように思われる。

この時(3.5)の積分はP=Qで対数的特異性を(3.6)は代数的特異性をもつ点に注意すべきである。

付録

$$I_n^m = \frac{1}{4\pi}\int_0^{2\pi}\frac{\cos^m\theta}{R^n}d\theta, \quad \cdots \quad (A.1)$$

必要なものは $n=1,3,5$ で $m \leq n$

$$I_n^m - I_n^{m+2} = \frac{1}{4\pi}\int_0^{2\pi}\frac{\sin^2\theta\cos^m\theta}{R^n}d\theta = \frac{1}{4\pi(n-2)}\int_0^{2\pi}\frac{\cos^{m+1}\theta - m\cos^{m-1}\theta\sin^2\theta}{R^{n-2}}d\theta$$

$$= \frac{1}{n-2}\left[(m+1)I_{n-2}^{m+1} - m\,I_{n-2}^{m-1}\right], \quad (A.2)$$

$$I_n^{m+1} = \frac{1}{4\pi}\int_0^{2\pi}\frac{\cos^{m+1}\theta\,d\theta}{R^n} = \frac{1}{4\pi}\int_0^{2\pi}\left[\frac{m\cos^{m-1}\theta\sin^2\theta}{R^n} + \frac{n\sin^2\theta\cos^{m}\theta}{R^{n+2}}\right]d\theta$$

$$= m\left(I_n^{m-1} - I_n^{m+1}\right) + n\left(I_{n+2}^m - I_{n+2}^{m+2}\right), \quad (A.2')$$

$$\because \quad R^2 = x^2 + r^2 + r'^2 - 2rr'\cos\theta$$

$$\therefore \cos\theta = -\frac{1}{2rr'}\left(R^2 - x^2 - r^2 - r'^2\right) = -\frac{R^2}{2rr'} + \frac{r_1^2+r_2^2}{rr'}$$

$$= -\frac{R^2}{2rr'} + \frac{4(r_1^2+r_2^2)}{k^2 r_2^2}$$

但し $r_1^2 = x^2 + (r+r')^2, \quad r_2^2 = x^2 + (r-r')^2,$

$$k^2 = \frac{4rr'}{r_2^2}, \quad k'^2 = \frac{r_1^2}{r_2^2}.$$

$$\therefore I_n^m = \frac{1}{4\pi}\int_0^{2\pi}\frac{d\theta}{R^2}\cos^{m-1}\theta\left[\frac{4(1+k'^2)}{k^2} - \frac{R^2}{2rr'}\right]$$

$$= \left(\frac{r_1^2+r_2^2}{rr'}\right)I_n^{m-1} - \frac{1}{2rr'}I_{n-2}^{m-1}, \quad \cdots \quad (A.3)$$

$$I_1^{0} = \frac{1}{4\pi}\int_0^{2\pi}\frac{d\theta}{R} = \frac{1}{4\pi r_2}\int_0^{2\pi}\frac{d\theta}{\sqrt{1-k^2\sin^2\frac{\theta}{2}}} = \frac{1}{\pi r_2}\int_0^{\pi/2}\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}},$$

$$R^2 = x^2+r^2+r'^2-2rr'\cos\theta = r_2^2\left(1-k^2\sin^2\frac{\theta}{2}\right).$$

$$I_1^0 = \frac{1}{\pi r_2}K(k), \qquad \cdots \qquad (A\cdot 4)$$

$$I_3^0 = \frac{1}{\pi r_2^3}\int_0^{\frac{\pi}{2}}\frac{d\varphi}{(1-k^2\sin^2\varphi)^{\frac{3}{2}}} = \frac{1}{\pi r_2^3}\frac{E}{k'^2}$$

$(1-2$ $\qquad -2\left(\sin^2\varphi-\frac{1}{k^2}\right)+\frac{2}{k^2}$

$$I_3^1 = \frac{1}{\pi r_2^3}\int_0^{\frac{\pi}{2}}\frac{(1-2\sin^2\varphi)\,d\varphi}{(1-k^2\sin^2\varphi)^{\frac{3}{2}}} = \frac{1}{\pi r_2^3}\int_0^{\frac{\pi}{2}}\frac{\left[1-\frac{2}{k^2}+\frac{2}{k^2}(1-k^2\sin^2\varphi)\right]d\varphi}{(1-k^2\sin^2\varphi)^{\frac{3}{2}}}$$

$$= \left(1-\frac{2}{k^2}\right)I_3^0 + \frac{2}{k^2r_2^2}I_1^0, \qquad \cdots \qquad (A\cdot 5)$$

$$I_1^1 = \frac{1}{\pi r_2}\int_0^{\frac{\pi}{2}}\frac{(1-2\sin^2\varphi)\,d\varphi}{\sqrt{1-k^2\sin^2\varphi}} = \left(1-\frac{2}{k^2}\right)I_1^0 + \frac{2}{k^2r_2}\cdot E, \qquad \cdots \qquad (A\cdot 6)$$

$$I_5^0 = \frac{1}{\pi r_2^5}\int_0^{\frac{\pi}{2}}\frac{d\varphi}{(1-k^2\sin^2\varphi)^{\frac{5}{2}}}$$

$$\frac{\partial}{\partial k}\int_0^{\frac{\pi}{2}}\frac{d\varphi}{(1-k^2\sin^2\varphi)^{\frac{3}{2}}} = +3k\int_0^{\frac{\pi}{2}}\frac{\sin^2\varphi\,d\varphi}{(\qquad)^{\frac{5}{2}}}$$

$$= \frac{\partial}{\partial k}\left(\frac{E}{1-k^2}\right) = \frac{E-K}{k(1-k^2)} + \frac{2Ek}{(1-k^2)^2}$$

$$-\frac{1}{k^2} + \frac{1}{k'^2}$$

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2\varphi\, d\varphi}{(1-k^2\sin^2\varphi)^{\frac{5}{2}}} = \frac{\frac{2}{3}E}{(1-k^2)^2} + \frac{E-K}{k^2(1-k^2)},$$

$$= \frac{1}{k^2}\int_0^{\frac{\pi}{2}} \frac{d\varphi}{(\quad)^{\frac{5}{2}}} - \frac{1}{k^2}\int_0^{\frac{\pi}{2}} \frac{d\varphi}{(\quad)^{\frac{3}{2}}}.$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{d\varphi}{(\quad)^{\frac{5}{2}}} = \frac{E}{1-k^2} + \frac{2E}{3(1-k^2)^2} + \frac{E-K}{k^2(1-k^2)}, \quad (A.7)$$

$$I_5^0 = \frac{1}{\pi\gamma_2^{5}}\left[\frac{E}{k'^2}\left(1 + \frac{2}{3k'^2} + \frac{1}{k^2}\right) - \frac{K}{k^2 k'^2}\right], \quad (A.8)$$