

V

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軸対称スレーク流れ・流れ定数

副付三割

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附録A 極値の核 $A=1m^2$

参考文献

- 1) 鳥羽 "スレーク流れの最小抵抗問題" 58.57.13
- 2) Payne, E.E. & Pell, W.H. "The Stokes flow problem for a class of axially symmetric bodies"
J. E.M. vol. 2, 1959(?)

摘要

今までは最終的にふた元物質を取り扱う意図が"ある"
直交座標で定式化に果た¹⁾が、輻射熱の流れでは
やはり流れ関数を導入しておく方が便利である。

その流れ関数はポテンシャル流れにおける²⁾
の流れ関数の定義をそのまま使えばよい²⁾ようである。

特に直交座標による速度ベクトル表現では輻射熱
流れへの変換が複雑なので、それらの式のチェックを
兼ねて別法として流れ関数表示の境界方程式を
導いて見た。

§1 は流れ関数の定義と環境数による表示
^{前置きとして}
した点だけであり意味はない。

§2 では流れ関数についていう定理を等温
境界における圧力と温度によって流れ関数が
表現出来る事を示し、§3と附録Aで核周
数の具体的な表現を求め前報告と比較
しての妥当性を確かめた。

$$\text{運動方程式} \quad \frac{\partial^2 x}{\partial t^2} - \frac{\partial^2 y}{\partial t^2} = 0$$

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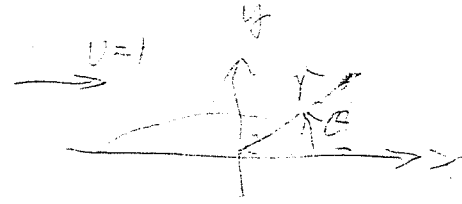
1. 自由振動の方程式

運動方程式 (1.2) は、自由振動の方程式と見做す

定義は

$$U = \frac{1}{y} \frac{\partial L}{\partial y}$$

$$V = -\frac{1}{y} \frac{\partial L}{\partial x}$$



満足は

$$\delta = \frac{\partial U}{\partial x} - \frac{\partial V}{\partial y} = -\frac{1}{y} L(x), \quad (1.2)$$

2.1.2

$$L \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{1}{y} \frac{\partial}{\partial y} = \frac{\partial^2}{\partial x^2} + y \frac{\partial}{\partial y} \left[\frac{1}{y} \frac{\partial}{\partial y} \right]$$

ポテンシャルエネルギーは

(1.3)

$$\delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1}{y} \frac{\partial}{\partial y} - \frac{1}{y^2}, \quad (1.4)$$

δ は調和振動数なので

$$\Delta \delta = \frac{1}{y} L(y\delta) = 0, \quad \dots \quad (1.5)$$

したがって (1.2) より

$$L^2 \psi = 0, \quad \dots \quad (1.6)$$

ここで極座標を導入しよう。

$$x = r \cos \theta, \quad y = r \sin \theta, \quad \mu = \cos \theta$$

とすると

$$\frac{\partial}{\partial x} = \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} = \mu \frac{\partial}{\partial r} - \frac{1-\mu^2}{r} \frac{\partial}{\partial \mu}$$

$$\frac{\partial}{\partial y} = \sin \theta \frac{\partial}{\partial r} + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta} = \sqrt{1-\mu^2} \frac{\partial}{\partial r} - \frac{\mu \sqrt{1-\mu^2}}{r} \frac{\partial}{\partial \mu} \quad (1.7)$$

$$L \equiv \frac{\partial^2}{\partial r^2} + \frac{1-\mu^2}{r^2} \frac{\partial^2}{\partial \mu^2}$$

* Uは(1.2)のポテンシャルは

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1}{y} \frac{\partial}{\partial y} \right] U = 0$$

$$\Delta u = u_{xx} - u_{yy} + \frac{u_x}{y} - \frac{u}{y^2} = -\frac{1}{y} \frac{\partial}{\partial x} \left(\frac{\partial^2 u}{\partial y^2} \right) - u_{yy} + \frac{u_x}{y} - \frac{u}{y^2}$$

$$= u_{xy} + \frac{u_x}{y} - \frac{u}{y^2} - v_{xy} - \frac{-x}{y} = \frac{\partial}{\partial y} (u_y - vx) + \frac{1}{y} (u_x - u)$$

$$= -\frac{1}{y} \frac{\partial}{\partial y} (yS) = -\frac{\partial P}{\partial x} - \frac{u}{y^2}$$

$$\Delta v = v_{xx} + v_{yy} + \frac{v_x}{y} - \frac{v}{y^2} = v_{xx} - u_{xy} = \frac{\partial^2}{\partial x^2} = \frac{\partial P}{\partial y}$$

$$v + yv_x = \frac{\partial}{\partial y} (y v) = -y u_x$$

$$2v_y + y v_{xy} = -u_x - y u_{xy}, \quad v_{yy} + \frac{v_x}{y} = -\frac{v_y}{y} - \frac{u_x}{y} - \frac{u_{xy}}{y}$$

$$v_{yy} + \frac{v_x}{y} - \frac{v}{y^2} = -\frac{1}{y^2} \frac{\partial}{\partial y} (y v) - \frac{u_x}{y} - u_{xy} = -u_{xy}$$

$$\therefore \frac{\partial P}{\partial x} = -\frac{1}{y} \frac{\partial}{\partial y} (yS), \quad \frac{\partial P}{\partial y} = \frac{1}{y} \frac{\partial}{\partial x} (yS)$$

$$\frac{\partial P}{\partial S} = \frac{1}{y} \frac{\partial}{\partial x} (yS)$$

レジヤントル) $\equiv \frac{1}{r} \frac{d}{d\mu} \left((1-\mu^2) \frac{dP_n}{d\mu} \right) + n(n+1)P_n(\mu) = 0$

$$\frac{d}{d\mu} \left((1-\mu^2) \frac{dP_n}{d\mu} \right) + n(n+1)P_n(\mu) = 0 \quad (1.8)$$

左辺を

$$f = \frac{(1-\mu^2)}{r^n} \frac{dP_n}{d\mu}$$

とかくと

$$L f = 0,$$

左辺

$$L \left((1-\mu^2)^{n+1/2} \frac{dP_n}{d\mu} \right) = 0 \quad (1.9)$$

で"右辺" $\frac{1}{r^n} \frac{dP_n}{d\mu}$ $n=1$ では

$$f = \frac{\mu^2}{2}$$

は一般流れで $n \geq 1$ では $\frac{1}{r^n} \frac{dP_n}{d\mu}$ としておける

さて

$$h = (1-\mu^2)^{2-n} \frac{dP_n}{d\mu} \quad (1.11)$$

とかくと

$$L h = -2(2n+1) \frac{(1-\mu^2)}{r^n} \frac{dP_n}{d\mu} \quad (1.12)$$

右辺から

$$L^2 h = 0 \quad (1.14)$$

したがって一般に流れ座標は一般流れを含めて

$$\chi = \frac{\mu^2}{2} + (1-\mu^2) \sum_{n \neq 0}^{\infty} (A_n r^{2n} + B_n r^{-n}) \frac{dP_n}{d\mu} \quad (1.15)$$

のように表わされる。

速度は

$$U = \frac{1}{r} \frac{\partial \psi}{\partial \eta} = \frac{1}{r} \frac{\partial}{\partial \eta} - \frac{\mu}{r^2} \frac{\partial}{\partial \mu}$$

$$= 1 + (-\mu)^n \sum_{n=1}^{\infty} \left(\frac{(2-n)}{r^n} A_n - \frac{n B_n}{r^{n+2}} \right) \frac{d P_n}{d \mu}$$

$$+ \mu \sum_{n=1}^{\infty} \left(\frac{A_n}{r^n} - \frac{B_n}{r^{n+2}} \right) n(n+1) P_n(\mu)$$

$$V = -\frac{1}{r} \frac{\partial \psi}{\partial x} = \left(-\frac{\mu}{r \sqrt{1-\mu^2}} \frac{\partial}{\partial \eta} - \frac{\sqrt{1-\mu^2}}{r^2} \frac{\partial}{\partial \mu} \right)$$

$$= -\mu \sqrt{1-\mu^2} \sum_{n=1}^{\infty} \left(\frac{(2-n)}{r^n} A_n - \frac{n B_n}{r^{n+2}} \right) \frac{d P_n}{d \mu}$$

$$+ \sqrt{1-\mu^2} \sum_{n=1}^{\infty} \left(\frac{A_n}{r^n} + \frac{B_n}{r^{n+2}} \right) n(n+1) P_n(\mu)$$

$$y \zeta = L \pi = -2(1-\mu^2) \sum_{n=1}^{\infty} \frac{(2n-1) A_n}{r^n} \frac{d P_n}{d \mu}$$

ただし

$$y = 2 \sum_{n=1}^{\infty} \frac{(2n-1) A_n}{r^{n+1}} P_n(\mu) \quad \dots (17)$$

$$\text{但し } P_n^m(\mu) = (-1)^m (1-\mu^2)^{\frac{m}{2}} \frac{d^m}{d\mu^m} P_n(\mu) \quad \dots (18)$$

すなわち

$$P_0(\mu) = 1, \quad P_1(\mu) = \mu, \quad P_2(\mu) = \frac{1}{2}(3\mu^2 - 1)$$

$$P_3(\mu) = \frac{1}{2}(5\mu^3 - 3\mu), \quad P_4 = \frac{1}{8}(35\mu^4 - 30\mu^2 + 3)$$

$$\frac{d P_0}{d \mu} = 0, \quad \frac{d P_1}{d \mu} = 1, \quad \frac{d P_2}{d \mu} = 3\mu$$

境界条件は

$$u = v = 0 \quad \text{on } C$$

$$\text{on } C$$

$$C = \partial D$$

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \quad \text{on } C$$

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \quad \text{on } C$$

である。又一般の形は

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \quad \text{on } C$$

$$(1.21)$$

である。

(1.16) より充分条件は

$$u = 1 + \frac{A_1}{r} (1 + \mu^2)$$

$$v = \frac{A_1}{r} \mu \sqrt{1 - \mu^2}$$

$$(1.22)$$

～を抗力を D とすると

$$u - 1 = \frac{-D}{8\pi r} (1 + \mu^2)$$

$$v = \frac{-D}{8\pi r} \mu \sqrt{1 - \mu^2}$$

$$(1.23)$$

$$A_1 = \frac{-D}{8\pi}$$

$$(1.24)$$

よって、求めるべきは

(1.24) の境界条件の方が便利である。

と。

$$\frac{r^2}{2} + r^2 \sum_{n=1}^{\infty} \left(\frac{A_n}{r^n} + \frac{B_n}{r^{n+2}} \right) \frac{dP_n}{du} = 0$$

(1.25)

$$r + r \sum_{n=1}^{\infty} \left(\frac{(2-n)}{r^n} A_n - \frac{n B_n}{r^{n+2}} \right) \frac{dP_n}{du} = 0$$

これは一般の π に対して成り立つから、 $\pi = 1$ とおくと
より簡単に

$$\sum_{n=1}^{\infty} (n A_n + (n+2) B_n) \frac{dP_n}{du} = 0$$

から

$$B_n = -\frac{n}{n+2} A_n$$

これを式(1)に代入して、 $\frac{dP_n}{du}$ の係数を 0 とおくと

$$A_1 = -3B_1 = -\frac{3}{4}$$

$$A_n = B_n = 0 \quad \text{for } n \geq 2$$

(1.26)

$$\therefore D = 6\pi,$$

球に近い物体以外では一般に (1.15) は境界上で
収束するとは言えないので 境界上に特異性をおく方法
が望ましい。

2. 互逆定理

ϕ, ψ は $z = x + iy$ の領域 D の境界 C 上で $\phi = \psi = 0$ のとき、
 $\int_D [L(\phi)L(\psi) - \psi L(\phi)] \frac{dx dy}{y} = 0$ の値分

$$I = \int_D [L(\phi)L(\psi) - \psi L(\phi)] \frac{dx dy}{y}, \quad (2.1) \quad \phi, \psi \in D$$

部分積分をやる

$$I = \int_C \left[\frac{2}{y} \frac{\partial \phi}{\partial n} - \frac{\phi}{y} \frac{\partial \psi}{\partial n} \right] ds, \quad (2.2)$$

$$L(\psi) = L(\phi) = 0 \text{ のとき } (2.3)$$

$$\int_C \left[\frac{2}{y} \frac{\partial \phi}{\partial n} - \phi \frac{\partial \psi}{\partial n} \right] \frac{ds}{y} = C, \quad (2.4)$$

さて かつ一般的に (2.2) は成立つから、まず (2.4) (2.2)

の ϕ の代わりに $L(\psi)$ を代入すると

$$\iint_D [L(\psi)L(\psi) - \psi L^2(\psi)] \frac{dx dy}{y} = \int_C \left[\psi \frac{\partial L(\psi)}{\partial n} - L(\psi) \frac{\partial \psi}{\partial n} \right] \frac{ds}{y}, \quad (2.5)$$

ψ のかわりに ψ , ϕ のかわりに $L(\psi)$ を代入すると

$$\iint_D [L(\psi)L(\psi) - \psi L^2(\psi)] \frac{dx dy}{y} = \int_C \left[\psi \frac{\partial L(\psi)}{\partial n} - L(\psi) \frac{\partial \psi}{\partial n} \right] \frac{ds}{y}, \quad (2.6)$$

2.2 と 2.6 から

$$L^2(\psi) = L^2(\psi) = 0, \quad (2.7)$$

よって

$$\iint_D [3L^2(\psi) - \psi L^2(\psi)] \frac{dx dy}{y} = 0 = \int_C \left[\psi \frac{\partial L(\psi)}{\partial n} - L(\psi) \frac{\partial \psi}{\partial n} - \frac{3}{y} \frac{\partial L(\psi)}{\partial n} + L(\psi) \frac{\partial \psi}{\partial n} \right] \frac{ds}{y}, \quad (2.8)$$

次の形に整理できる。

(1.2)より $L(\psi) = -\frac{y}{2}$ である。

$$\int_C \left[\gamma \frac{\partial L(\psi)}{\partial n} - L(\psi) \frac{\partial \gamma}{\partial n} + \gamma \frac{\partial (y\psi)}{\partial n} - y\psi \frac{\partial \gamma}{\partial n} \right] \frac{ds}{\gamma}, \quad (2.7)$$

又さらに γ の代わりに $F_X(\psi, Q)$ を代入すると

$$L_\psi [X] = - \int_C F_X(\psi, Q) \quad (2.10)$$

$$- \int_C \frac{\partial F}{\partial n} \frac{ds}{\gamma} = 1 \quad (2.11)$$

したがって

$$\psi(Q) = \int_C \left[\gamma \frac{\partial F}{\partial n} + \frac{\partial \gamma}{\partial n} F + \frac{\partial F(\psi, Q)}{\partial n} X - y\psi \frac{\partial X}{\partial n} \right] \frac{ds}{\gamma}, \quad (2.12)$$

を得る。

境界条件は

$$\begin{aligned} \psi(D) &= -\frac{y^2}{2}, \\ \frac{\partial \psi}{\partial n} &= -y \frac{\partial \psi}{\partial n}, \end{aligned} \quad \text{on } C, \quad (2.13)$$

したがって Q が領域 D の点ならば明らかに

$$\int_C \left[\frac{y^2}{2} \frac{\partial F}{\partial n} - y \frac{\partial \psi}{\partial n} F \right] \frac{ds}{\gamma} = 0, \quad (2.14)$$

よって (2.13) を満たす場合は

$$\psi(Q) = \int_C \left[\frac{\partial F(\psi)}{\partial n} X - y\psi \frac{\partial X}{\partial n} \right] \frac{ds}{\gamma}, \quad (2.15)$$

また、 $\vec{p} \equiv \vec{\lambda} \cdot \vec{T}$ と

$$\frac{\partial p}{\partial s} = \frac{1}{y} \frac{\partial (y s)}{\partial n}, \quad (2.15)$$

より

$$\gamma(Q) = \int_C \left[\frac{\partial p}{\partial s} X - y \frac{\partial X}{\partial n} \right] ds, \quad (2.16)$$

あるいは部分積分して

$$\gamma(Q) = - \int_C \left[p \frac{\partial X}{\partial s} + y \frac{\partial X}{\partial n} \right] ds, \quad (2.17)$$

さらに

$$\left. \begin{aligned} T_x &= -p \frac{\partial X}{\partial n} - y \frac{\partial y}{\partial n} \\ T_y &= -p \frac{\partial y}{\partial n} + y \frac{\partial X}{\partial n} \end{aligned} \right\} \quad (2.18)$$

とすれば

$$\begin{aligned} p \frac{\partial X}{\partial s} + y \frac{\partial X}{\partial n} &= \left(p \frac{\partial X}{\partial s} + y \frac{\partial X}{\partial n} \right) \frac{\partial X}{\partial X} + \left(p \frac{\partial y}{\partial s} + y \frac{\partial y}{\partial n} \right) \frac{\partial X}{\partial y} \\ &= T_y \frac{\partial X}{\partial X} - T_x \frac{\partial X}{\partial y}, \end{aligned}$$

となる

$$\gamma(Q) = \int_C \left[T_x \frac{\partial X}{\partial y} - T_y \frac{\partial X}{\partial X} \right] ds, \quad (2.19)$$

また

$$u = \frac{1}{y'} \frac{\partial \gamma}{\partial y'} = \frac{1}{y'} \int_C \left[T_x \frac{\partial^2 X}{\partial y \partial y'} - T_y \frac{\partial^2 X}{\partial X \partial y'} \right] ds, \quad (2.20)$$

$$v = \frac{1}{y'} \frac{\partial \gamma}{\partial x'} = \frac{1}{y'} \int_C \left[T_x \frac{\partial^2 X}{\partial y \partial x'} - T_y \frac{\partial^2 X}{\partial X \partial x'} \right] ds, \quad (2.21)$$

$$V_1^{(1)} = - \frac{1}{g y'} \frac{\partial^2 X}{\partial y \partial y'}$$

$$V_y^{(1)} = \frac{1}{g y'} \frac{\partial^2 X}{\partial x \partial y'}$$

$$V_1^{(4)} = \frac{1}{g y'} \frac{\partial^2 X}{\partial y \partial x'}$$

$$V_y^{(4)} = - \frac{1}{g y'} \frac{\partial^2 X}{\partial x \partial x'}$$

(2.22)

$$u = - \int [F_x V_1^{(4)} + F_y V_y^{(4)}] y ds$$

$$v = - \int_c [F_x V_1^{(4)} + F_y V_y^{(4)}] y ds$$

(2.23)

3. 核関数

$$X = X_1 + X_2 \quad (3.1)^*$$

$$X_1 = -\frac{y y'}{2} \int_0^\infty e^{-t(x-y)} J_0(y) J_0(y') dt \quad (3.2)$$

$$X_2 = -\frac{y y'}{2} \int_0^\infty e^{-t(x+y)} J_1(y) J_1(y') dt \quad (3.3)$$

と $x = y = y' = 0$ のとき

$$L_p = \frac{\partial^2}{\partial x^2} + y \frac{\partial}{\partial y} \frac{1}{y} \frac{\partial}{\partial y}, \quad -\Delta = \frac{\partial^2}{\partial x^2} + y \frac{\partial}{\partial y} \left(\frac{1}{y} \frac{\partial}{\partial y} \right)$$

と L_q

$$\begin{aligned} L_p[X_1] &= -\frac{y y'}{2} \int_0^\infty e^{-t x} J_0(y) J_0(y') dt = G(p, q), \\ &= -\frac{y y'}{4\pi} \int_0^{2\pi} \frac{\cos \theta}{R} d\theta, \quad R^2 = x^2 + y^2 + y'^2 = 2 y y' \cos \theta \end{aligned} \quad (3.4)$$

$$L_q[X_1] = 0,$$

$$L_p[X_2] = 0,$$

$$L_q[X_2] = -\frac{y y'}{2} \int_0^\infty e^{-t x} J_1(y) J_1(y') dt = -G(p, q) \quad (3.5)$$

($x = y = y' = 0$ と $x > 0$ と $x < 0$ とで異なる)

$$G(p, q) \xrightarrow{p \rightarrow 0} -\frac{y}{2\pi} \log(\sqrt{x^2 + y^2}), \quad (3.6)$$

$$y \frac{\partial}{\partial y} \left[\frac{1}{y} \frac{\partial}{\partial y} (y J_0(y)) \right] = - y t \frac{\partial}{\partial y} \left[\frac{1}{y} \frac{\partial}{\partial y} (y J_0(y)) \right] = \frac{1}{y} J_1 = \frac{1}{y} J_1' (y) \\ = \frac{1}{y} J_1 - t^2 J_0$$

$$L_y [J_0(y)] = \frac{1}{y} J_1 //$$

$$y \frac{\partial}{\partial y} \left[\frac{1}{y} \frac{\partial}{\partial y} (y^2 J_0(y)) \right] = z \frac{d}{dz} \left[\frac{1}{z} \frac{d}{dz} (z^2 J_0) \right] = z \frac{d}{dz} [2J_0 - z J_1] \\ = -2z J_1 - z^2 J_0$$

$$\therefore L [y^2 J_0(y)] = -2y t J_1(y) //$$

$$y \frac{\partial}{\partial y} \left[\frac{1}{y} \frac{\partial}{\partial y} (y J_1(y)) \right] = t z \frac{d}{dz} \left[\frac{1}{z} \frac{d}{dz} (z J_1) \right] = t z \frac{d}{dz} J_0 = -z J_1 (y)$$

$$\therefore L [y J_1(y)] = 0 //$$

$$y \frac{\partial}{\partial y} \left[\frac{1}{y} \frac{\partial}{\partial y} (y^2 J_1(y)) \right] = z \frac{d}{dz} \left[\frac{1}{z} \frac{d}{dz} (z^2 J_1) \right] = z \frac{d}{dz} z J_1 \\ = z^2 J_0 = z^2 \left(\frac{2}{3} J_1 - J_2 \right)$$

$$\therefore L (y^2 J_1(y)) = 2y t J_1(y) //$$

$$\frac{\partial}{\partial y} J_0(y) = -t J_1(y), \quad \frac{\partial}{\partial y} [y J_1(y)] = y t J_0(y)$$

$$\frac{\partial}{\partial y} [y^2 J_2(y)] = y^2 t J_1(y), \quad \frac{\partial}{\partial y} [y^2 J_0(y)] = y [2J_0 - y t J_1]$$

$$J_2(y) = \frac{2}{y t} J_1(y) - J_0(y)$$

求微分

$$\frac{1}{y} \frac{\partial X_1}{\partial y} = \frac{y}{2} \int_0^\infty e^{-tx} \frac{\partial}{\partial y} \left(\frac{1}{J_0(tx)} J_0(tx) \right) \frac{dt}{t} - \frac{y}{4} \int_0^\infty e^{-tx} \frac{\partial}{\partial y} \left(\frac{1}{J_0(tx)} J_0(tx) \right) dt$$

$$= \frac{3}{y^2} X_1 + \frac{1}{2} G$$

$$\frac{1}{y} \frac{\partial X_1}{\partial x} = - \frac{y}{4} \int_0^\infty e^{-tx} \frac{\partial}{\partial x} \left(\frac{1}{J_0(tx)} J_0(tx) \right) dt = - \frac{y}{2} \frac{\partial}{\partial x} \left(\frac{1}{J_0(tx)} J_0(tx) \right)$$

$$\frac{\partial X_1}{\partial x} = - \frac{\partial X_1}{\partial x}$$

$$\frac{1}{y} \frac{\partial X_1}{\partial y} = \frac{y}{2} \int_0^\infty e^{-tx} \frac{\partial}{\partial y} \left(\frac{1}{J_0(tx)} J_0(tx) \right) dt = \frac{y}{2} S$$

$$\frac{1}{y} \frac{\partial^2 X_1}{\partial y^2} = S + \frac{y}{2} \frac{\partial S}{\partial y} = 1$$

$$\frac{1}{y} \frac{\partial^2 X_1}{\partial x \partial x} = - \frac{y}{2} \frac{\partial^2 T}{\partial x^2} = - \frac{y}{2} \frac{\partial S}{\partial x} = \frac{y}{2} \frac{\partial S}{\partial x} = \frac{y}{2} \frac{\partial S}{\partial x}$$

$$\begin{aligned} \frac{1}{y} \frac{\partial^2 X_1}{\partial x \partial y} &= - \frac{1}{y} \frac{\partial}{\partial y} \left[\frac{y^2}{2} T \right] = - \frac{y}{2} T - \frac{y}{2} \frac{\partial T}{\partial y} \\ &= - \frac{y}{2} T + \frac{y}{2} \frac{\partial P}{\partial x} \end{aligned}$$

$$\frac{1}{y} \frac{\partial^2 X_1}{\partial x \partial x} = - \frac{y}{2} \frac{\partial^2 T}{\partial x^2} = \frac{y}{2} \frac{\partial S}{\partial x} =$$

$$\begin{aligned}
 \frac{1}{y} \frac{\partial^2 X_1}{\partial y^2} &= -\frac{y^{1/2}}{4} \int_0^\infty e^{-tx} \frac{1}{T_0+y} T_0+y \, dt \\
 &= -\frac{y'}{2} \int_0^\infty e^{-tx} \frac{1}{T_0+y} T_0+y \frac{dt}{t} + \frac{y'}{4} \int_0^\infty e^{-tx} T_0+y T_0+y \, dt \\
 &= \frac{y^{1/2}}{2} P + \frac{1}{4} T;
 \end{aligned}$$

$$\begin{aligned}
 \frac{1}{y} \frac{\partial^2 X_2}{\partial y^2} &= -\frac{1}{y} \frac{\partial X_2}{\partial X'} = \frac{y^{1/2}}{4} \int_0^\infty e^{-tx} \frac{1}{T_0+y} T_0+y \, dt \\
 &= \frac{y'}{2} \int_0^\infty e^{-tx} \frac{1}{T_0+y} T_0+y \frac{dt}{t} - \frac{y^{1/2}}{4} \int_0^\infty e^{-tx} T_0+y T_0+y \, dt \\
 &= \frac{1}{y} G_A - \frac{y^{1/2}}{2y} T,
 \end{aligned}$$

$$\frac{1}{y'} \frac{\partial^2 X_2}{\partial y'} = -\frac{y y'}{4} \int_0^\infty e^{-tx} T_0+y T_0+y \, dt = \frac{1}{2} G,$$

$$\frac{1}{y y'} \frac{\partial^2 X_2}{\partial y' \partial y} = \frac{1}{2 y} \frac{\partial G}{\partial y} = \frac{1}{2} y' \frac{\partial S}{\partial y},$$

$$\frac{1}{y y'} \frac{\partial^2 X_2}{\partial y' \partial X} = \frac{1}{2 y} \frac{\partial G}{\partial X} = -\frac{1}{2} \frac{\partial T}{\partial y} = -\frac{y'}{2} \frac{\partial F}{\partial y}$$

$$\frac{1}{y y' \partial X' \partial y} = \frac{y'}{2} \frac{\partial S}{\partial X'} + \frac{1}{y'} T$$

$$\frac{1}{y y'} \frac{\partial^2 X_2}{\partial X' \partial X'} = \frac{1}{y y'} G - \frac{y'}{2 y} \frac{\partial T}{\partial X'} = +P + \frac{y'}{2} \frac{\partial S}{\partial y},$$

証明

$$\frac{1}{y y'} \frac{\partial^2 X}{\partial y \partial y'} = S = \frac{1}{2} \left(y \frac{\partial S}{\partial y} + y' \frac{\partial S}{\partial y'} \right),$$

$$\frac{1}{y y'} \frac{\partial^2 X}{\partial y' \partial x} = \frac{1}{2} \left[y \frac{\partial P}{\partial x} - y' \frac{\partial P}{\partial x} \right], \quad (2.11)$$

$$\frac{1}{y y'} \frac{\partial^2 X}{\partial x' \partial y} = \frac{1}{2} \left[y \frac{\partial P}{\partial x} - y' \frac{\partial S}{\partial x} \right],$$

$$\frac{1}{y y'} \frac{\partial^2 X}{\partial x' \partial x} = P = \frac{1}{2} \left[y' \frac{\partial S}{\partial y} + y \frac{\partial S}{\partial y'} \right],$$

よって, (2.22) を通し 附録 (A.9) に一致する。

附録 A 種口ノ核

$$S^*(P, Q) = \frac{1}{4\pi} \int_0^{2\pi} R ds = \frac{R E(1/2)}{\pi} \quad (A.1)$$

$$S = \frac{1}{4\pi} \int_0^{2\pi} \frac{ds}{R} = \frac{K(k)}{\pi r_2} = \frac{1}{2} \int_0^{\infty} \frac{e^{-t|x|}}{J_0(ty) J_0(ty')} dt, \quad (A.2)$$

$$P = \frac{1}{4\pi} \int_0^{2\pi} \frac{\cos \delta}{R} ds = \frac{1}{2} \int_0^{\infty} \frac{e^{-t|x|}}{J_1(ty) J_1(ty')} dt, \quad (A.3)$$

$$T = \frac{y}{2} \int_0^{\infty} \frac{e^{-t|x|}}{J_1(ty) J_0(ty')} dt, \quad (A.4)$$

$$\tilde{T} = \frac{y'}{2} \int_0^{\infty} \frac{e^{-t|x|}}{J_0(ty) J_1(ty')} dt,$$

$$T(x; y, y') = \tilde{T}(x; y', y),$$

$$\frac{\partial \tilde{T}}{\partial x} = \frac{\tilde{T}}{x}, \quad \frac{\tilde{T}}{1A} = -\frac{y'}{2} \int_0^{\infty} \frac{e^{-t|x|}}{J_0(ty) J_1(ty')} \frac{dt}{t},$$

$$\left[\begin{array}{l} \text{如ク外ノラ 読書ニハ} \\ S = \tilde{T}, \quad S_A = \tilde{T}_A, \text{ 対称ニテハ} \end{array} \quad G = -yy'P, \right]$$

E, K は 楕円積分で $T, \tilde{T}$ は 尤も 種 楕円積分也;

$$(k')^2 = r_1^2/r_2^2, \quad k^2 = 1-k'^2 = \frac{4yy'}{r_2^2},$$

$$r_1^2 = x^2 + (y-y')^2, \quad r_2^2 = x^2 + (y+y')^2,$$

§ 5"

$$\frac{\partial S^+}{\partial x} = -\frac{\partial S^+}{\partial x'} = x S,$$

$$\frac{\partial S^+}{\partial y} = y S - y' P,$$

$$\frac{\partial S^+}{\partial y'} = y' S - y P,$$

(A.5-)

$$\left. \begin{aligned} \frac{\partial T}{\partial x} &= \frac{1}{y} \frac{\partial G}{\partial y} = -\frac{y}{y'} \frac{\partial (y'P)}{\partial y} = y \frac{\partial S}{\partial x}, \\ \frac{\partial T}{\partial y'} &= \frac{1}{y} \frac{\partial G}{\partial x'} = -\frac{y}{y'} \frac{\partial P}{\partial x}, \quad \frac{\partial T}{\partial y} = -\frac{y}{y'} \frac{\partial S}{\partial x} \end{aligned} \right\}$$

$$\frac{\partial \tilde{T}}{\partial x} = \frac{1}{y} \frac{\partial G}{\partial y} = -\frac{y'}{y} \frac{\partial (y'P)}{\partial y} = y' \frac{\partial S}{\partial y'},$$

$$\frac{\partial \tilde{T}}{\partial y'} = -\frac{y'}{y} \frac{\partial S}{\partial x}, \quad \frac{\partial \tilde{T}}{\partial y} = -\frac{1}{y} \frac{\partial G}{\partial x} = -y' \frac{\partial P}{\partial x},$$

$$\frac{\partial}{\partial x} S^* = x S = -\frac{1}{2} \int_0^\infty J_0(y) J_0(y') \frac{\partial^2}{\partial t^2} dt$$

$$= \frac{1}{2} - (T + \tilde{T}), \quad \dots (A.7)$$

$$\frac{\partial^2}{\partial x \partial x'} S^* = - \left(\frac{\partial T}{\partial x} + \frac{\partial \tilde{T}}{\partial x} \right) = -y \frac{\partial S}{\partial y} - \frac{y'}{y} \frac{\partial S}{\partial y'},$$

$$\frac{\partial^2}{\partial x \partial y} S^* = - \frac{\partial}{\partial y} (T + \tilde{T}) = y \frac{\partial S}{\partial x} - y' \frac{\partial P}{\partial x},$$

$$\frac{\partial^2}{\partial x \partial y'} S^* = y' \frac{\partial S}{\partial x} - y \frac{\partial P}{\partial x},$$

$$\frac{\partial^2}{\partial y \partial y'} S^* = y \frac{\partial S}{\partial y'} - \frac{\partial (y'P)}{\partial y'} = -P + y \frac{\partial S}{\partial y'} - y' \frac{\partial P}{\partial y'},$$

$$V_1^{(1)} = \frac{1}{2} S_{xx}^* - S = - \left[S + \frac{1}{2} (y S_y + y' S_{y'}) \right],$$

$$V_2^{(1)} = \frac{1}{2} S_{xy}^* = \frac{1}{2} y S_x - y' P_x,$$

$$V_2^{(2)} = -\frac{1}{2} S_{xy'}^* = \frac{1}{2} (y P_x - y' S_x)$$

$$V_2^{(2)} = -P - \frac{1}{2} S_{yy'}^* = -\frac{1}{2} P - \frac{1}{2} (y S_{y'} - y' P_{y'})$$

$$= -P - \frac{1}{2} (y S_{y'} + y' S_{y'})$$

同様 - 射影に r を r_1 とし、 r_2 の代りに r_2 を r_2 とする。

微分を $\frac{\partial}{\partial r_1}$ と $\frac{\partial}{\partial r_2}$ の方が楽である。(例 r_1, r_2 の代りに r_1, r_2)

極限積分の微分を、

$$\frac{\partial K}{\partial r_1} = \frac{E}{r_1} - K,$$

$$\frac{\partial E}{\partial r_2} = E - K,$$

から

$$\frac{\partial S^*}{\partial x} = x S,$$

$$\frac{\partial S^*}{\partial y} = \frac{S^*}{2y} - \left\{ \frac{r_2^2}{2y} - (y+y') \right\} S^*,$$

$$\frac{\partial S}{\partial x} = - \frac{x}{(r_1 r_2)^2} S^*,$$

$$\frac{\partial S}{\partial y} = - \frac{S}{2y} + \frac{1}{r_1^2} \left(\frac{1}{2y} - \frac{y+y'}{r_2^2} \right) S^*,$$

$$\frac{\partial^2 S^*}{\partial x^2} = S - \frac{x^2}{(r_1 r_2)^2} S^*,$$

$$\frac{\partial^2 S^*}{\partial x \partial y} = - \frac{x}{2y} S + \frac{x}{r_1^2} \left(\frac{1}{2y} - \frac{y+y'}{r_2^2} \right) S^*,$$

$$\frac{\partial^2 S^*}{\partial x \partial y'} = - \frac{x}{2y'} S + \frac{x}{r_1^2} \left(\frac{1}{2y'} - \frac{y+y'}{r_2^2} \right) S^*,$$

$$\frac{\partial^2 S^*}{\partial y \partial y'} = - \left(\frac{y^2 + y'^2}{2y y'} \right) S + \frac{\{ x^2 (y^2 + y'^2) + (y^2 - y'^2) \}}{2y y' (r_1 r_2)^2} S^*$$

$$X_1 = \frac{(y')^2}{8} \int_0^{\infty} e^{-tx} J_2(xy) [J_0(xy) + J_2(xy)] dt$$

$$X_2 = -\frac{(y')^2}{8} \int_0^{\infty} e^{-tx} [J_0(xy) + J_2(xy)] J_2(xy) dt$$

$$\therefore X = \frac{(y')^2}{8} \int_0^{\infty} e^{-tx} [J_0(xy)J_0(xy) - J_2(xy)J_2(xy)] dt$$

$$J = \frac{1}{4\pi} \int_0^{2\pi} \frac{d\theta}{R} = \frac{K(\theta)}{\pi r_2} = \frac{1}{2} \int_0^{\infty} e^{-tx} J_0(xy) J_0(xy) dt$$

$$\frac{1}{R} = \int_0^{\infty} e^{-tx} dt [J_0(xy)J_0(xy) + 2 \sum_{n=1}^{\infty} J_n(xy)J_n(xy) \cos n\theta]$$

$$\therefore I = \frac{1}{2\pi} \int_0^{2\pi} \frac{\cos \theta}{R} d\theta = \int_0^{\infty} e^{-tx} J_2(xy) J_2(xy) dt$$

$$\therefore X = \frac{(y')^2}{4} J - \frac{(y')^2}{8} I$$

$$\left\{ \begin{aligned} r_1^2 &= x^2 + (y-y')^2, & r_2^2 &= x^2 + (y+y')^2, \\ r_1' &= \left(\frac{y}{r_2}\right)^2, & r_2' &= 1 - r_1'^2 \end{aligned} \right\} \text{ by (2)}$$

$$R = r_2 \sqrt{1 - k^2 \sin^2 \varphi}, \quad 0 \leq \varphi, \quad \theta = \pi - 2\varphi, \quad \cos \theta = -\cos 2\varphi$$

$$\therefore \cos 2\varphi = 1 - 8k^2 \sin^2 \varphi \cos^2 \varphi$$

$$I = \frac{2}{\pi r_2} \int_0^{\frac{\pi}{2}} \frac{1 - 8k^2 \sin^2 \varphi \cos^2 \varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} d\varphi$$

$$\text{By (2)} \quad \int_0^{\frac{\pi}{2}} \frac{k^2 \sin^2 \varphi \cos^2 \varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} d\varphi = \frac{1}{2k^2} \int_0^{\frac{\pi}{2}} \frac{1 - k^2 \sin^2 \varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} \cos 2\varphi d\varphi$$

$$= \frac{1}{4k^2} \int_0^{2\pi} \frac{\left(1 - \frac{2-k^2}{k^2}\right) + \frac{2-k^2}{k^2} (1 - k^2 \sin^2 \varphi)}{\sqrt{1 - k^2 \sin^2 \varphi}} d\varphi$$

$$= -\frac{k^2}{2k^4} K(k) + \frac{1+k^2}{2k^2} E(k)$$

$$\therefore I = \frac{2(1-k^2)}{\pi f_2 k} [(1+k^2)K - 2E],$$

$$\therefore X = \frac{(yy')^2}{4} \frac{K}{\pi f_2} - \frac{(yy')^2(1+k^2)}{4\pi f_2 k} [(1+k^2)K - 2E]$$

$$= \frac{(yy')^2}{4\pi f_2 k} [-4k^2 K + 2(1+k^2)E]$$

$$= \frac{(yy')^2(1+k^2)}{2\pi f_2 k^2} \left[E - \frac{2k^2}{1+k^2} K \right] //$$

$$X = \frac{(yy')^2}{16\pi} \int_0^{2\pi} \frac{1 - \cos 2\epsilon}{R} d\epsilon = \frac{(yy')^2}{8\pi} \int_0^{2\pi} \frac{\sin^2 \epsilon d\epsilon}{R},$$

軸対称スロー流れの流線関数

別添割

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附録 A 種々の核 A-123

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概要

今までは最終的にふたえ物体を取扱う意図が"ある"で直交座標で定式化にきた"が軸対称流れではやはり流れ関数を導入しておく方が便利である。

その流れ関数はポテンシャル流れにおけるストークスの流れ関数の定義をそのまま使えば"よい"ようである。²⁾

特に直交座標による速度ベクトル表現では軸対称流れへの変換が複雑なので、これらの式のチェックをも兼ねて別法として流れ関数表示の境界方程式を導いて見た。

§1 は、~~流れ関数の定義と~~ ^{前置きとして} 環関数による展開しただけであまり意味はない。

§2 では流れ関数についての互逆定理を導き境界における圧力と渦度によって流れ関数が表現出来る事と示し、§3と附録Aで核関数の具体的表現を求め前報告と比較しその正当性を確かめた。

連続の式 $\frac{\partial(yu)}{\partial x} + \frac{\partial(yv)}{\partial y} = 0$

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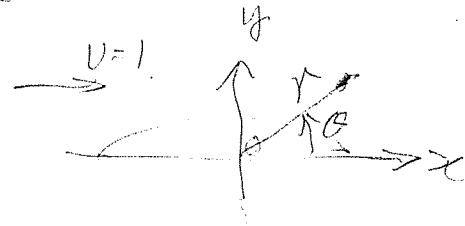
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1. 流れ関数の導出

速度を u, v , 流れ関数を χ とおくと

定義は

$$\left. \begin{aligned} u &= \frac{1}{y} \frac{\partial \chi}{\partial y} \\ v &= -\frac{1}{y} \frac{\partial \chi}{\partial x} \end{aligned} \right\} (1.1)$$



渦度は

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\frac{1}{y} L\chi, \quad (1.2)$$

2.1.2

$$L \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{1}{y} \frac{\partial}{\partial y} = \frac{\partial^2}{\partial x^2} + y \frac{\partial}{\partial y} \left[\frac{1}{y} \frac{\partial}{\partial y} \right], \quad (1.3)$$

なおラプラスアン^{*} Δ は

$$\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1}{y} \frac{\partial}{\partial y} - \frac{1}{y^2}, \quad (1.4)$$

ζ は 渦度関数 なので

$$\Delta \zeta = \frac{1}{y} L(y\zeta) = 0, \quad (1.5)$$

したがって (1.2) から

$$L^2 \chi = 0, \quad (1.6)$$

そこで極座標を導入しよう。

$$x = r \cos \theta, \quad y = r \sin \theta, \quad \mu = \cos \theta$$

とすると

$$\left. \begin{aligned} \frac{\partial}{\partial x} &= \cos \theta \frac{\partial}{\partial r} - \frac{\mu \sin \theta}{r} \frac{\partial}{\partial \theta} = \mu \frac{\partial}{\partial r} + \frac{1-\mu^2}{r} \frac{\partial}{\partial \mu} \\ \frac{\partial}{\partial y} &= \sin \theta \frac{\partial}{\partial r} + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta} = \sqrt{1-\mu^2} \frac{\partial}{\partial r} - \frac{\mu \sqrt{1-\mu^2}}{r} \frac{\partial}{\partial \mu} \end{aligned} \right\} (1.7)$$

$$L \equiv \frac{\partial^2}{\partial r^2} + \frac{1-\mu^2}{r^2} \frac{\partial^2}{\partial \mu^2}, \quad \dots$$

* u は 2.1.2 のラプラスアンは

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1}{y} \frac{\partial}{\partial y} \right] u = 0$$

$$\begin{aligned}\Delta u &= u_{xx} + u_{yy} + \frac{u_y}{y} - \frac{u}{y^2} = -\frac{1}{y} \frac{\partial}{\partial x} \left(\frac{\partial (y u)}{\partial y} \right) + u_{yy} + \frac{u_y}{y} - \frac{u}{y^2} \\ &= u_{yy} + \frac{u_y}{y} - \frac{u}{y^2} - v_{xy} - \frac{v_x}{y} = \frac{\partial}{\partial y} (u_y - v_x) + \frac{1}{y} (u_y - v_x) \\ &= -\frac{1}{y} \frac{\partial}{\partial y} (y S) = -\frac{\partial p}{\partial x} - \frac{u}{y^2}\end{aligned}$$

$$\Delta v = v_{xx} + v_{yy} + \frac{v_y}{y} - \frac{v}{y^2} = v_{xx} - u_{xy} = \frac{\partial^2}{\partial x^2} = \frac{\partial p}{\partial y}$$

$$v + y v_y = \frac{\partial}{\partial y} (y v) = -y u_x$$

$$2 v_y + y v_{yy} = -u_x + y u_{xy}, \quad v_{yy} + \frac{v_y}{y} = -\frac{v_x}{y} - \frac{u_x}{y} - u_{xy}$$

$$v_{yy} + \frac{v_y}{y} - \frac{v}{y^2} = -\frac{1}{y^2} \frac{\partial}{\partial y} (y v) - \frac{u_x}{y} - u_{xy} = -u_{xy}$$

$$\frac{\partial p}{\partial x} = -\frac{1}{y} \frac{\partial}{\partial y} (y S), \quad \frac{\partial p}{\partial y} = \frac{1}{y} \frac{\partial}{\partial x} (y S)$$

$$\frac{\partial p}{\partial S} = \frac{1}{y} \frac{\partial}{\partial x} (y S)$$

ルジャンドル多項式 $P_n(\mu)$ を導入すると、

$$\frac{d}{d\mu} \left((1-\mu^2) \frac{dP_n}{d\mu} \right) + n(n+1)P_n(\mu) = 0, \quad (1.8)$$

なるから

$$f = \frac{(1-\mu^2)}{r^n} \frac{dP_n}{d\mu} \quad \left. \begin{array}{l} \text{とおくと} \\ Lf = 0, \end{array} \right\} \quad (1.9)$$

また、

$$L \left((1-\mu^2)^{n+1/2} \frac{dP_n}{d\mu} \right) = 0, \quad (1.10)$$

で「ある」が、 $n=1$ では

$$f = \frac{y^2}{z} \quad (1.11)$$

は一般流れて $n \geq 1$ では一般流れておらず、

さて

$$h = (1-\mu^2)^{2-n} \frac{dP_n}{d\mu} \quad (1.11)$$

とおくと

$$Lh = -2(2n+1) \frac{(1-\mu^2)}{r^{2n}} \frac{dP_n}{d\mu}, \quad (1.12)$$

となるから

$$L^2 h = 0, \quad (1.14)$$

よって一般に流れていない一般流れておらず、

$$\chi = \frac{y^2}{z} + (1-\mu^2) \sum_{n=1}^{\infty} (A_n r^{2n+1} + B_n r^{-2n-1}) \frac{dP_n}{d\mu}, \quad (1.15)$$

の形式で表わされる。

速度は

$$\begin{aligned}
 U &= \frac{1}{y} \frac{\partial}{\partial y} \chi = \left(\frac{1}{y} \frac{\partial}{\partial y} - \frac{\mu}{r^2} \frac{\partial}{\partial \mu} \right) \chi \\
 &= 1 + (1-\mu^2) \sum_{n=1}^{\infty} \left(\frac{(2n-1)}{r^n} A_n - \frac{n B_n}{r^{n+2}} \right) \frac{dP_n}{d\mu} \\
 &\quad + \mu \sum_{n=1}^{\infty} \left(\frac{A_n}{r^n} + \frac{B_n}{r^{n+2}} \right) n(n+1) P_n(\mu),
 \end{aligned}$$

$$\begin{aligned}
 V &= -\frac{1}{y} \frac{\partial \chi}{\partial x} = \left(-\frac{\mu}{r\sqrt{1-\mu^2}} \frac{\partial}{\partial r} - \frac{\sqrt{1-\mu^2}}{r^2} \frac{\partial}{\partial \mu} \right) \chi \\
 &= -\mu\sqrt{1-\mu^2} \sum_{n=1}^{\infty} \left(\frac{(2n-1)}{r^n} A_n - \frac{n B_n}{r^{n+2}} \right) \frac{dP_n}{d\mu} \\
 &\quad + \sqrt{1-\mu^2} \sum_{n=1}^{\infty} \left(\frac{A_n}{r^n} + \frac{B_n}{r^{n+2}} \right) n(n+1) P_n, \quad (1.16)
 \end{aligned}$$

$$\mathcal{L} \chi = L \chi = -2(1-\mu^2) \sum_{n=1}^{\infty} \frac{(2n-1) A_n}{r^n} \frac{dP_n}{d\mu}$$

また、

$$\mathcal{L} = 2 \sum_{n=1}^{\infty} \frac{(2n-1) A_n}{r^{n+1}} P_n'(\mu), \quad (1.17)$$

$$P_n^{(m)}(\mu) = (-1)^m (1-\mu^2)^{\frac{m}{2}} \frac{d^m}{d\mu^m} P_n(\mu), \quad (1.18)$$

よって

$$P_0(\mu) = 1, \quad P_1(\mu) = \mu, \quad P_2(\mu) = \frac{1}{2}(3\mu^2 - 1),$$

$$P_3(\mu) = \frac{1}{2}(5\mu^3 - 3\mu), \quad P_4(\mu) = \frac{1}{8}(35\mu^4 - 30\mu^2 + 3), \quad (1.19)$$

$$\frac{dP_0}{d\mu} = 0, \quad \frac{dP_1}{d\mu} = 1, \quad \frac{dP_2}{d\mu} = 3\mu$$

境界条件は

$$u = v = 0 \quad \text{on } C, \quad (1.20)$$

であるが、もしも

$$\frac{\partial \chi}{\partial n} = \frac{\partial \chi}{\partial s} = 0,$$

でもよく、又、一般の円周上で

$$\frac{\partial \chi}{\partial n} = \chi = 0 \quad \text{on } C, \quad (1.21)$$

でもいい。

(1.1) より、分速 ω では

$$u \doteq 1 + \frac{A_1}{r} (1 + \mu^2) +$$

$$v \doteq \frac{A_1}{r} \mu \sqrt{1 - \mu^2}$$

$$\left. \begin{array}{l} \dots (1.22) \end{array} \right\}$$

～方、抵抗力を D とすると、

$$u - 1 \doteq \frac{-D}{8\pi r} (1 + \mu^2)$$

$$v = \frac{-D}{8\pi r} \mu \sqrt{1 - \mu^2}$$

$$\left. \begin{array}{l} \dots (1.23) \end{array} \right\}$$

$$\therefore A_1 = \frac{-D}{8\pi} \quad (1.24)$$

これから考えて見よう。

(1.21) の境界条件の方が便利で、それを書くと、

$$\left. \begin{aligned} \frac{r^2}{2} + r^2 \sum_{n=1}^{\infty} \left(\frac{A_n}{r^n} + \frac{B_n}{r^{n+2}} \right) \frac{dP_n}{d\mu} &= 0 \\ r + r \sum_{n=1}^{\infty} \left(\frac{(2-n)}{r^n} A_n - \frac{n B_n}{r^{n+2}} \right) \frac{dP_n}{d\mu} &= 0 \end{aligned} \right\} \quad (1.25)$$

これは一般の π 形状でもよいが、今は $r=1$ とおくと
より簡単に

$$\sum (n A_n + (n+2) B_n) \frac{dP_n}{d\mu} = 0$$

から

$$B_n = -\frac{n}{n+2} A_n.$$

これを式(1)に代入して、 $\frac{dP_n}{d\mu}$ の係数を 0 とおくと、

$$\left. \begin{aligned} A_1 = -3B_1 &= -\frac{3}{4} \\ A_n = B_n &= 0 \quad \text{for } n \geq 2 \end{aligned} \right\} \quad (1.26)$$

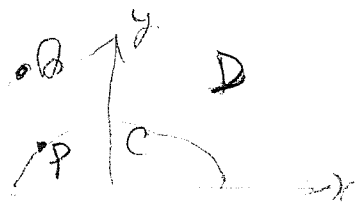
$$\therefore D = 6\pi,$$

球に近い物体以外では一般に (1.15) は境界上で
収束するとは言えないので 境界上に特異性をおこす
が望ましい。

2. 互逆定理

ϕ, ψ なる 2つのスカラー関数の偏微分係数を考えよう。
 について 2次の積分

$$I = \iint_D [\phi L(\psi) - \psi L(\phi)] \frac{dx dy}{y}, \quad (2.1)$$



部分積分によつて

$$I = \int_C \left[\frac{\psi}{y} \frac{\partial \phi}{\partial n} - \frac{\phi}{y} \frac{\partial \psi}{\partial n} \right] ds, \quad (2.2)$$

$$L(\psi) = L(\phi) = 0 \text{ の仮定は. } (2.3)$$

$$\int_C \left[\psi \frac{\partial \phi}{\partial n} - \phi \frac{\partial \psi}{\partial n} \right] \frac{ds}{y} = 0, \quad (2.4)$$

さて かつと一般的に (2.2) は成立つから, また (2.1), (2.2) の ϕ の代わりに $L(\psi)$ を代入して置くと

$$\iint_D [L(\psi)L(\psi) - \psi L^2(\psi)] \frac{dx dy}{y} = \int_C \left[\psi \frac{\partial L(\psi)}{\partial n} - L(\psi) \frac{\partial \psi}{\partial n} \right] \frac{ds}{y}, \quad (2.5)$$

ψ のかわりに ψ , ϕ のかわりに $L(\psi)$ を代入すると

$$\iint_D [L(\psi)L(\psi) - \psi L^2(\psi)] \frac{dx dy}{y} = \int_C \left[\psi \frac{\partial L(\psi)}{\partial n} - L(\psi) \frac{\partial \psi}{\partial n} \right] \frac{ds}{y}, \quad (2.6)$$

2つを相引いて

$$L^2(\psi) = L^2(\psi) = 0, \quad \dots (2.7)$$

よおしく

$$\iint_D [\psi L^2(\psi) - \psi L^2(\psi)] \frac{dx dy}{y} = 0 = \int_C \left[\psi \frac{\partial L(\psi)}{\partial n} - L(\psi) \frac{\partial \psi}{\partial n} - \psi \frac{\partial L(\psi)}{\partial n} + L(\psi) \frac{\partial \psi}{\partial n} \right] \frac{ds}{y}, \quad (2.8)$$

なる相互定理を得る。

$$(1.2)より \quad L(\psi) = -\psi\zeta \quad \text{であるから}$$

$$\int_C \left[\psi \frac{\partial L(\zeta)}{\partial n} - L(\zeta) \frac{\partial \psi}{\partial n} + \zeta \frac{\partial (\psi\zeta)}{\partial n} - \psi\zeta \frac{\partial \zeta}{\partial n} \right] \frac{ds}{\gamma}, \quad (2.9)$$

又(2.5)の右辺に核 $X(P, Q)$ を代入すると

$$L_p[X(\zeta)] = -G(P, Q) \quad \dots (2.10)$$

$$- \int_C \frac{\partial G}{\partial n} \frac{ds}{\gamma} = 1, \quad \dots (2.11)$$

を得る。

$$\psi(Q) = \int_C \left[\psi \frac{\partial G}{\partial n} + \frac{\partial \psi}{\partial n} G + \frac{\partial (\psi\zeta)}{\partial n} X - \psi\zeta \frac{\partial X}{\partial n} \right] \frac{ds}{\gamma}, \quad (2.12)$$

を得る。

境界条件は

$$\left. \begin{aligned} \psi(P) &= -\frac{\gamma^2}{2}, \\ \frac{\partial \psi}{\partial n} &= -\gamma \frac{\partial \psi}{\partial n} \end{aligned} \right\} \text{ on } C, \quad (2.13)$$

ここで Q 点が領域 D の点でない限りから

$$\int_C \left[\frac{\gamma^2}{2} \frac{\partial G}{\partial n} - \gamma \frac{\partial \psi}{\partial n} G \right] \frac{ds}{\gamma} = 0, \quad \dots (2.14)$$

よって (2.13) を満たす場合は

$$\psi(Q) = \int_C \left[\frac{\partial (\psi\zeta)}{\partial n} X - \psi\zeta \frac{\partial X}{\partial n} \right] \frac{ds}{\gamma}, \quad (2.15)$$

さらに圧力 p を加えると.

$$\frac{\partial p}{\partial s} = \frac{1}{y} \frac{\partial (\gamma s)}{\partial n}, \quad \dots (2.16)$$

したがって

$$\gamma(Q) = \int_C \left[\frac{\partial p}{\partial s} X - \gamma \frac{\partial X}{\partial n} \right] ds, \quad \dots (2.17)$$

あるいは部分積分して

$$\gamma(Q) = - \int_C \left[p \frac{\partial X}{\partial s} + \gamma \frac{\partial X}{\partial n} \right] ds, \quad (2.18)$$

さらに

$$\left. \begin{aligned} T_x &= -p \frac{\partial X}{\partial n} - \gamma \frac{\partial y}{\partial n} \\ T_y &= -p \frac{\partial y}{\partial n} + \gamma \frac{\partial X}{\partial n} \end{aligned} \right\} \dots (2.19)$$

とすれば

$$\begin{aligned} p \frac{\partial X}{\partial s} + \gamma \frac{\partial X}{\partial n} &= (p \frac{\partial X}{\partial s} + \gamma \frac{\partial X}{\partial n}) \frac{\partial X}{\partial x} + (p \frac{\partial y}{\partial s} + \gamma \frac{\partial y}{\partial n}) \frac{\partial X}{\partial y} \\ &= T_y \frac{\partial X}{\partial x} - T_x \frac{\partial X}{\partial y}, \end{aligned}$$

したがって

$$\gamma(Q) = \int_C \left[T_x \frac{\partial X}{\partial y} - T_y \frac{\partial X}{\partial x} \right] ds, \quad (2.20)$$

また

$$u = \frac{1}{y'} \frac{\partial y}{\partial y'} = \frac{1}{y'} \int_C \left[T_x \frac{\partial^2 X}{\partial y \partial y'} - T_y \frac{\partial^2 X}{\partial x \partial y'} \right] ds, \quad \left. \begin{aligned} & \\ & \end{aligned} \right\} (2.21)$$

$$v = -\frac{1}{y'} \frac{\partial y}{\partial x'} = -\frac{1}{y'} \int_C \left[T_x \frac{\partial^2 X}{\partial y \partial x'} - T_y \frac{\partial^2 X}{\partial x \partial x'} \right] ds,$$

$$\begin{aligned}
 V_1^{(1)} &= -\frac{1}{y y'} \frac{\partial^2 X}{\partial y \partial y'} \\
 V_y^{(1)} &= \frac{1}{y y'} \frac{\partial^2 X}{\partial x \partial y'} \\
 V_1^{(4)} &= \frac{1}{y y'} \frac{\partial^2 X}{\partial y \partial x'} \\
 V_y^{(4)} &= -\frac{1}{y y'} \frac{\partial^2 X}{\partial x \partial x'}
 \end{aligned}
 \quad \left. \vphantom{\begin{aligned} V_1^{(1)} \\ V_y^{(1)} \\ V_1^{(4)} \\ V_y^{(4)} \end{aligned}} \right\} \quad (2.22)$$

$$\begin{aligned}
 u &= - \int [\tau_x V_1^{(1)} + \tau_y V_y^{(1)}] y ds \\
 v &= - \int_c [\tau_x V_1^{(4)} + \tau_y V_y^{(4)}] y ds
 \end{aligned}
 \quad \left. \vphantom{\begin{aligned} u \\ v \end{aligned}} \right\} \quad (2.23)$$

3. 核関数

$$X = X_1 + X_2, \quad (3.1)^*$$

$$X_1 = \frac{yy'}{4} \int_0^\infty e^{-t|x-x'|} J_0(ty) J_1(ty') \frac{dt}{t}, \quad (3.2)$$

$$X_2 = -\frac{yy'^2}{4} \int_0^\infty e^{-t|x-x'|} J_1(ty) J_2(ty') \frac{dt}{t}, \quad (3.3)$$

とある(これは \$x\$ の変換より)

$$L_P \equiv \frac{\partial^2}{\partial x^2} + y \frac{\partial}{\partial y} \left(\frac{1}{y} \frac{\partial}{\partial y} \right), \quad L_Q \equiv \frac{\partial^2}{\partial x'^2} + y' \frac{\partial}{\partial y'} \left(\frac{1}{y'} \frac{\partial}{\partial y'} \right)$$

より

$$\begin{aligned} L_P[X_1] &= -\frac{yy'}{2} \int_0^\infty e^{-t|x-x'|} J_0(ty) J_1(ty') \frac{dt}{t} = G(p, Q), \\ &= \frac{-yy'}{4\pi} \int_0^{2\pi} \frac{\cos \theta}{R} d\theta, \quad R^2 = x^2 + y^2 + y'^2 = 2yy' \cos \theta \end{aligned} \quad (3.4)$$

$$L_Q[X_1] = 0,$$

$$L_P[X_2] = 0,$$

$$L_Q[X_2] = -\frac{yy'}{2} \int_0^\infty e^{-t|x-x'|} J_1(ty) J_1(ty') \frac{dt}{t} = G(p, Q) \quad (3.5)$$

(\$x\$ については \$x' = 0, x > 0\$ と置ける)

$$G(p, Q) \xrightarrow{p \rightarrow Q} -\frac{y}{2\pi} \log(\sqrt{x^2 + (yy')^2}), \quad (3.6)$$

$$y \frac{\partial}{\partial y} \left[\frac{1}{y} \frac{\partial}{\partial y} (y J_0(xy)) \right] = -yt \frac{\partial}{\partial y} \left[\frac{J_1(xy)}{y} \right] = \frac{t}{y} J_1 - t^2 J_1'(xy) \\ = \frac{2t}{y} J_1 - t^2 J_0$$

$$L_y [J_0(xy)] = \frac{2t}{y} J_1 //$$

$$y \frac{\partial}{\partial y} \left[\frac{1}{y} \frac{\partial}{\partial y} (y^2 J_0(xy)) \right] = z \frac{d}{dz} \left[\frac{1}{z} \frac{d}{dz} (z^2 J_0) \right] = z \frac{d}{dz} [2J_0 - zJ_1] \\ = -2zJ_1 - z^2 J_0$$

$$\therefore L[y^2 J_0(xy)] = -2yt J_1(yt) //$$

$$y \frac{\partial}{\partial y} \left[\frac{1}{y} \frac{\partial}{\partial y} (y J_1(xy)) \right] = t z \frac{d}{dz} \left[\frac{1}{z} \frac{d}{dz} (z J_1) \right] = t z \frac{d}{dz} J_0 = -z J_1(yt)$$

$$\therefore L[y J_1(xy)] = 0 //$$

$$y \frac{\partial}{\partial y} \left[\frac{1}{y} \frac{\partial}{\partial y} (y^2 J_2(xy)) \right] = z \frac{d}{dz} \left[\frac{1}{z} \frac{d}{dz} (z^2 J_2) \right] = z \frac{d}{dz} (z J_1) \\ = z^2 J_0 = z^2 \left(\frac{2}{z} J_1 - J_2 \right)$$

$$\therefore L[y^2 J_2(xy)] = 2yt J_1(yt) //$$

$$\frac{\partial}{\partial y} J_0(xy) = -t J_1(xy), \quad \frac{\partial}{\partial y} [y J_1(xy)] = yt J_0(xy)$$

$$\frac{\partial}{\partial y} [y^2 J_2(xy)] = y^2 t J_1(xy), \quad \frac{\partial}{\partial y} [y^2 J_0(xy)] = y [2J_0 - yt J_1]$$

$$J_2(xy) = \frac{2}{yt} J_1(yt) - J_0(yt)$$

求微分

$$\frac{1}{y} \frac{\partial X_1}{\partial y} = \frac{y y'}{2} \int_0^{\infty} e^{-tx} \frac{J_0(xy) J_1(xy')}{t} dt - \frac{y y'}{4} \int_0^{\infty} e^{-tx} J_1(xy) J_1(xy') dt$$

$$= \frac{3}{y^2} X_1 + \frac{1}{2} G$$

$$\frac{1}{y} \frac{\partial X_1}{\partial x} = - \frac{y y'}{4} \int_0^{\infty} e^{-tx} \frac{J_0(xy) J_1(xy')}{t} dt = - \frac{y}{2} T$$

$$\frac{\partial X_1}{\partial x} = - \frac{\partial X_1}{\partial x},$$

$$\frac{1}{y'} \frac{\partial X_1}{\partial y'} = \frac{y^2}{4} \int_0^{\infty} e^{-tx} \frac{J_0(xy) J_0(xy')}{t} dt = \frac{y^2}{2} S$$

$$\frac{1}{y'} \frac{\partial^2 X_1}{\partial y \partial y'} = S + \frac{y}{2} \frac{\partial S}{\partial y} = 1$$

$$\frac{1}{y'} \frac{\partial^2 X_1}{\partial x \partial y'} = - \frac{y}{2 y'} \frac{\partial T}{\partial y'} = - \frac{y}{2} \frac{\partial S}{\partial x} = \frac{y}{2} \frac{\partial S}{\partial x}$$

$$\begin{aligned} \frac{1}{y'} \frac{\partial^2 X_1}{\partial x \partial y'} &= - \frac{1}{y'} \frac{\partial}{\partial y'} \left[\frac{y^2}{2} T \right] = - \frac{y}{y'} T - \frac{y}{2 y'} \frac{\partial T}{\partial y'} \\ &= - \frac{y}{y'} T + \frac{y}{2} \frac{\partial P}{\partial x} \end{aligned} \quad (38)$$

$$\frac{1}{y'} \frac{\partial^2 X_1}{\partial x \partial y'} = - \frac{y}{2 y'} \frac{\partial T}{\partial x} = \frac{y}{2} \frac{\partial S}{\partial y} =$$

$$\begin{aligned}
 \frac{1}{y} \frac{\partial X_2}{\partial y} &= -\frac{y^{1/2}}{4} \int_0^\infty e^{-tx} J_0(ty) J_2(ty') dt \\
 &= -\frac{y'}{2} \int_0^\infty e^{-tx} J_0(ty) J_1(ty') \frac{dt}{t} + \frac{y^{1/2}}{4} \int_0^\infty e^{-tx} J_0(ty) J_0(ty') dt \\
 &= \frac{y^{1/2}}{2} S + \frac{2}{P_A};
 \end{aligned}$$

$$\begin{aligned}
 \frac{1}{y} \frac{\partial X_2}{\partial x} &= -\frac{1}{y} \frac{\partial X_2}{\partial x'} = \frac{y^{1/2}}{4} \int_0^\infty e^{-tx} J_1(tx) J_2(ty') dt \\
 &= \frac{y'}{2} \int_0^\infty e^{-tx} J_1(ty) J_1(ty') \frac{dt}{t} - \frac{y^{1/2}}{4} \int_0^\infty e^{-tx} J_1(ty) J_0(ty') dt \\
 &= \frac{1}{y} G_A - \frac{y^{1/2}}{2y} T,
 \end{aligned}$$

$$\frac{1}{y'} \frac{\partial X_2}{\partial y'} = -\frac{yy'}{4} \int_0^\infty e^{-tx} J_1(tx) J_1(ty) dt = \frac{1}{2} G,$$

$$\frac{1}{yy'} \frac{\partial^2 X_2}{\partial y' \partial y} = \frac{1}{2y} \frac{\partial G}{\partial y} = \frac{1}{2} y' \frac{\partial S}{\partial y},$$

$$\frac{1}{yy'} \frac{\partial^2 X_2}{\partial y' \partial x} = \frac{1}{2y} \frac{\partial G}{\partial x} = -\frac{1}{2} \frac{\partial T}{\partial y} = -\frac{y'}{2} \frac{\partial P}{\partial y}$$

$$\frac{1}{yy'} \frac{\partial^2 X_2}{\partial x' \partial y} = \frac{y'}{2} \frac{\partial S}{\partial x'} + \frac{1}{y'} \frac{2}{T}$$

$$\frac{1}{yy'} \frac{\partial^2 X_2}{\partial x' \partial x'} = \frac{1}{yy'} G - \frac{y'}{2y} \frac{\partial T}{\partial x'} = +P + \frac{y'}{2} \frac{\partial S}{\partial y},$$

(3.10)

解高

$$\frac{1}{yy'} \frac{\partial^2 X}{\partial y \partial y'} = S + \frac{1}{2} \left(y \frac{\partial S}{\partial y} + y' \frac{\partial S}{\partial y'} \right),$$

$$\frac{1}{yy'} \frac{\partial^2 X}{\partial y' \partial x} = \frac{1}{2} \left[y \frac{\partial S}{\partial x} - y' \frac{\partial P}{\partial x} \right],$$

$$\frac{1}{yy'} \frac{\partial^2 X}{\partial x' \partial y} = \frac{1}{2} \left[y \frac{\partial P}{\partial x} - y' \frac{\partial S}{\partial x} \right],$$

$$\frac{1}{yy'} \frac{\partial^2 X}{\partial x' \partial x} = P + \frac{1}{2} \left[y' \frac{\partial S}{\partial y} + y \frac{\partial S}{\partial y'} \right],$$

(3.11)

とすべし, (2.22) を通し 附録(A.9)に一致す。

附録 A 種々の核

$$S^*(P, Q) = \frac{1}{4\pi} \int_0^{2\pi} R d\theta = \frac{R_2 E(R)}{\pi} \quad (A.1)$$

$$S = \frac{1}{4\pi} \int_0^{2\pi} \frac{d\theta}{R} = \frac{1}{\pi R_2} = \frac{1}{2} \int_0^\infty e^{-t|x|} J_0(ty) J_0(ty') dt, \quad (A.2)$$

$$P = \frac{1}{4\pi} \int_0^{2\pi} \frac{\cos \theta}{R} d\theta = \frac{1}{2} \int_0^\infty e^{-t|x|} J_1(ty) J_1(ty') dt, \quad (A.3)$$

$$T = \frac{y}{2} \int_0^\infty e^{-t|x|} J_1(ty) J_0(ty') dt,$$

$$\tilde{T} = \frac{y'}{2} \int_0^\infty e^{-t|x|} J_0(ty) J_1(ty') dt,$$

$$\frac{\partial \tilde{T}_A}{\partial x} = \tilde{T}, \quad \tilde{T}_A = -\frac{y'}{2} \int_0^\infty e^{-tx} J_0(ty) J_1(ty') \frac{dt}{t},$$

$$\left. \begin{array}{l} (A.4) \\ T(x; y, y') \\ = \tilde{T}(x; y', y) \end{array} \right\}$$

$$\left[\begin{array}{l} \text{対称な核では} \quad G = -yy'P, \\ S = \tilde{T}, \quad S_A = \tilde{T}_A, \text{ 成り立っている} \end{array} \right]$$

E, K は 楕円積分で T, $\tilde{T}$ は 2 種 楕円積分となる。

$$(R')^2 = R_1^2/R_2^2, \quad R^2 = 1 - R'^2 = \frac{4yy'}{R_2^2},$$

$$Y_1^2 = x^2 + (y-y')^2, \quad Y_2^2 = x^2 + (y+y')^2,$$

対称

$$\frac{\partial S^*}{\partial x} = -\frac{\partial S^*}{\partial x'} = x S,$$

$$\frac{\partial S^*}{\partial y} = y S - y' P,$$

$$\frac{\partial S^*}{\partial y'} = y' S - y P,$$

$$(A.5)$$

$$\left. \begin{aligned}
 \frac{\partial T}{\partial x} &= \frac{1}{y'} \frac{\partial G}{\partial y'} = -\frac{y}{y'} \frac{\partial (yP)}{\partial y'} = y \frac{\partial S}{\partial y}, \\
 \frac{\partial T}{\partial y'} &= \frac{1}{y'} \frac{\partial G}{\partial x'} = y \frac{\partial P}{\partial x'}, \quad \frac{\partial T}{\partial y} = -y \frac{\partial S}{\partial x}, \\
 \frac{\partial \tilde{T}}{\partial x} &= \frac{1}{y} \frac{\partial G}{\partial y} = -\frac{y'}{y} \frac{\partial (yP)}{\partial y} = y' \frac{\partial S}{\partial y}, \\
 \frac{\partial \tilde{T}}{\partial y'} &= -y' \frac{\partial S}{\partial x}, \quad \frac{\partial \tilde{T}}{\partial y} = -\frac{1}{y} \frac{\partial G}{\partial x} = y' \frac{\partial P}{\partial x},
 \end{aligned} \right\} \quad (A.6)$$

$$\begin{aligned}
 \frac{\partial}{\partial x} S^* &= x S = -\frac{1}{2} \int_0^\infty J_0(y) J_0(y') \frac{\partial \rho}{\partial t} dt \\
 &= \frac{1}{2} - (T + \tilde{T}), \quad (A.7)
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial^2}{\partial x \partial x} S^* &= -\left(\frac{\partial T}{\partial x} + \frac{\partial \tilde{T}}{\partial x} \right) = -y \frac{\partial S}{\partial y} - y' \frac{\partial S}{\partial y'}, \\
 \frac{\partial^2}{\partial x \partial y} S^* &= -\frac{\partial}{\partial y} (T + \tilde{T}) = y \frac{\partial S}{\partial x} - y' \frac{\partial P}{\partial x}, \\
 \frac{\partial^2}{\partial x \partial y'} S^* &= y' \frac{\partial S}{\partial x} - y \frac{\partial P}{\partial x}, \\
 \frac{\partial^2}{\partial y \partial y'} S^* &= y \frac{\partial S}{\partial y'} - \frac{\partial (yP)}{\partial y'} = -P + y \frac{\partial S}{\partial y'} - y' \frac{\partial P}{\partial y'}, \\
 V_1^{(1)} &= \frac{1}{2} S_{xx}^* - S = -\left[S + \frac{1}{2} (y S_y + y' S_{y'}) \right], \\
 V_2^{(1)} &= \frac{1}{2} S_{xy}^* = \frac{1}{2} (y S_x - y' P_x), \\
 V_2^{(2)} &= -\frac{1}{2} S_{xy'}^* = \frac{1}{2} (y P_x - y' S_x), \\
 V_2^{(2)} &= -P - \frac{1}{2} S_{xy'}^* = -\frac{1}{2} P - \frac{1}{2} (y S_{y'} - y' P_{y'}) \\
 &= -P - \frac{1}{2} (y S_{y'} + y' P_{y'})
 \end{aligned} \quad (A.8)$$

(A.9)

実際、計算には (A.10), (A.3) の手前分より、 S^* の微分を計算する方が楽である。(以下 (A.11) を同じ)
~~Append D~~

$$\left. \begin{aligned} \frac{dK}{dr} &= \frac{E}{r^2} - K, \\ \frac{dE}{dr} &= E - K, \end{aligned} \right\} \text{--- (A.10)}$$

から

$$\left. \begin{aligned} \frac{\partial}{\partial x} S^* &= \chi S, \\ \frac{\partial}{\partial y} S^* &= \frac{S^*}{2y} - \left\{ \frac{r_2^2}{2y} - (y+y') \right\} S, \\ \frac{\partial S}{\partial x} &= - \frac{\chi}{(r_1 r_2)^2} S^*, \\ \frac{\partial S}{\partial y} &= - \frac{S'}{2y} + \frac{1}{r_1^2} \left(\frac{1}{2y} - \frac{y+y'}{r_2^2} \right) S^*, \\ \frac{\partial^2}{\partial x^2} S^* &= S - \frac{\chi^2}{(r_1 r_2)^2} S^*, \\ \frac{\partial^2}{\partial x \partial y} S^* &= - \frac{\chi}{2y} S' + \frac{\chi}{r_1^2} \left(\frac{1}{2y} - \frac{y+y'}{r_2^2} \right) S^*, \\ \frac{\partial^2}{\partial x \partial y'} S^* &= - \frac{\chi}{2y'} S + \frac{\chi}{r_1^2} \left(\frac{1}{2y'} - \frac{y+y'}{r_2^2} \right) S^*, \\ \frac{\partial^2}{\partial y \partial y'} S^* &= - \left(\frac{y^2 + y'^2}{2y y'} \right) S + \frac{\{ \chi^2 (y^2 + y'^2) + (y^2 - y'^2)^2 \}}{2y y' (r_1 r_2)^2} S^* \end{aligned} \right\} \text{(A.11)}$$

$$X_1 = \frac{(yy')^2}{8} \int_0^\infty e^{-tx} J_0(xy) \{J_0(xy') + J_2(xy')\} dt$$

$$X_2 = -\frac{(yy')^2}{8} \int_0^\infty e^{-tx} \{J_0(xy) + J_2(xy)\} J_2(xy') dt$$

$$\therefore X = \frac{(yy')^2}{8} \int_0^\infty e^{-tx} [J_0(xy)J_0(xy') - J_2(xy)J_2(xy')] dt$$

$$S = \frac{1}{4\pi} \int_0^{2\pi} \frac{d\theta}{R} = \frac{K(\theta)}{\pi r_2} = \frac{1}{2} \int_0^\infty e^{-tx} J_0(xy) J_0(xy') dt$$

$$\frac{1}{R} = \int_0^\infty e^{-tx} dt \left[J_0(xy)J_0(xy') + 2 \sum_{n=1}^\infty J_n(xy)J_n(xy') \cos n\theta \right]$$

$$\therefore I = \frac{1}{2\pi} \int_0^{2\pi} \frac{\cos^2 \theta}{R} d\theta = \int_0^\infty e^{-tx} J_2(xy) J_2(xy') dt$$

$$\therefore X = \frac{(yy')^2}{4} S - \frac{(yy')^2}{8} I$$

$$\left[\begin{aligned} r_1^2 &= x^2 + (y-y')^2, & r_2^2 &= x^2 + (y+y')^2, \\ k'^2 &= \left(\frac{r_1}{r_2}\right)^2, & k^2 &= 1 - k'^2. \end{aligned} \right] \text{ c.f. c.}$$

$$R = r_2 \sqrt{1 - k'^2 \sin^2 \varphi}, \quad \text{if } L, \quad \theta = \pi - 2\varphi, \quad \cos \theta = -\cos 2\varphi$$

$$\therefore \cos 2\theta = 1 - 8 \sin^2 \varphi \cos^2 \varphi$$

$$I = \frac{2}{\pi r_2} \int_0^{\frac{\pi}{2}} \frac{1 - 8 \sin^2 \varphi \cos^2 \varphi}{\sqrt{1 - k'^2 \sin^2 \varphi}} d\varphi$$

$$\text{By } \int_0^{\frac{\pi}{2}} \frac{\sin^2 \varphi \cos^2 \varphi}{\sqrt{1 - k'^2 \sin^2 \varphi}} d\varphi = \frac{1}{2k'^2} \int_0^{\frac{\pi}{2}} \frac{1 - k'^2 \sin^2 \varphi}{\sqrt{1 - k'^2 \sin^2 \varphi}} \cos 2\varphi d\varphi$$

$$= \frac{1}{4k'^2} \int_0^{2\pi} \frac{(1 - \frac{2-k'^2}{k'^2}) + \frac{2-k'^2}{k'^2} (1 - k'^2 \sin^2 \varphi)}{\sqrt{1 - k'^2 \sin^2 \varphi}} d\varphi$$

$$= -\frac{k'^2}{2k'^4} K(k) + \frac{1+k'^2}{2k'^2} E(k),$$

$$\therefore I = \frac{2(1+k'^2)}{\pi b k^4} [(1+k'^2)K - 2E],$$

$$\begin{aligned} \therefore X &= \frac{(yy')^2}{4} \frac{K}{\pi v_2} - \frac{(yy')^2(1+k'^2)}{4\pi b k^4} [(1+k'^2)K - 2E] \\ &= \frac{(yy')^2}{4\pi b k^4} [-4k'^2 K + 2(1+k'^2)E] \\ &= \frac{(yy')^2(1+k'^2)}{2\pi b k^4} \left[E - \frac{2k'^2}{1+k'^2} K \right] \quad \text{, } // \end{aligned}$$

$$X = \frac{(yy')^2}{16\pi} \int_0^{2\pi} \frac{1 - \cos 2\theta}{R} d\theta = \frac{(yy')^2}{8\pi} \int_0^{2\pi} \frac{\sin^2 \theta}{R} d\theta,$$