11　　V　　M-106

No.

Date 58.5.15

軸対称ストークス流れの流れ関数

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内容

参考文献

1) 別府 "軸対称流れの最小抵抗問題" 58.5.13

2) Payne, L. E. & Pell, W. H. "The Stokes flow problem for a class of axially symmetric bodies" J. F. M. vol. 2, 1959(?)

概要

今までは最終的に三次元物体を取扱う意図があるので直交座標で定式化して来たが軸対称流れではやはり流れ関数を導入しておく方が便利である。

その流れ関数はポテンシャル流れにおけるストークスの流れ関数の定義をそのまま使えばよいようである。2)

特に直交座標による速度ベクトル表現では軸対称流れへの変換が複雑なので、これらの式のチェックをも兼ねて別法として流れ関数表示の境界方程式を導いて見た。

§1は前置きとして流れ関数の定義と球関数による表示をしただけであまり意味はない。

§2では流れ関数についての相反定理を導き境界における圧力と渦度によって流れ関数が表現出来る事を示し、§3と附録Ⅱで核関数の具体的表現を求め前報告と比較しその妥当性を確かめた。

運動の式 [illegible]

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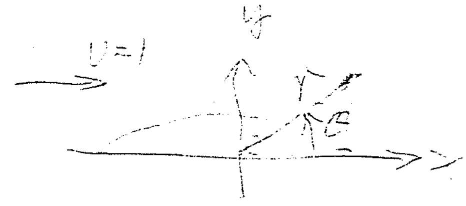
1 流れ関数と渦度

速度を (u, v), 流れ関数を χ とおくと

定義は
$$u = \frac{1}{y}\frac{\partial \chi}{\partial y}$$
$$v = -\frac{1}{y}\frac{\partial \chi}{\partial x}$$



渦度は
$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\frac{1}{y} L \chi, \quad (1.2)$$

ここに
$$L \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{1}{y}\frac{\partial}{\partial y} = \frac{\partial^2}{\partial x^2} + y\frac{\partial}{\partial y}\left[\frac{1}{y}\frac{\partial}{\partial y}\right], \quad (1.3)$$

なお ラプラシアン* Δ は
$$\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1}{y}\frac{\partial}{\partial y} - \frac{1}{y^2}, \quad (1.4)$$

ζ は調和関数なので
$$\Delta \zeta = \frac{1}{y} L(y\zeta) = 0, \quad (1.5)$$

それ故 (1.2) から
$$L^2 \chi = 0, \quad (1.6)$$

ここで極座標を導入しよう。
$$x = r\cos\theta, \quad y = r\sin\theta, \quad \mu = \cos\theta$$

とすると
$$\frac{\partial}{\partial x} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} = \mu\frac{\partial}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial}{\partial \mu}$$
$$\frac{\partial}{\partial y} = \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta} = \sqrt{1-\mu^2}\frac{\partial}{\partial r} - \frac{\mu\sqrt{1-\mu^2}}{r}\frac{\partial}{\partial \mu} \quad (1.7)$$

$$L \equiv \frac{\partial^2}{\partial r^2} + \frac{1-\mu^2}{r^2}\frac{\partial^2}{\partial \mu^2}, \quad (1.8)$$

* u についてのラプラシアンは $\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1}{y}\frac{\partial}{\partial y}\right] u = 0$

$$\Delta u = u_{xx} + u_{yy} + \frac{u_y}{y} \cancel{} = -\frac{1}{y}\frac{\partial}{\partial x}\left(\frac{\partial (yv)}{\partial y}\right) + u_{yy} + \frac{u_y}{y} - \frac{u}{y^2}$$

$$= u_{yy} + \frac{u_y}{y} - \frac{u}{y^2} - v_{xy} - \frac{v_x}{y} = \frac{\partial}{\partial y}(u_y - v_x) + \frac{1}{y}(u_y - v_x)$$

$$= -\frac{1}{y}\frac{\partial}{\partial y}(y\zeta) = \frac{\partial p}{\partial x} \qquad - \frac{u}{y^2}$$

$v + u_y$

$$\Delta v = v_{xx} + v_{yy} + \frac{v_y}{y} - \frac{v}{y^2} = v_{xx} - u_{xy} = \frac{\partial \zeta}{\partial x} = \frac{\partial p}{\partial y}$$

$$v + yv_y = \frac{\partial}{\partial y}(yv) = -y u_x ,$$

$$2v_y + y v_{yy} = -u_x - y u_{xy} , \quad v_{yy} + \frac{v_y}{y} = -\frac{v_y}{y} - \frac{u_x}{y} - u_{xy}$$

$$v_{yy} + \frac{v_y}{y} - \frac{v}{y^2} = -\frac{1}{y^2}\frac{\partial}{\partial y}(yv) - \frac{u_x}{y} - u_{xy} = -u_{xy}$$

$$\therefore \quad \frac{\partial p}{\partial x} = -\frac{1}{y}\frac{\partial}{\partial y}(y\zeta) , \quad \frac{\partial p}{\partial y} = \frac{1}{y}\frac{\partial}{\partial x}(y\zeta) .$$

$$\frac{\partial p}{\partial s} = \frac{1}{y}\frac{\partial}{\partial n}(y\zeta)$$

ルジャンドル関数 $P_n(\mu)$ を導入すると

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dP_n}{d\mu}\right\} + n(n+1)P_n(\mu) = 0 \quad , \quad \cdots \quad (1.8)$$

すなわち

$$f = \frac{(1-\mu^2)}{r^n}\frac{dP_n}{d\mu}$$

とおくと

$$Lf = 0 , \quad (1.9)$$

また

$$L\left((1-\mu^2)r^{n+1}\frac{dP_n}{d\mu}\right) = 0 \quad , \quad \cdots \quad (1.10)$$

であるが特に $n=1$ では

$$f = \frac{y^2}{2} \quad , \quad \cdots \quad (1.11)$$

は一様流れで $n \geq 1$ では上式だけ考えておけばよい

さて

$$h = (1-\mu^2)r^{2-n}\frac{dP_n}{d\mu} \quad , \quad \cdots \quad (1.11)$$

とおくと

$$Lh = -2(2n+1)\frac{(1-\mu^2)}{r^n}\frac{dP_n}{d\mu} \quad , \quad (1.12)$$

となるから

$$L^2 h = 0 \quad , \quad \cdots \quad (1.14)$$

それ故一般に流れ関数は一様流れを含めて

$$\chi = \frac{y^2}{2} + (1-\mu^2)\sum_{n=p}^{\infty}\left(A_n r^{2-n} + B_n r^{-n}\right)\frac{dP_n}{d\mu} \quad , \quad (1.15)$$

のように表わされる。

速度に

$$u = \frac{1}{r}\frac{\partial}{\partial\mu}\psi = \left(\frac{1}{r}\frac{\partial}{\partial r} - \frac{\mu}{r^2}\frac{\partial}{\partial\mu}\right)\chi$$

$$= 1 + (1-\mu^2)\sum_{n=1}^{\infty}\left(\frac{(2-n)}{r^n}A_n - \frac{nB_n}{r^{n+2}}\right)\frac{dP_n}{d\mu}$$

$$+ \mu\sum_{n=1}^{\infty}\left(\frac{A_n}{r^n} - \frac{B_n}{r^{n+2}}\right)n(n+1)P_n(\mu) \quad ,$$

$$v = -\frac{1}{r}\frac{\partial\psi}{\partial r} = \left(-\frac{\mu}{r\sqrt{1-\mu^2}}\frac{\partial}{\partial r} - \frac{\sqrt{1-\mu^2}}{r^2}\frac{\partial}{\partial\mu}\right)\chi$$

$$= -\mu\sqrt{1-\mu^2}\sum_{n=1}^{\infty}\left(\frac{(2-n)}{r^n}A_n - \frac{nB_n}{r^{n+2}}\right)\frac{dP_n}{d\mu}$$

$$+ \sqrt{1-\mu^2}\sum_{n=1}^{\infty}\left(\frac{A_n}{r^n} + \frac{B_n}{r^{n+2}}\right)n(n+1)P_n \quad ,$$

$$y\zeta = L\chi = -2(1-\mu^2)\sum_{n=1}\frac{(2n-1)A_n}{r^n}\frac{dP_n}{d\mu} .$$

あるいは

$$\zeta = 2\sum_{n=1}^{\infty}\frac{(2n-1)A_n}{r^{n+1}}P_n^1(\mu) \quad , \quad \cdots (1.17)$$

但し

$$P_n^m(\mu) = (-)^m(1-\mu^2)^{\frac{m}{2}}\frac{d^m}{d\mu^m}P_n(\mu) \quad , \quad \cdots (1.18)$$

なお

$$P_0(\mu) = 1, \quad P_1(\mu) = \mu, \quad P_2(\mu) = \frac{1}{2}(3\mu^2 - 1),$$

$$P_3(\mu) = \frac{1}{2}(5\mu^3 - 3\mu), \quad P_4 = \frac{1}{8}(35\mu^4 - 30\mu^2 + 3),$$

$$\frac{dP_0}{d\mu} = 0, \quad \frac{dP_1}{d\mu} = 1, \quad \frac{dP_2}{d\mu} = 3\mu$$

境界条件は

$$u = v = 0 \quad \text{on } C \qquad (1.20)$$

であるが、これは $\frac{\partial \psi}{\partial n} = \frac{\partial \psi}{\partial s} = 0$,

でもよく、又一般の形式では

$$\frac{\partial}{\partial n}\psi = \psi = 0 \quad \text{on } C, \qquad (1.21)$$

でよい。

(1.16)より遠方では

$$\left.\begin{aligned} u &\fallingdotseq 1 + \frac{A_1}{r}(1+\mu^2) \\ v &\fallingdotseq \frac{A_1}{r}\mu\sqrt{1-\mu^2} \end{aligned}\right\} \cdots (1.22)$$

一方抗力をDとすると、

$$\left.\begin{aligned} u-1 &\fallingdotseq \frac{-D}{8\pi r}(1+\mu^2) \\ v &= \frac{-D}{8\pi r}\mu\sqrt{1-\mu^2} \end{aligned}\right\} \cdots (1.23)$$

$$\therefore \quad A_1 = \frac{-D}{8\pi}, \quad \cdots \cdots (1.24)$$

球について考えて見よう。

(1.21)の境界条件の方が便利でこれを書き下すと、

$$\frac{r^2}{2} + r^2 \sum_{n=1} \left(\frac{A_n}{r^n} + \frac{B_n}{r^{n+2}} \right) \frac{dP_n}{d\mu} = 0$$

$$r + r \sum_{n=1}^{\infty} \left(\frac{(2-n)}{r^n} A_n - \frac{n B_n}{r^{n+2}} \right) \frac{dP_n}{d\mu} = 0 \qquad (1.25)$$

これは一般の形式でもよいが、今 $r=1$ とおくと

辺々相引いて

$$\sum (n A_n + (n+2) B_n) \frac{dP_n}{d\mu} = 0$$

から $$B_n = -\frac{n}{n+2} A_n$$

これを第1式に代入して $\frac{dP_n}{d\mu}$ の係数を 0 とおくと

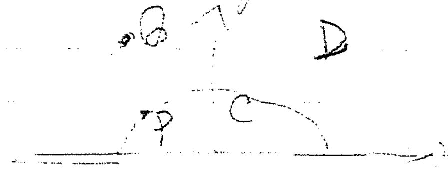
$$\left.\begin{aligned} & A_1 = -3B_1 = -\frac{3}{4}, \\ & A_n = B_n = 0 \quad \text{for } n \geq 2 \\ & \therefore D = 6\pi, \end{aligned}\right\} \qquad (1.27)$$

球に近い物体以外では一般に (1.18) は境界上で収束すると体言えないので境界上に特異性をおく方法が望ましい。

2. 可逆定理

Φ, Ψ を [illegible] の関数とし，
において次の積分

$$I=\iint_D\left[\Phi L(\Psi)-\Psi L(\Phi)\right]\frac{dxdy}{y}, \quad (2.1)$$



部分積分によって

$$I=\int_C\left[\frac{\Psi}{y}\frac{\partial\Phi}{\partial n}-\frac{\Phi}{y}\frac{\partial\Psi}{\partial n}\right]ds, \quad (2.2)$$

$$L(\Psi)=L(\Phi)=0 \text{ の時は.} \quad (2.3)$$

$$\int_C\left[\Psi\frac{\partial\Phi}{\partial n}-\Phi\frac{\partial\Psi}{\partial n}\right]\frac{ds}{y}=0, \quad (2.4)$$

さて もっと一般的に (2.2)が成立つから，まず (2.1) (2.2) の Φ の所に $L\xi$ を代入してみると

$$\iint_D\left[L(\xi)L(\Psi)-\Psi L^2(\xi)\right]\frac{dxdy}{y}=\int_C\left[\Psi\frac{\partial L(\xi)}{\partial n}-L(\xi)\frac{\partial\Psi}{\partial n}\right]\frac{ds}{y}, \quad (2.5)$$

Ψ のかわりに ξ，Φ のかわりに $L(\Psi)$ を入れると

$$\iint_D\left[L(\Psi)L(\xi)-\xi L^2(\Psi)\right]\frac{dxdy}{y}=\int_C\left[\xi\frac{\partial L(\Psi)}{\partial n}-L(\Psi)\frac{\partial\xi}{\partial n}\right]\frac{ds}{y}, \quad (2.6)$$

辺々相引いて $L^2(\xi)=L^2(\Psi)=0, \quad (2.7)$
とおくと

$$\iint_D\left[\xi L^2(\Psi)-\Psi L^2(\xi)\right]\frac{dxdy}{y}=0=\int_C\left[\Psi\frac{\partial L(\xi)}{\partial n}-L(\xi)\frac{\partial\Psi}{\partial n}-\xi\frac{\partial L(\Psi)}{\partial n}+L(\Psi)\frac{\partial\xi}{\partial n}\right]\frac{ds}{y}, \quad (2.8)$$

なる相反定理を得る。

(1.2)より　$L(\psi) = -y\zeta$　であるので

$$\int_C \left[\psi \frac{\partial L(G)}{\partial n} - L(G)\frac{\partial \psi}{\partial n} + \zeta \frac{\partial (y\zeta)}{\partial n} - y\zeta \frac{\partial \zeta}{\partial n} \right] \frac{ds}{y}, \quad (2.9)$$

又さらに ζ の代りに 核 $X(P,Q)$ を代入すると

$$L_P[X(P)] = -G(P,Q) \quad \cdots \quad (2.10)$$

$$-\int_\varepsilon \frac{\partial G}{\partial n}\frac{ds}{y} = 1 \quad , \quad \cdots \quad (2.11)$$

なので

$$\psi(Q) = \int_C \left[\psi \frac{\partial G}{\partial n} + \frac{\partial \psi}{\partial n} G + \frac{\partial (y\zeta)}{\partial n} X - y\zeta \frac{\partial X}{\partial n} \right] \frac{ds}{y}, \quad (2.12)$$

を得る。

境界条件は

$$\psi(P) = -\frac{y^2}{2}, \quad \frac{\partial \psi}{\partial n} = -y\frac{\partial y}{\partial n}, \quad \text{on } C, \quad (2.13)$$

ここで Q点が領域Dの点ならば明らかに

$$\int_C \left[\frac{y^2}{2}\frac{\partial G}{\partial n} - y\frac{\partial y}{\partial n} G \right] \frac{ds}{y} = 0, \quad \cdots \quad (2.14)$$

よって (2.13) を満たす場合は

$$\psi(Q) = \int_C \left[\frac{\partial (y\zeta)}{\partial n} X - y\zeta \frac{\partial X}{\partial n} \right] \frac{ds}{y}, \quad (2.15)$$

さらに圧力 p を導入すると、

$$\frac{\partial p}{\partial s} = \frac{1}{y}\frac{\partial (y\zeta)}{\partial n}, \qquad \cdots (2.16)$$

であるから

$$\Psi(Q) = \int_C \left[\frac{\partial p}{\partial s}X - \zeta\frac{\partial X}{\partial n}\right] ds, \qquad \cdots (2.17)$$

あるいは部分積分して

$$\Psi(Q) = -\int_C \left[p\frac{\partial X}{\partial s} + \zeta\frac{\partial X}{\partial n}\right] ds, \qquad (2.18)$$

さらに

$$\left.\begin{aligned} \tau_x &= -p\frac{\partial x}{\partial n} - \zeta\frac{\partial y}{\partial n} \\ \tau_y &= -p\frac{\partial y}{\partial n} + \zeta\frac{\partial x}{\partial n} \end{aligned}\right\} \qquad \cdots (2.19)$$

とおけば

$$p\frac{\partial X}{\partial s} + \zeta\frac{\partial X}{\partial n} = \left(p\frac{\partial x}{\partial s} + \zeta\frac{\partial x}{\partial n}\right)\frac{\partial X}{\partial x} + \left(p\frac{\partial y}{\partial s} + \zeta\frac{\partial y}{\partial n}\right)\frac{\partial X}{\partial y}$$
$$= \tau_y\frac{\partial X}{\partial x} - \tau_x\frac{\partial X}{\partial y},$$

となって

$$\Psi(Q) = \int_C \left[\tau_x\frac{\partial X}{\partial y} - \tau_y\frac{\partial X}{\partial x}\right] ds, \qquad (2.20)$$

それ故

$$\left.\begin{aligned} u &= \frac{1}{y'}\frac{\partial \Psi}{\partial y'} = \frac{1}{y'}\int_C \left[\tau_x\frac{\partial^2 X}{\partial y\partial y'} - \tau_y\frac{\partial^2 X}{\partial x\partial y'}\right] ds, \\ v &= -\frac{1}{y'}\frac{\partial \Psi}{\partial x'} = -\frac{1}{y'}\int_C \left[\tau_x\frac{\partial^2 X}{\partial y\partial x'} - \tau_y\frac{\partial^2 X}{\partial x\partial x'}\right] ds, \end{aligned}\right\} \qquad (2.21)$$

$$
\left.
\begin{aligned}
&\left.
\begin{aligned}
V_1^{(1)} &= -\frac{1}{g^{y'}} \frac{\partial^2 X}{\partial y \partial y'} \\
V_y^{(1)} &= \frac{1}{g^{y'}} \frac{\partial^2 X}{\partial x \partial y'}
\end{aligned}
\right\} \\
&\left.
\begin{aligned}
V_1^{(y)} &= \frac{1}{g^{y'}} \frac{\partial^2 X}{\partial y \partial x'} \\
V_y^{(y)} &= -\frac{1}{g^{y'}} \frac{\partial^2 X}{\partial x \partial x'}
\end{aligned}
\right\}
\end{aligned}
\right\} \qquad (2.22)
$$

$$
\left.
\begin{aligned}
u &= -\int \left[\tau_x V_1^{(1)} + \tau_y V_y^{(1)} \right] y \, ds \\
v &= -\int_c \left[\tau_x V_1^{(y)} + \tau_y V_y^{(y)} \right] y \, ds
\end{aligned}
\right\} \cdots\cdots (2.23)
$$

3. 核関数

$$X = X_1 + X_2 \qquad (3.1)^{*}$$

$$X_1 = \frac{y^2 y'}{4}\int_0^\infty e^{-t|x-x'|} J_0(ty) J_1(ty')\frac{dt}{t}, \qquad (3.2)$$

$$X_2 = -\frac{y y'^2}{4}\int_0^\infty e^{-t|x-x'|} J_1(ty) J_0(ty')\frac{dt}{t}, \qquad (3.3)$$

とおくと二つの式より

$$L_P \equiv \frac{\partial^2}{\partial x^2} + y\frac{\partial}{\partial y}\frac{1}{y}\frac{\partial}{\partial y}, \quad L_Q \equiv \frac{\partial^2}{\partial x'^2} + y'\frac{\partial}{\partial y'}\left(\frac{1}{y'}\frac{\partial}{\partial y'}\right)$$

として、

$$\left.\begin{aligned} L_P[X_1] &= -\frac{yy'}{2}\int_0^\infty e^{-tx} J_1(yt) J_1(ty')\,dt = G(P,Q), \\ &= -\frac{yy'}{4\pi}\int_0^{2\pi}\frac{\cos\theta}{R}d\theta, \quad R^2 = x^2+y^2+y'^2-2yy'\cos\theta \\ L_Q[X_1] &= 0, \end{aligned}\right\} (3.4)$$

$$\left.\begin{aligned} L_P[X_2] &= 0, \\ L_Q[X_2] &= -\frac{yy'}{2}\int_0^\infty e^{-tx} J_1(ty) J_1(ty')\,dt = G(P,Q) \end{aligned}\right\} (3.5)$$

(以下では $x'=0$, $x>0$ として表す)

$$G(P,Q) \xrightarrow[P\to Q]{} -\frac{y}{2\pi}\log\left(\sqrt{x^2+(y-y')^2}\right), \qquad (3.6)$$

* 楕円積分表示は附録A

$$y\frac{\partial}{\partial y}\left[\frac{1}{y}\frac{\partial}{\partial y}J_0(ty)\right] = -yt\frac{\partial}{\partial y}\left[\frac{J_1(ty)}{y}\right] = \frac{t}{y}J_1 - t^2 J_1'(ty)$$

$$= \frac{2t}{y}J_1 - t^2 J_0$$

$$L_y\left[J_0(ty)\right] = \frac{2t}{y}J_1 \quad //$$

$$y\frac{\partial}{\partial y}\left[\frac{1}{y}\frac{\partial}{\partial y}y^2 J_0(ty)\right] = z\frac{d}{dz}\left[\frac{1}{z}\frac{d}{dz}z^2 J_0\right] = z\frac{d}{dz}\left[2J_0 - zJ_1\right]$$

$$= -2zJ_1 - z^2 J_0$$

$$\therefore\ L\left[y^2 J_0(ty)\right] = -2yt J_1(yt) \quad //$$

$$y\frac{\partial}{\partial y}\left[\frac{1}{y}\frac{\partial}{\partial y}(yJ_1)\right] = t z\frac{d}{dz}\left[\frac{1}{z}\frac{d}{dz}(zJ_1)\right] = t z\frac{d}{dz}J_0 = -zJ_1$$

$$\therefore\ L\left[yJ_1(yt)\right] = 0 \quad //$$

$$y\frac{\partial}{\partial y}\left[\frac{1}{y}\frac{\partial}{\partial y}(y^2 J_2)\right] = z\frac{d}{dz}\left[\frac{1}{z}\frac{d}{dz}z^2 J_2\right] = z\frac{d}{dz}zJ_1$$

$$= z^2 J_0 = z^2\left(\frac{2}{z}J_1 - J_2\right).$$

$$\therefore\ L\left(y^2 J_2(ty)\right) = 2yt J_1(yt) \quad //$$

$$\frac{\partial}{\partial y}J_0(ty) = -tJ_1(ty), \quad \frac{\partial}{\partial y}\left[yJ_1(ty)\right] = yt J_0(ty)$$

$$\frac{\partial}{\partial y}\left[y^2 J_2(ty)\right] = y^2 t J_1(ty), \quad \frac{\partial}{\partial y}\left[y^2 J_0(ty)\right] = y\left[2J_0 - ytJ_1\right]$$

$$J_2(ty) = \frac{2}{yt}J_1(yt) - J_0(yt)$$

さらに微分すると

$$\frac{1}{y}\frac{\partial X_1}{\partial y} = \frac{yy'}{2}\int_0^\infty e^{-tx} J_1(ty) J_1(ty') \frac{dt}{t} - \frac{yy'}{4}\int_0^\infty e^{-tx} J_1(ty) J_1(ty')\, dt$$

$$= \frac{2}{y^2}X_1 + \frac{y'}{2} G$$

$$\frac{1}{y}\frac{\partial X_1}{\partial x} = -\frac{yy'}{4}\int_0^\infty e^{-tx} J_0(ty) J_1(ty')\, dt = -\frac{yy'}{2}\tilde{T},$$

$$\frac{\partial X_1}{\partial x'} = -\frac{\partial X_1}{\partial x},$$

$$\frac{1}{y'}\frac{\partial X_1}{\partial y'} = \frac{y^2}{4}\int_0^\infty e^{-tx} J_0(ty) J_0(ty')\, dt = \frac{y^2}{2} S,$$

$$\frac{1}{yy'}\frac{\partial^2 X_1}{\partial y \partial y'} = S + \frac{y}{2}\frac{\partial S}{\partial y} =$$

$$\frac{1}{yy'}\frac{\partial^2 X_1}{\partial y' \partial x} = -\frac{y}{2y'}\frac{\partial \tilde{T}}{\partial y'} = -\frac{y}{2}\frac{\partial S}{\partial x'} = \frac{y}{2}\frac{\partial S}{\partial x} =$$

$$\frac{1}{yy'}\frac{\partial^2 X_1}{\partial x' \partial y} = \frac{-1}{yy'}\frac{\partial}{\partial y}\left[\frac{y^2}{2}\tilde{T}\right] = -\frac{1}{y'}\tilde{T} - \frac{y}{2y'}\frac{\partial \tilde{T}}{\partial y}$$

$$= -\frac{1}{y'}\tilde{T} + \frac{y}{2}\frac{\partial P}{\partial x}$$

$$\frac{1}{yy'}\frac{\partial^2 X_1}{\partial x \partial x'} = -\frac{y}{2y'}\frac{\partial \tilde{T}}{\partial x'} = \frac{y}{2}\frac{\partial S}{\partial y'} =$$

$$\frac{1}{y}\frac{\partial X_2}{\partial y} = -\frac{y'^2}{4}\int_0^\infty e^{-tx} J_2(ty) J_2(ty')\, dt$$

$$= -\frac{y'}{2}\int_0^\infty e^{-tx} J_2(ty) J_1(y't)\frac{dt}{t} + \frac{y'^2}{4}\int_0^\infty e^{-tx} J_1(ty) J_2(ty')\, dt$$

$$= \frac{y'^2}{2} S + \tilde{P}_A ;$$

$$\frac{1}{y}\frac{\partial X_2}{\partial x} = -\frac{1}{y}\frac{\partial X_2}{\partial x'} = \frac{y'^2}{4}\int_0^\infty e^{-tx} J_1(ty) J_2(ty')\, dt$$

$$= \frac{y'}{2}\int_0^\infty e^{-tx} J_1(ty) J_1(ty')\frac{dt}{t} - \frac{y'^2}{4}\int_0^\infty e^{-tx} J_1(ty) J_0(ty')\, dt$$

$$= \frac{1}{y} G_A - \frac{y'^2}{2y} T ,$$

$$\frac{1}{yy'}\frac{\partial X_2}{\partial y'} = -\frac{yy'}{4}\int_0^\infty e^{-tx} J_1(ty) J_1(ty')\, dt = \frac{1}{2} G ,$$

$$\frac{1}{yy'}\frac{\partial^2 X_2}{\partial y'\partial y} = \frac{1}{2y}\frac{\partial G}{\partial y} = \frac{1}{2} y' \frac{\partial S}{\partial y'} ,$$

$$\frac{1}{yy'}\frac{\partial^2 X_2}{\partial y'\partial x} = \frac{1}{2y}\frac{\partial G}{\partial x} = -\frac{1}{2}\frac{\partial T}{\partial y} = -\frac{y'}{2}\frac{\partial P}{\partial x'} \qquad (3.10)$$

$$\frac{1}{yy'}\frac{\partial^2 X_2}{\partial x'\partial y} = \frac{y'}{2}\frac{\partial S}{\partial x'} + \frac{1}{y'}\tilde{T}$$

$$\frac{1}{yy'}\frac{\partial^2 X_2}{\partial x\partial x'} = \frac{-1}{yy'} G - \frac{y'}{2y}\frac{\partial T}{\partial x'} = +P + \frac{y'}{2}\frac{\partial S}{\partial y} ,$$

結局

$$\frac{1}{yy'}\frac{\partial^2 X}{\partial y \partial y'} = P + \frac{1}{2}\left(y\frac{\partial S}{\partial y} + y'\frac{\partial S}{\partial y'}\right),$$

$$\frac{1}{yy'}\frac{\partial^2 X}{\partial y' \partial x} = \frac{1}{2}\left[y\frac{\partial S}{\partial x} - y'\frac{\partial P}{\partial x}\right], \qquad (3.11)$$

$$\frac{1}{yy'}\frac{\partial^2 X}{\partial x' \partial y} = \frac{1}{2}\left[y\frac{\partial P}{\partial x} - y'\frac{\partial S}{\partial x}\right],$$

$$\frac{1}{yy'}\frac{\partial^2 X}{\partial x' \partial x} = P + \frac{1}{2}\left[y'\frac{\partial S}{\partial y} + y\frac{\partial S}{\partial y'}\right],$$

となって、(2.22)を通じ附録(A.9)に一致する。

附録 A 種々な核

$$S^*(p,q) = \frac{1}{4\pi}\int_0^{2\pi} R\,d\theta = \frac{r_2 E(k)}{\pi} \quad , \qquad \cdots (A\cdot1)$$

$$S = \frac{1}{4\pi}\int_0^{2\pi}\frac{d\theta}{R} = \frac{K(k)}{\pi r_2} = \frac{1}{2}\int_0^\infty e^{-t|x|} J_0(ty) J_0(ty')\,dt, \quad (A\cdot2)$$

$$P = \frac{1}{4\pi}\int_0^{2\pi}\frac{\cos\theta}{R}\,d\theta = \frac{1}{2}\int_0^\infty e^{-t|x|} J_1(ty) J_1(ty')\,dt, \quad (A\cdot3)$$

$$\left.\begin{aligned}
T &= \frac{y}{2}\int_0^\infty e^{-t|x|} J_1(ty) J_0(ty')\,dt, \\
\tilde{T} &= \frac{y'}{2}\int_0^\infty e^{-t|x|} J_0(ty) J_1(ty')\,dt, \\
\frac{d\tilde{T}_A}{dx} &= \tilde{T}, \quad \tilde{T}_A = -\frac{y'}{2}\int_0^\infty e^{-tx} J_0(ty) J_1(ty')\frac{dt}{t},
\end{aligned}\right\} \quad (A\cdot4) \quad \begin{aligned} &T(x; y, y') \\ &= \tilde{T}(x; y', y), \end{aligned}$$

[ダクトペラ理論では $G = -yy'P$,
$S = \tilde{T}$, $S_A = \tilde{T}_A$, が対応している]

E, Kは楕円積分で $T, \tilde{T}$は第3種楕円積分など。

$$(k')^2 = r_1^2/r_2^2, \quad k^2 = 1 - k'^2 = \frac{4yy'}{r_2^2},$$

$$r_1^2 = x^2 + (y-y')^2, \quad r_2^2 = x^2 + (y+y')^2,$$

まず

$$\left.\begin{aligned}
\frac{\partial}{\partial x}S^* &= -\frac{\partial}{\partial x'}S^* = xS, \\
\frac{\partial S^*}{\partial y} &= yS - y'P, \\
\frac{\partial S^*}{\partial y'} &= y'S - yP,
\end{aligned}\right\} \quad (A\cdot5)$$

$$\frac{\partial T}{\partial x} = \frac{1}{y'}\frac{\partial G}{\partial y'} = \frac{-y}{y'}\frac{\partial (y'P)}{\partial y'} = y\frac{\partial S}{\partial y'},$$

$$\frac{\partial T}{\partial y'} = -\frac{1}{y'}\frac{\partial G}{\partial x} = y\frac{\partial P}{\partial x}, \quad \frac{\partial T}{\partial y} = -y\frac{\partial S}{\partial x},$$

$$\frac{\partial \tilde{T}}{\partial x} = \frac{1}{y}\frac{\partial G}{\partial y} = \frac{-y'}{y}\frac{\partial (yP)}{\partial y} = y'\frac{\partial S}{\partial y'},$$

$$\frac{\partial \tilde{T}}{\partial y'} = -y'\frac{\partial S}{\partial x}, \quad \frac{\partial \tilde{T}}{\partial y} = -\frac{1}{y}\frac{\partial G}{\partial x} = -y'\frac{\partial P}{\partial x}, \qquad \text{(A.6)}$$

$$\frac{\partial}{\partial x} S^* = x S = -\frac{1}{2}\int_0^\infty J_0(ty)J_0(ty')\frac{\partial e^{-tx}}{\partial t}\,dt$$

$$= \frac{1}{2} - (T + \tilde{T}), \qquad \text{---(A.7)}$$

$$\frac{\partial^2}{\partial x \partial x} S^* = -\left(\frac{\partial T}{\partial x} + \frac{\partial \tilde{T}}{\partial x}\right) = -y\frac{\partial S}{\partial y} - y'\frac{\partial S}{\partial y'},$$

$$\frac{\partial^2 S^*}{\partial x \partial y} = -\frac{\partial}{\partial y}(T + \tilde{T}) = y\frac{\partial S}{\partial x} - y'\frac{\partial P}{\partial x},$$

$$\frac{\partial^2 S^*}{\partial x \partial y'} = y'\frac{\partial S}{\partial x} - y\frac{\partial P}{\partial x}, \qquad \text{(A.8)}$$

$$\frac{\partial^2 S^*}{\partial y \partial y'} = y\frac{\partial S}{\partial y'} - \frac{\partial (y'P)}{\partial y'} = -P + y\frac{\partial S}{\partial y'} - y'\frac{\partial P}{\partial y'}$$

$$V_1^{(1)} = \frac{1}{2}S^*_{xx} - S = -\left[S + \frac{1}{2}(y S_y + y' S_{y'})\right],$$

$$V_2^{(1)} = \frac{1}{2}S^*_{xy} = \frac{1}{2}(y S_x - y' P_x),$$

$$V^{(2)} = -\frac{1}{2}S^*_{xy'} = \frac{1}{2}(y P_x - y' S_x) \qquad \text{(A.9)}$$

$$V_2^{(2)} = -P - \frac{1}{2}S^*_{yy'} = -\frac{P}{2} - \frac{1}{2}(y S_{y'} - y' P_{y'}),$$

$$= -P - \frac{1}{2}(y S_{..} + y' S_{..})$$

実際の計算には（2.11），(A-1)の積分を用いずに微分を計算する方が楽である。（以下 [illegible]）

楕円積分の微分式

$$k\frac{dK}{dk} = \frac{E}{k'^2} - K,$$

$$k\frac{dE}{dk} = E - K,$$

から

$$\frac{\partial}{\partial x} S^* = x S,$$

$$\frac{\partial}{\partial y} S^* = \frac{S^*}{2y} - \left\{\frac{r_2^2}{2y} - (y+y')\right\} S,$$

$$\frac{\partial S}{\partial x} = -\frac{x}{(r_1 r_2)^2} S^*,$$

$$\frac{\partial S}{\partial y} = -\frac{S}{2y} + \frac{1}{r_1^2}\left(\frac{1}{2y} - \frac{y+y'}{r_2^2}\right) S^*,$$

$$\frac{\partial^2}{\partial x^2} S^* = S - \frac{x^2}{(r_1 r_2)^2} S^*,$$

$$\frac{\partial^2}{\partial x \partial y} S^* = -\frac{x}{2y} S + \frac{x}{r_1^2}\left(\frac{1}{2y} - \frac{y+y'}{r_2^2}\right) S^*,$$

$$\frac{\partial^2}{\partial x \partial y'} S^* = -\frac{x}{2y'} S + \frac{x}{r_1^2}\left(\frac{1}{2y'} - \frac{y+y'}{r_2^2}\right) S^*,$$

$$\frac{\partial^2}{\partial y \partial y'} S^* = -\left(\frac{y^2+y'^2}{2yy'}\right) S + \frac{\left\{x^2(y^2+y'^2) + (y^2-y'^2)^2\right\}}{2yy'(r_1 r_2)^2} S^*$$

$$X_1 = \frac{(yy')^2}{8}\int_0^\infty e^{-tx} J_2(ty)\{J_0(ty') + J_2(ty')\}\,dt ,$$

$$X_2 = -\frac{(yy')^2}{8}\int_0^\infty e^{-tx}\{J_0(ty) + J_2(ty)\} J_2(ty')\,dt ,$$

$$\therefore\ X = \frac{(yy')^2}{8}\int_0^\infty e^{-tx}\left[J_0(ty)J_0(ty') - J_2(ty)J_2(ty')\right] dt ,$$

$$S = \frac{1}{4\pi}\int_0^{2\pi}\frac{d\theta}{R} = \frac{K(k)}{\pi r_2} = \frac{1}{2}\int_0^\infty e^{-tx} J_0(ty)J_0(ty')\,dt ,$$

$$\frac{1}{R} = \int_0^\infty e^{-tx}dt\left[J_0(ty)J_0(ty') + 2\sum_{n=1}^{\infty} J_n(ty)J_n(ty')\cos n\theta\right].$$

$$\therefore\ I = \frac{1}{2\pi}\int_0^{2\pi}\frac{\cos 2\theta}{R}\,d\theta = \int_0^\infty e^{-tx} J_2(ty)J_2(ty')\,dt ,$$

$$\therefore\ X = \frac{(yy')^2}{4} S - \frac{(yy')^2}{8} I ,$$

$$\left.\begin{array}{l} r_1^2 = x^2 + (y-y')^2, \quad r_2^2 = x^2 + (y+y')^2, \\ k'^2 = \left(\dfrac{r_1}{r_2}\right)^2, \quad k^2 = 1 - k'^2 \end{array}\right\} \text{とおく}$$

$R = r_2\sqrt{1-k^2\sin^2\varphi}$, ただし, $\theta = \pi - 2\varphi$, $\cos\theta = -\cos 2\varphi$

$$\therefore\ \cos 2\theta = 1 - 8\sin^2\varphi\cos^2\varphi$$

$$I = \frac{2}{\pi r_2}\int_0^{\frac{\pi}{2}}\frac{1 - 8\sin^2\varphi\cos^2\varphi}{\sqrt{1-k^2\sin^2\varphi}}\,d\varphi$$

ここで $\displaystyle\int_0^{\frac{\pi}{2}}\frac{\sin^2\varphi\cos^2\varphi}{\sqrt{}}\,d\varphi = \frac{1}{2k^2}\int_0^{\frac{\pi}{2}}\sqrt{1-k^2\sin^2\varphi}\,\cos 2\varphi\,d\varphi$

$$= \frac{1}{4k^2}\int_0^{2\pi}\frac{\left(1 - \frac{2-k^2}{k^2}\right) + \frac{2-k^2}{k^2}(1-k^2\sin^2\varphi)}{\sqrt{1-k^2\sin^2\varphi}}\,d\varphi$$

$$= -\frac{k'^2}{2k^4}K(k) + \frac{1+k'^2}{2k^4}E(k) ,$$

$$\therefore\quad I = \frac{2(1+k'^2)}{\pi r_2 k^2}\left[(1+k'^2)K - 2E\right],$$

$$\therefore\quad X = \frac{(yy')^2}{4}\frac{K}{\pi r_2} - \frac{(yy')^2(1+k'^2)}{4\pi r_2 k^4}\left[(1+k'^2)K - 2E\right]$$

$$= \frac{(yy')^2}{4\pi r_2 k^2}\left[-4k'^2 K + 2(1+k'^2)E\right]$$

$$= \frac{(yy')^2(1+k'^2)}{2\pi r_2 k^2}\left[E - \frac{2k'^2}{1+k'^2}K\right],$$

$$X = \frac{(yy')^2}{16\pi}\int_0^{2\pi}\frac{1-\cos 2\theta}{R}\,d\theta = \frac{(yy')^2}{8\pi}\int_0^{2\pi}\frac{\sin^2\theta\,d\theta}{R},$$

No.

Date 58. 8. 15

軸対称ストークス流れの流れ関数

別府正則

内容

参考文献

1) 別府 "ストークス流れの最小抵抗問題" 58.5.13

2) Payne, L. E. & Pell, W. H. "The Stokes flow problem for a class of axially symmetric bodies" J. F. M. vol. 2, 1959(?)

概要

今までは最終的に3次元物体を取扱かう意図があるので直交座標で定式化してきた[1]が軸対称流れではやはり流れ関数を導入しておく方が便利である。

その流れ関数はポテンシャル流れにおけるストークスの流れ関数の定義をそのまゝ使えばよいようである。[2]

特に直交座標による速度ベクトル表現では軸対称流れへの変換が複雑なので，それらの式のチェックをも兼ねて別法として流れ関数表示の境界方程式を導いて見た。

§1 は流れ関数の定義と球関数による展開した〈前置きとして〉だけであまり意味は無い。

§2 では流れ関数についての相反定理を導き境界における圧力と渦度によって流れ関数が表現出来る事を示し，§3と附録Aで核関数の具体的表現を求め前報告と比較しその正当性を確かめた。

連続の式 $\frac{\partial (yu)}{\partial x} + \frac{\partial}{\partial y}(yv) = 0$

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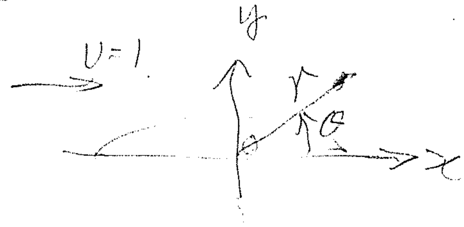
1. 流れ関数の表現

速度を u, v, 流れ関数を ψ とおくと



定義は
$$u = \frac{1}{y}\frac{\partial \psi}{\partial y}, \quad v = -\frac{1}{y}\frac{\partial \psi}{\partial x} \qquad (1.1)$$

渦度は
$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\frac{1}{y} L\psi, \qquad (1.2)$$

ここで
$$L \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{1}{y}\frac{\partial}{\partial y} = \frac{\partial^2}{\partial x^2} + y\frac{\partial}{\partial y}\left[\frac{1}{y}\frac{\partial}{\partial y}\right]. \qquad (1.3)$$

なおラプラシアン* Δ は
$$\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1}{y}\frac{\partial}{\partial y} - \frac{1}{y^2}, \qquad (1.4)$$

ζ は調和関数なので
$$\Delta \zeta = \frac{1}{y} L(y\zeta) = 0, \quad \ldots \qquad (1.5)$$

それ故 (1.2) から
$$L^2 \psi = 0, \quad \ldots \qquad (1.6)$$

ついで極座標を導入しよう。
$$x = r\cos\theta, \quad y = r\sin\theta, \quad \mu = \cos\theta$$

とすると
$$\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} = \mu\frac{\partial}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial}{\partial \mu}$$
$$\frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta} = \sqrt{1-\mu^2}\frac{\partial}{\partial r} - \frac{\mu\sqrt{1-\mu^2}}{r}\frac{\partial}{\partial \mu} \qquad (1.7)$$

$$L \equiv \frac{\partial^2}{\partial r^2} + \frac{1-\mu^2}{r^2}\frac{\partial^2}{\partial \mu^2}, \quad \ldots$$

* uについてのラプラシアンは $\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1}{y}\frac{\partial}{\partial y}\right] u = 0$

$$\Delta u = u_{xx} + u_{yy} + \frac{u_y}{y} - \frac{u}{y^2} = -\frac{1}{y}\frac{\partial}{\partial x}\left(\frac{\partial (yv)}{\partial y}\right) + u_{yy} + \frac{u_y}{y} - \frac{u}{y^2}$$

$$= u_{yy} + \frac{u_y}{y} - \frac{u}{y^2} - v_{xy} - \frac{v_x}{y} = \frac{\partial}{\partial y}(u_y - v_x) + \frac{1}{y}(u_y - v_x)$$

$$= -\frac{1}{y}\frac{\partial}{\partial y}(y\zeta) = \frac{\partial p}{\partial x}$$

$$\Delta v = v_{xx} + \underbrace{v_{yy} + \frac{v_y}{y} - \frac{v}{y^2}}_{} = v_{xx} - u_{xy} = \frac{\partial \zeta}{\partial x} = \frac{\partial p}{\partial y}$$

$$v + yv_y = \frac{\partial}{\partial y}(yv) = -y u_x,$$

$$2v_y + y v_{yy} = -u_x - y u_{xy}, \quad v_{yy} + \frac{v_y}{y} = -\frac{v_y}{y} - \frac{u_x}{y} - u_{xy}$$

$$v_{yy} + \frac{v_y}{y} - \frac{v}{y^2} = -\frac{1}{y^2}\frac{\partial}{\partial y}(yv) - \frac{u_x}{y} - u_{xy} = -u_{xy}$$

$$\therefore \quad \underline{\frac{\partial p}{\partial x} = -\frac{1}{y}\frac{\partial}{\partial y}(y\zeta), \quad \frac{\partial p}{\partial y} = \frac{1}{y}\frac{\partial}{\partial x}(y\zeta).}$$

$$\frac{\partial p}{\partial s} = \frac{1}{y}\frac{\partial}{\partial n}(y\zeta)$$

ルジャンドル関数 $P_n(\mu)$ を導入すると、

$$\frac{d}{d\mu}\left((1-\mu^2)\frac{dP_n}{d\mu}\right) + n(n+1)P_n(\mu) = 0, \quad \cdots (1.8)$$

なるゆえ

$$f = \frac{(1-\mu^2)}{r^n}\frac{dP_n}{d\mu}$$

とおくと

$$Lf = 0, \qquad (1.9)$$

なお、

$$L\left((1-\mu^2) r^{n+1}\frac{dP_n}{d\mu}\right) = 0, \quad \cdots (1.10)$$

であるが特に $n=1$ では

$$f = \frac{y^2}{2}, \quad \cdots (1.11)$$

は一様流れで $n \geq 1$ ではそれぞれ有意であり

さて

$$h = (1-\mu^2) r^{2-n}\frac{dP_n}{d\mu}, \quad \cdots (1.11)$$

とおくと

$$Lh = -2(2n+1)\frac{(1-\mu^2)}{r^n}\frac{dP_n}{d\mu}, \quad (1.12)$$

となるから

$$L^2 h = 0, \quad \cdots (1.14)$$

それ故一般に流れ関数は一様流れを含めて

$$\chi = \frac{y^2}{2} + (1-\mu^2)\sum_{n=0}^{\infty}\left(A_n r^{2-n} + B_n r^{-n}\right)\frac{dP_n}{d\mu}, \quad (1.15)$$

のように表わされる。

速度は

$$u = \frac{1}{y}\frac{\partial}{\partial y}\chi = \left(\frac{1}{r}\frac{\partial}{\partial r} - \frac{\mu}{r^2}\frac{\partial}{\partial \mu}\right)\chi$$

$$= 1 + (1-\mu^2)\sum_{n=1}^{\infty}\left(\frac{(2-n)}{r^n}A_n - \frac{n B_n}{r^{n+2}}\right)\frac{dP_n}{d\mu}$$

$$+ \mu \sum_{n=1}^{\infty}\left(\frac{A_n}{r^n} + \frac{B_n}{r^{n+2}}\right) n(n+1) P_n(\mu) ,$$

$$v = -\frac{1}{y}\frac{\partial \chi}{\partial x} = \left(-\frac{\mu}{r\sqrt{1-\mu^2}}\frac{\partial}{\partial r} - \frac{\sqrt{1-\mu^2}}{r^2}\frac{\partial}{\partial \mu}\right)\chi$$

$$= -\mu\sqrt{1-\mu^2}\sum_{n=1}^{\infty}\left(\frac{(2-n)}{r^n}A_n - \frac{n B_n}{r^{n+2}}\right)\frac{dP_n}{d\mu}$$

$$+ \sqrt{1-\mu^2}\sum_{n=1}^{\infty}\left(\frac{A_n}{r^n} + \frac{B_n}{r^{n+2}}\right) n(n+1) P_n , \qquad (1.16)$$

$$y\zeta = L\chi = -2(1-\mu^2)\sum_{n=1}\frac{(2n-1)A_n}{r^n}\frac{dP_n}{d\mu} .$$

あるいは

$$\zeta = 2\sum_{n=1}^{\infty}\frac{(2n-1)A_n}{r^{n+1}} P_n^1(\mu) , \quad \cdots (1.17)$$

但し

$$P_n^m(\mu) = (-)^m (1-\mu^2)^{\frac{m}{2}} \frac{d^m}{d\mu^m} P_n(\mu) , \quad \cdots (1.18)$$

なお

$$P_0(\mu) = 1, \quad P_1(\mu) = \mu, \quad P_2(\mu) = \frac{1}{2}(3\mu^2 - 1),$$

$$P_3(\mu) = \frac{1}{2}(5\mu^3 - 3\mu), \quad P_4 = \frac{1}{8}(35\mu^4 - 30\mu^2 + 3),$$

$$\frac{dP_0}{d\mu} = 0, \quad \frac{dP_1}{d\mu} = 1, \quad \frac{dP_2}{d\mu} = 3\mu \qquad (1.19)$$

境界条件は

$$u = v = 0 \quad \text{on } C, \qquad (1.20)$$

であるがもちろん $\frac{\partial \psi}{\partial n} = \frac{\partial \psi}{\partial s} = 0$,

でもよく又一般の形態では

$$\frac{\partial \psi}{\partial n} = \psi = 0 \quad \text{on } C, \qquad (1.21)$$

である。

(1.16)より充分遠方では

$$\left.\begin{aligned} u &\fallingdotseq 1 + \frac{A_1}{r}(1+\mu^2) + \\ v &\fallingdotseq \frac{A_1}{r}\mu\sqrt{1-\mu^2} \end{aligned}\right\} \cdots (1.22)$$

一方抗力を D とすると、

$$\left.\begin{aligned} u-1 &\fallingdotseq \frac{-D}{8\pi r}(1+\mu^2) \\ v &= \frac{-D}{8\pi r}\mu\sqrt{1-\mu^2} \end{aligned}\right\} \cdots (1.23)$$

$$\therefore \quad A_1 = \frac{-D}{8\pi}, \qquad (1.24)$$

球について考えて見よう。

(1.21)の境界条件の方が便利でそれを書き下すと、

$$
\left.
\begin{aligned}
&\frac{r^2}{2} + r^2 \sum_{n=1} \left(\frac{A_n}{r^n} + \frac{B_n}{r^{n+2}} \right) \frac{dP_n}{d\mu} = 0 \\
&r + r \sum_{n=1}^{\infty} \left(\frac{(2-n)}{r^n} A_n - \frac{n B_n}{r^{n+2}} \right) \frac{dP_n}{d\mu} = 0
\end{aligned}
\right\} \quad (1.25)
$$

これは一般の形状でもよいが，今は $r=1$ とおくと

辺々相引いて

$$\sum \left(n A_n + (n+2) B_n \right) \frac{dP_n}{d\mu} = 0$$

から
$$B_n = -\frac{n}{n+2} A_n .$$

これを第1式に代入して．$\frac{dP_n}{d\mu}$ の係数を 0 とおくと．

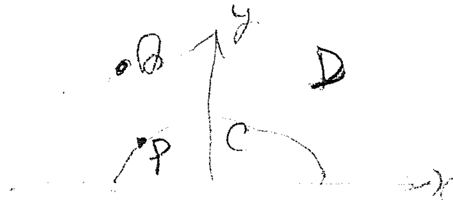
$$
\left.
\begin{aligned}
&A_1 = -3B_1 = -\frac{3}{4} \quad , \cdots\cdots \\
&A_n = B_n = 0 \quad \text{for } n \geq 2 \\
&\therefore \ D = 6\pi ,
\end{aligned}
\right\} \quad (1.26)
$$

球に近い物体以外では 一般に (1.15) は境界上で収束するとは言えないので 境界上に特異性をおく方法が望ましい。

2. 可逆定理

φ, ψ なる2つのストークスの流れ関数について次の積分を考えよう。

$$I=\iint_D\left[\phi L(\psi)-\psi L(\phi)\right]\frac{dxdy}{y}, \quad (2.1)$$



部分積分によって

$$I=\int_C\left[\frac{\psi}{y}\frac{\partial\phi}{\partial n}-\frac{\phi}{y}\frac{\partial\psi}{\partial n}\right]ds, \quad (2.2)$$

$$L(\psi)=L(\phi)=0 \quad \text{の時は.} \quad (2.3)$$

$$\int_C\left[\psi\frac{\partial\phi}{\partial n}-\phi\frac{\partial\psi}{\partial n}\right]\frac{ds}{y}=0, \quad (2.4)$$

さて もっと一般的に (2.2)は成立つから、まず (2.1), (2.2) の中の φ に $L\xi$ を代入して見ると

$$\iint_D\left[L(\xi)L(\psi)-\psi L^2(\xi)\right]\frac{dxdy}{y}=\int_C\left[\psi\frac{\partial L(\xi)}{\partial n}-L(\xi)\frac{\partial\psi}{\partial n}\right]\frac{ds}{y}, \quad (2.5)$$

ψのかわりにξ、φのかわりに $L(\psi)$ を入れると

$$\iint_D\left[L(\psi)L(\xi)-\xi L^2(\psi)\right]\frac{dxdy}{y}=\int_C\left[\xi\frac{\partial L(\psi)}{\partial n}-L(\psi)\frac{\partial\xi}{\partial n}\right]\frac{ds}{y}, \quad (2.6)$$

辺々相引いて

$$L^2(\xi)=L^2(\psi)=0, \quad (2.7)$$

とおくと

$$\iint_D\left[\xi L^2(\psi)-\psi L^2(\xi)\right]\frac{dxdy}{y}=0=\int_C\left[\psi\frac{\partial L(\xi)}{\partial n}-L(\xi)\frac{\partial\psi}{\partial n}-\xi\frac{\partial L(\psi)}{\partial n}+L(\psi)\frac{\partial\xi}{\partial n}\right]\frac{ds}{y}, \quad (2.8)$$

なる相反定理を得る。

(1.2)より　　$L(\psi) = -y\zeta$ 　であるから

$$\int_C \left[\psi \frac{\partial L(\zeta)}{\partial n} - L(\zeta)\frac{\partial \psi}{\partial n} + \zeta \frac{\partial (y\zeta)}{\partial n} - y\zeta \frac{\partial \zeta}{\partial n} \right] \frac{ds}{y}, \quad (2.9)$$

又さらに　ζの代りに　核 $X(P,Q)$ を代入すると

$$L_p[X(\ )] = -\ G(P,Q) \quad \cdots \quad (2.10)$$

$$-\int_\varepsilon \frac{\partial G}{\partial n}\frac{ds}{y} = 1 \quad , \quad \cdots \quad (2.11)$$

なるは、

$$\psi(Q) = \int_C \left[\psi \frac{\partial G}{\partial n} + \frac{\partial \psi}{\partial n} G + \frac{\partial (y\zeta)}{\partial n} X - y\zeta \frac{\partial X}{\partial n} \right] \frac{ds}{y}, \quad (2.12)$$

を得る。

境界条件は

$$\left.\begin{aligned} \psi(P) &= -\frac{y^2}{2}, \\ \frac{\partial \psi}{\partial n} &= -y\frac{\partial y}{\partial n}, \end{aligned}\right\} \text{on } C, \quad (2.13)$$

そして Q点が 領域Dの点ならば 明らかに

$$\int_C \left[\frac{y^2}{2}\frac{\partial G}{\partial n} - y\frac{\partial y}{\partial n} G \right]\frac{ds}{y} = 0, \quad \cdots (2.14)$$

よって (2.13)を満たす場合は

$$\psi(Q) = \int_C \left[\frac{\partial (y\zeta)}{\partial n} X - y\zeta \frac{\partial X}{\partial n} \right] \frac{ds}{y}, \quad (2.15)$$

さらに圧力pを導入すると、

$$\frac{\partial p}{\partial s} = \frac{1}{y}\frac{\partial (y\zeta)}{\partial n}, \quad \cdots (2.16)$$

であるから

$$\Psi(Q) = \int_C \left[\frac{\partial p}{\partial s} X - \zeta \frac{\partial X}{\partial n}\right] ds, \quad \cdots (2.17)$$

あるいは部分積分して

$$\Psi(Q) = -\int_C \left[p\frac{\partial X}{\partial s} + \zeta\frac{\partial X}{\partial n}\right] ds, \quad (2.18)$$

さらに

$$\left.\begin{aligned} \tau_x &= -p\frac{\partial x}{\partial n} - \zeta\frac{\partial y}{\partial n} \\ \tau_y &= -p\frac{\partial y}{\partial n} + \zeta\frac{\partial x}{\partial n} \end{aligned}\right\} \cdots (2.19)$$

とおけば

$$\begin{aligned} p\frac{\partial X}{\partial s} + \zeta\frac{\partial X}{\partial n} &= \left(p\frac{\partial x}{\partial s} + \zeta\frac{\partial x}{\partial n}\right)\frac{\partial X}{\partial x} + \left(p\frac{\partial y}{\partial s} + \zeta\frac{\partial y}{\partial n}\right)\frac{\partial X}{\partial y} \\ &= \tau_y\frac{\partial X}{\partial x} - \tau_x\frac{\partial X}{\partial y}, \end{aligned}$$

となって

$$\Psi(Q) = \int_C \left[\tau_x\frac{\partial X}{\partial y} - \tau_y\frac{\partial X}{\partial x}\right] ds, \quad (2.20)$$

それ故

$$\left.\begin{aligned} u &= \frac{1}{y'}\frac{\partial \Psi}{\partial y'} = \frac{1}{y'}\int_C \left[\tau_x\frac{\partial^2 X}{\partial y\partial y'} - \tau_y\frac{\partial^2 X}{\partial x\partial y'}\right] ds, \\ v &= \frac{-1}{y'}\frac{\partial \Psi}{\partial x'} = \frac{-1}{y'}\int_C \left[\tau_x\frac{\partial^2 X}{\partial y\partial x'} - \tau_y\frac{\partial^2 X}{\partial x\partial x'}\right] ds, \end{aligned}\right\} (2.21)$$

$$\therefore \quad \left.\begin{aligned} V_1^{(1)} &= -\frac{1}{yy'}\frac{\partial^2 X}{\partial y \partial y'}, \\ V_y^{(1)} &= \frac{1}{yy'}\frac{\partial^2 X}{\partial x \partial y'}, \\ V_1^{(4)} &= \frac{1}{yy'}\frac{\partial^2 X}{\partial y \partial x'} \\ V_y^{(4)} &= -\frac{1}{yy'}\frac{\partial^2 X}{\partial x \partial x'} \end{aligned}\right\} \quad (2.22)$$

$$\left.\begin{aligned} u &= -\int_C \left[T_x V_1^{(1)} + T_y V_y^{(1)} \right] y\, ds \\ v &= -\int_C \left[T_x V_1^{(4)} + T_y V_y^{(4)} \right] y\, ds \end{aligned}\right\} \cdots (2.23)$$

3. 核関数

$$X = X_1 + X_2 , \qquad (3.1)^*$$

$$X_1 = \frac{yy'^2}{4}\int_0^\infty e^{-t|x-x'|} J_0(ty) J_1(ty') \frac{dt}{t} , \qquad (3.2)$$

$$X_2 = -\frac{yy'^2}{4}\int_0^\infty e^{-t|x-x'|} J_1(ty) J_2(ty') \frac{dt}{t} , \qquad (3.3)$$

とおくと二次元の表より

$$L_P = \frac{\partial^2}{\partial x^2} + y\frac{\partial}{\partial y}\left(\frac{1}{y}\frac{\partial}{\partial y}\right) , \quad L_Q = \frac{\partial^2}{\partial x'^2} + y'\frac{\partial}{\partial y'}\left(\frac{1}{y'}\frac{\partial}{\partial y'}\right)$$

として

$$\left.\begin{aligned} L_P[X_1] &= -\frac{yy'}{2}\int_0^\infty e^{-tx} J_1(yt) J_1(ty')\,dt = G(P,Q), \\ &= -\frac{yy'}{4\pi}\int_0^{2\pi}\frac{\cos\theta}{R}\,d\theta , \quad R^2 = x^2+y^2+y'^2-2yy'\cos\theta , \\ L_Q[X_1] &= 0 , \end{aligned}\right\} \quad (3.4)$$

$$\left.\begin{aligned} L_P[X_2] &= 0 , \\ L_Q[X_2] &= -\frac{yy'}{2}\int_0^\infty e^{-tx} J_1(ty) J_1(ty')\,dt = G(P,Q) \end{aligned}\right\} \quad (3.5)$$

(以下では $x'=0$, $x>0$ として記す)

$$G(P,Q) \xrightarrow[P\to Q]{} -\frac{y}{2\pi}\log\left(\sqrt{x^2+(y-y')^2}\right) , \qquad (3.6)$$

---

* 楕円積分表示は附録A

$$y\frac{\partial}{\partial y}\left[\frac{1}{y}\frac{\partial}{\partial y}J_0(ty)\right] = -yt\frac{\partial}{\partial y}\left[\frac{J_1(ty)}{y}\right] = \frac{t}{y}J_1 - t^2 J_1'(ty)$$
$$= \frac{2t}{y}J_1 - t^2 J_0$$

$$L_y\left[J_0(ty)\right] = \frac{2t}{y}J_1 \quad //$$

$$y\frac{\partial}{\partial y}\left[\frac{1}{y}\frac{\partial}{\partial y}y^2 J_0(ty)\right] = z\frac{d}{dz}\left(\frac{1}{z}\frac{d}{dz}(z^2 J_0)\right) = z\frac{d}{dz}\left[2J_0 - zJ_1\right]$$
$$= -2zJ_1 - z^2 J_0$$

$$\therefore\ L\left[y^2 J_0(ty)\right] = -2ytJ_1(yt) \quad //$$

$$y\frac{\partial}{\partial y}\left[\frac{1}{y}\frac{\partial}{\partial y}(yJ_1)\right] = tz\frac{d}{dz}\left[\frac{1}{z}\frac{d}{dz}(zJ_1)\right] = tz\frac{d}{dz}J_0 = -zJ_1(yt)$$

$$\therefore\ L\left[yJ_1(yt)\right] = 0 \quad //$$

$$y\frac{\partial}{\partial y}\left[\frac{1}{y}\frac{\partial}{\partial y}(y^2 J_2)\right] = z\frac{d}{dz}\left[\frac{1}{z}\frac{d}{dz}(z^2 J_2)\right] = z\frac{d}{dz}(zJ_1)$$
$$= z^2 J_0 = z^2\left(\frac{2}{z}J_1 - J_2\right).$$

$$\therefore\ L\left(y^2 J_2(ty)\right) = 2ytJ_1(y,t) \quad //$$

$$\frac{\partial}{\partial y}J_0(ty) = -tJ_1(ty), \quad \frac{\partial}{\partial y}\left[yJ_1(ty)\right] = ytJ_0(ty)$$

$$\frac{\partial}{\partial y}\left[y^2 J_2(ty)\right] = y^2 t J_1(ty), \quad \frac{\partial}{\partial y}\left[y^2 J_0(ty)\right] = y\left[2J_0 - ytJ_1\right].$$

$$J_2(ty) = \frac{2}{yt}J_1(yt) - J_0(yt)$$

先に 微分すると

$$\frac{1}{y}\frac{\partial X_1}{\partial y} = \frac{y'}{2}\int_0^\infty e^{-tx} J_0(ty)J_1(ty')\frac{dt}{t} - \frac{yy'}{4}\int_0^\infty e^{-tx} J_1(ty)J_1(ty')\,dt$$

$$= \frac{2}{y^2}X_1 + \frac{1}{2}G \,,$$

$$\frac{1}{y}\frac{\partial X_1}{\partial x} = -\frac{yy'}{4}\int_0^\infty e^{-tx} J_0(ty)J_1(ty')\,dt = -\frac{y}{2}\tilde{T} \,,$$

$$\frac{\partial X_1}{\partial x'} = -\frac{\partial X_1}{\partial x} \,,$$

$$\frac{1}{y'}\frac{\partial X_1}{\partial y'} = \frac{y^2}{4}\int_0^\infty e^{-tx} J_0(ty)J_0(ty')\,dt = \frac{y^2}{2}S \,,$$

(3.7)

$$\frac{1}{yy'}\frac{\partial^2 X_1}{\partial y\,\partial y'} = S + \frac{y}{2}\frac{\partial S}{\partial y} =$$

$$\frac{1}{yy'}\frac{\partial^2 X_1}{\partial y'\,\partial x} = -\frac{y}{2y'}\frac{\partial \tilde{T}}{\partial y'} = -\frac{y}{2}\frac{\partial S}{\partial x'} = \frac{y}{2}\frac{\partial S}{\partial x} =$$

$$\frac{1}{yy'}\frac{\partial^2 X_1}{\partial x\,\partial y} = \frac{-1}{yy'}\frac{\partial}{\partial y}\left[\frac{y^2}{2}\tilde{T}\right] = -\frac{1}{y'}\tilde{T} - \frac{y}{2y'}\frac{\partial \tilde{T}}{\partial y} \,, \quad (38)$$

$$= -\frac{1}{y'}\tilde{T} + \frac{y}{2}\frac{\partial P}{\partial x}$$

$$\frac{1}{yy'}\frac{\partial^2 X_1}{\partial x\,\partial x'} = -\frac{y}{2y'}\frac{\partial \tilde{T}}{\partial x'} = \frac{y}{2}\frac{\partial S}{\partial y'} =$$

$$\frac{1}{y}\frac{\partial X_2}{\partial y} = -\frac{y'^2}{4}\int_0^\infty e^{-tx} J_0(ty) J_2(ty')\,dt$$

$$= -\frac{y'}{2}\int_0^\infty e^{-tx} J_0(ty) J_1(y't)\frac{dt}{t} + \frac{y'^2}{4}\int_0^\infty e^{-tx} J_0(ty) J_0(ty')\,dt$$

$$= \frac{y'^2}{2} S + \tilde{P}_A ;$$

$$\frac{1}{y}\frac{\partial X_2}{\partial x} = -\frac{1}{y}\frac{\partial X_2}{\partial x'} = \frac{y'^2}{4}\int_0^\infty e^{-tx} J_1(ty) J_2(ty')\,dt$$

$$= \frac{y'}{2}\int_0^\infty e^{-tx} J_1(ty) J_1(ty')\frac{dt}{t} - \frac{y'^2}{4}\int_0^\infty e^{-tx} J_1(ty) J_0(ty')\,dt \qquad (3.9)$$

$$= \frac{1}{y} G_A - \frac{y'^2}{2y} T ,$$

$$\frac{1}{y'}\frac{\partial X_2}{\partial y'} = -\frac{yy'}{4}\int_0^\infty e^{-tx} J_1(ty) J_1(ty)\,dt = \frac{1}{2} G ,$$

$$\left.\begin{aligned}
\frac{1}{yy'}\frac{\partial^2 X_2}{\partial y'\partial y} &= \frac{1}{2y}\frac{\partial G}{\partial y} = \frac{1}{2}y'\frac{\partial S}{\partial y'} , \\
\frac{1}{yy'}\frac{\partial^2 X_2}{\partial y'\partial x} &= \frac{1}{2y}\frac{\partial G}{\partial x} = -\frac{1}{2}\frac{\partial T}{\partial y} = -\frac{y'}{2}\frac{\partial P}{\partial y} \\
\frac{1}{yy'}\frac{\partial^2 X_2}{\partial x'\partial y} &= \frac{y'}{2}\frac{\partial S}{\partial x'} + \frac{1}{y'}\tilde{P} \\
\frac{1}{yy'}\frac{\partial^2 X_2}{\partial x\partial x'} &= \frac{-1}{yy'} G - \frac{y'}{2y}\frac{\partial T}{\partial x'} = +P + \frac{y'}{2}\frac{\partial S}{\partial y} ,
\end{aligned}\right\} (3.10)$$

結局

$$
\left.
\begin{aligned}
\frac{1}{yy'}\frac{\partial^2 X}{\partial y\,\partial y'} &= S + \frac{1}{2}\left(y\frac{\partial S}{\partial y} + y'\frac{\partial S}{\partial y'}\right), \\
\frac{1}{yy'}\frac{\partial^2 X}{\partial y'\,\partial x} &= \frac{1}{2}\left[y\frac{\partial S}{\partial x} - y'\frac{\partial P}{\partial x}\right], \\
\frac{1}{yy'}\frac{\partial^2 X}{\partial x'\,\partial y} &= \frac{1}{2}\left[y\frac{\partial P}{\partial x} - y'\frac{\partial S}{\partial x}\right], \\
\frac{1}{yy'}\frac{\partial^2 X}{\partial x'\,\partial x} &= P + \frac{1}{2}\left[y'\frac{\partial S}{\partial y} + y\frac{\partial S}{\partial y'}\right],
\end{aligned}
\right\}
\quad (3.11)
$$

となって，(2.22)を通じ附録(A.9)に一致する。

附録A 種々の核

$$S^*(P,Q) = \frac{1}{4\pi}\int_0^{2\pi} R\,d\theta = \frac{r_2 E(k)}{\pi} \quad , \qquad \cdots (A\cdot 1)$$

$$S = \frac{1}{4\pi}\int_0^{2\pi} \frac{d\theta}{R} = \frac{K(k)}{\pi r_2} = \frac{1}{2}\int_0^{\infty} e^{-t|x|} J_0(ty) J_0(ty')\,dt, \quad (A\cdot 2)$$

$$P = \frac{1}{4\pi}\int_0^{2\pi} \frac{\cos\theta}{R}\,d\theta = \frac{1}{2}\int_0^{\infty} e^{-t|x|} J_1(ty) J_1(ty')\,dt, \quad (A\cdot 3)$$

$$\left.\begin{aligned} T &= \frac{y}{2}\int_0^{\infty} e^{-t|x|} J_1(ty) J_0(ty')\,dt, \\ \tilde{T} &= \frac{y'}{2}\int_0^{\infty} e^{-t|x|} J_0(ty) J_1(ty')\,dt, \\ \frac{d\tilde{T}_A}{dx} &= \tilde{T}, \quad \tilde{T}_A = -\frac{y'}{2}\int_0^{\infty} e^{-tx} J_0(ty) J_1(ty')\frac{dt}{t}, \end{aligned}\right\} \begin{aligned} &(A\cdot 4) \\ &T(x; y, y') \\ &= \tilde{T}(x; y', y), \end{aligned}$$

[ダクトペラ覚書では　$G = -yy'P$,
$S = \tilde{T}$,　$S_A = \tilde{T}_A$, が対応している]

E, K は楕円積分で T, $\tilde{T}$ は ポラ 種 楕円積分となる。

$$(k')^2 = r_1^2/r_2^2, \quad k^2 = 1 - k'^2 = \frac{4yy'}{r_2^2},$$
$$r_1^2 = x^2 + (y-y')^2, \quad r_2^2 = x^2 + (y+y')^2,$$

まず

$$\left.\begin{aligned} \frac{\partial}{\partial x} S^* &= -\frac{\partial}{\partial x'} S^* = x S, \\ \frac{\partial S^*}{\partial y} &= yS - y'P, \\ \frac{\partial S^*}{\partial y'} &= y'S - yP, \end{aligned}\right\} \quad (A\cdot 5)$$

$$\frac{\partial T}{\partial x} = \frac{1}{y'}\frac{\partial G}{\partial y'} = \frac{-y}{y'}\frac{\partial (y'P)}{\partial y'} = y\frac{\partial S}{\partial y},$$

$$\frac{\partial T}{\partial y'} = \frac{1}{y'}\frac{\partial G}{\partial x'} = +y\frac{\partial P}{\partial x}, \quad \frac{\partial T}{\partial y} = -y\frac{\partial S}{\partial x},$$

$$\frac{\partial \tilde{T}}{\partial x} = \frac{1}{y}\frac{\partial G}{\partial y} = \frac{-y'}{y}\frac{\partial (yP)}{\partial y} = y'\frac{\partial S}{\partial y'},$$

$$\frac{\partial \tilde{T}}{\partial y'} = -y'\frac{\partial S}{\partial x}, \quad \frac{\partial \tilde{T}}{\partial y} = -\frac{1}{y}\frac{\partial G}{\partial x} = -y'\frac{\partial P}{\partial x}, \qquad (A.6)$$

$$\frac{\partial}{\partial x} S^* = x S = -\frac{1}{2}\int_0^\infty J_0(ty)J_0(ty')\frac{\partial e^{-tx}}{\partial t}\,dt$$

$$= \frac{1}{2} - (T + \tilde{T}), \qquad (A.7)$$

$$\frac{\partial^2}{\partial x \partial x} S^* = -\left(\frac{\partial T}{\partial x} + \frac{\partial \tilde{T}}{\partial x}\right) = -y\frac{\partial S}{\partial y} - y'\frac{\partial S}{\partial y'},$$

$$\frac{\partial^2 S^*}{\partial x \partial y} = -\frac{\partial}{\partial y}(T + \tilde{T}) = y\frac{\partial S}{\partial x} - y'\frac{\partial P}{\partial x},$$

$$\frac{\partial^2 S^*}{\partial x \partial y'} = y'\frac{\partial S}{\partial x} - y\frac{\partial P}{\partial x}, \qquad (A.8)$$

$+ y'\frac{\partial S}{\partial y}$

$$\frac{\partial^2 S^*}{\partial y \partial y'} = y\frac{\partial S}{\partial y'} - \frac{\partial (y'P)}{\partial y'} = -P + y\frac{\partial S}{\partial y'} - y'\frac{\partial P}{\partial y'},$$

$$V_1^{(1)} = \frac{1}{2}S^*_{xx} - S = -\left[S + \frac{1}{2}(y S_y + y' S_{y'})\right],$$

$$V_2^{(1)} = \frac{1}{2}S^*_{xy} = \frac{1}{2}(y S_x - y' P_x), \qquad (A.9)$$

$$V^{(2)} = -\frac{1}{2}S^*_{xy'} = \frac{1}{2}(y P_x - y' S_x)$$

$$V_2^{(2)} = -P - \frac{1}{2}S^*_{yy'} = -\frac{P}{2} - \frac{1}{2}(y S_{y'} - y' P_{y'}),$$

$$= -P - \frac{1}{2}(y S_{y'} + y' S_{y})$$

実際の計算には (A.2), (A.3) の手続きから、S^* の微分を計算する方が楽である。(以下 Ref. 1) Appendix D と同じ)

楕円積分の微分は、

$$\left.\begin{aligned} k\frac{dK}{dk} &= \frac{E}{k'^2} - K, \\ k\frac{dE}{dk} &= E - K, \end{aligned}\right\} \quad \cdots\cdots (A.10)$$

から

$$\frac{\partial}{\partial x} S^* = x S,$$

$$\frac{\partial}{\partial y} S^* = \frac{S^*}{2y} - \left\{\frac{r_2^2}{2y} - (y+y')\right\} S,$$

$$\frac{\partial S}{\partial x} = -\frac{x}{(r_1 r_2)^2} S^*,$$

$$\frac{\partial S}{\partial y} = -\frac{S}{2y} + \frac{1}{r_1^2}\left(\frac{1}{2y} - \frac{y+y'}{r_2^2}\right) S^*,$$

$$\frac{\partial^2}{\partial x^2} S^* = S - \frac{x^2}{(r_1 r_2)^2} S^*,$$

$$\frac{\partial^2}{\partial x \partial y} S^* = -\frac{x}{2y} S + \frac{x}{r_1^2}\left(\frac{1}{2y} - \frac{y+y'}{r_2^2}\right) S^*,$$

$$\frac{\partial^2}{\partial x \partial y'} S^* = -\frac{x}{2y'} S + \frac{x}{r_1^2}\left(\frac{1}{2y'} - \frac{y+y'}{r_2^2}\right) S^*,$$

$$\frac{\partial^2}{\partial y \partial y'} S^* = -\left(\frac{y^2 + y'^2}{2yy'}\right) S + \frac{\{x^2(y^2+y'^2) + (y^2 - y'^2)^2\}}{2yy'(r_1 r_2)^2} S^* \quad (A.11)$$

$$X_1 = \frac{(yy')^2}{8}\int_0^\infty e^{-tx} J_0(ty)\{J_0(ty') + J_2(ty')\}dt,$$

$$X_2 = -\frac{(yy')^2}{8}\int_0^\infty e^{-tx}\{J_0(ty)+J_2(ty)\}J_2(ty')dt,$$

$$\therefore\ X = \frac{(yy')^2}{8}\int_0^\infty e^{-tx}\left[J_0(ty)J_0(ty') - J_2(ty)J_2(ty')\right]dt,$$

$$S = \frac{1}{4\pi}\int_0^{2\pi}\frac{d\theta}{R} = \frac{K(k)}{\pi r_2} = \frac{1}{2}\int_0^\infty e^{-tx} J_0(ty)J_0(ty')dt,$$

$$\frac{1}{R} = \int_0^\infty e^{-tx}dt\left[J_0(ty)J_0(ty') + 2\sum_{n=1}^{\infty} J_n(ty)J_n(ty')\cos n\theta\right].$$

$$\therefore\ I = \frac{1}{2\pi}\int_0^{2\pi}\frac{\cos 2\theta\, d\theta}{R} = \int_0^\infty e^{-tx} J_2(ty)J_2(ty')dt,$$

$$\therefore\ X = \frac{(yy')^2}{4}S - \frac{(yy')^2}{8}I,$$

$$\left\{\begin{array}{ll} r_1^2 = x^2 + (y-y')^2, & r_2^2 = x^2 + (y+y')^2, \\ k'^2 = \left(\frac{r_1}{r_2}\right)^2, & k^2 = 1 - k'^2. \end{array}\right\}$$ とおくと

$R = r_2\sqrt{1-k^2\sin^2\varphi}$, 但し, $\theta = \pi - 2\varphi$, $\cos\theta = -\cos 2\varphi$.

$$\therefore\ \cos 2\theta = 1 - 8\sin^2\varphi\cos^2\varphi$$

$$I = \frac{2}{\pi r_2}\int_0^{\frac{\pi}{2}}\frac{1-8\sin^2\varphi\cos^2\varphi}{\sqrt{1-k^2\sin^2\varphi}}d\varphi,$$

所で
$$\int_0^{\frac{\pi}{2}}\frac{\sin^2\varphi\cos^2\varphi}{\sqrt{\quad}}d\varphi = \frac{1}{2k^2}\int_0^{\frac{\pi}{2}}\sqrt{1-k^2\sin^2\varphi}\,\cos 2\varphi\, d\varphi$$

$$= \frac{1}{4k^2}\int_0^{2\pi}\frac{\left(1-\frac{2-k^2}{k^2}\right) + \frac{2-k^2}{k^2}(1-k^2\sin^2\varphi)}{\sqrt{1-k^2\sin^2\varphi}}d\varphi$$

$$= -\frac{k'^2}{2k^4}K(k) + \frac{1+k'^2}{2k^4}E(k),$$

$$\therefore\quad I = \frac{2(1+k'^2)}{\pi r_2 k^4}\left[(1+k'^2)K - 2E\right],$$

$$\therefore\quad X = \frac{(yy')^2}{4}\frac{2K}{\pi r_2} - \frac{(yy')^2(1+k'^2)}{4\pi r_2 k^4}\left[(1+k'^2)K - 2E\right]$$

$$= \frac{(yy')^2}{4\pi r_2 k^4}\left[-4k'^2K + 2(1+k'^2)E\right]$$

$$= \frac{(yy')^2(1+k'^2)}{2\pi r_2 k^4}\left[E - \frac{2k'^2}{1+k'^2}K\right], //$$

$$X = \frac{(yy')^2}{16\pi}\int_0^{2\pi}\frac{1-\cos 2\theta}{R}\,d\theta = \frac{(yy')^2}{8\pi}\int_0^{2\pi}\frac{\sin^2\theta\,d\theta}{R},$$