WO

No. 0-1
Date 56.3.4

若い友人への便り（波の全反射について）

拝啓　お元気ですか。　私の方は一週間来風邪で
フラ・フラしながらも年度末の事とて学校へ出て学生のお相手
をしています。　微熱がとれず頭がボーとしているので
まとまった事は考えられず　いろんな事共がそれぞれに
頭の中をよぎっては消えてゆきます。

先日いろいろの話の中で短い波が船体で反射する
話がありましたね。　三菱の藤井さんの所の話を
聞いた時から私も気になっており，何年か前　川重の
山上さんもその改良を志しておられ　質問されたの
ですが　考えがまとまらない儘にお答え出来ません
でした。

最近　水の波の全反射または全透過浮体の
可能性と言う真面目だかふまじめだかわからない
テーマを追っていて気がついたのですが，そして
それが正に上記の事に関する気になっていた事の
答でもあるのですが，今まで水の全反射に
ついての法則がはっきりと書かれたものを見た

事がないと言う事です。

これは全く奇妙な事で光でも音でも他の波動現象ではテキストの最初に出て来る法則がないんです。

これ位の事なら少しボーとしていもコタツの中で出来ます。

どの道短波長の事ですから水深も吃水も無限大として、一様流速のない場合、ケルビン波のように一様速度で波系が前進する場合、一般の場合の順に以下考えました。

第1番目のは記す必要もないものです。

第2番目のはあまり意味がないように見えますが、コッチン関数に関するハスキンド・花岡の定理によって、これから短波長の時の造波抵抗が算出出来ます。

第3番目は所期の目的であった短波長の向波による抵抗増加が計算出来る事になります。

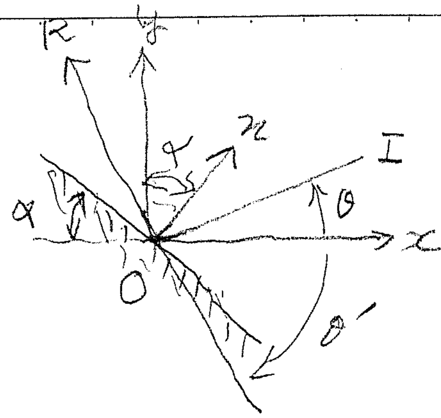
お気に召したら一つ試して見て下さい。

それではまた次にお会い出来る日を楽しみにしています。

乱筆多謝

匆々
別所拝

1. 前進速度のない場合



$$\Phi_I = e^{iK(x\cos\theta + y\sin\theta)}$$

$$\phi_R = e^{iK(x\cos\theta' - y\sin\theta')}$$

$$\frac{\partial \phi_I}{\partial n} + \frac{\partial \phi_R}{\partial n} = 0.$$

$$\frac{\partial}{\partial n}(x\cos\theta + y\sin\theta) = \frac{\partial x}{\partial n}\cos\theta + \frac{\partial y}{\partial n}\sin\theta = \sin(\alpha+\theta).$$

$$\therefore \quad \sin(\alpha+\theta) + \sin(\alpha-\theta') = 0.$$

$$\therefore \quad \sin(\alpha+\theta) = \sin(\theta'-\alpha)$$

$$\theta' = 2\alpha + \theta \ /\!/$$

これは幾何光学的反射法則である。

2. 定常波（ケルビン波）

$$\phi_I = e^{iK(x\cos\theta + y\sin\theta)\sec^2\theta}$$

$$\phi_R = a\, e^{iK(x\cos\theta' - y\sin\theta')\sec^2\theta'}$$

$$\zeta = \frac{U}{g}\frac{\partial \phi}{\partial x} \quad ; \text{surface elevation}$$

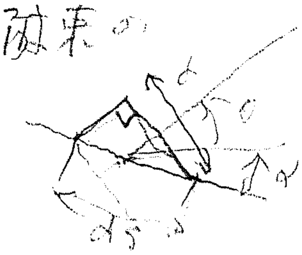
$$\frac{\partial \phi_I}{\partial n} + \frac{\partial \phi_R}{\partial n} = 0$$

故 $\sec^2\theta \sin(\theta+\alpha) = a \sec^2\theta' \sin(\theta'-\alpha)$ . . (1)

一方 エネルギーは入射されただけ反射するから

$$\frac{\rho g}{4}|\zeta_I|^2\, U\cos\theta \cdot \sin(\theta+\alpha) = \frac{\rho g}{4}|\zeta_R|^2\, U\cos\theta' \cdot \sin(\theta'-\alpha).$$

$\sin(\theta+\alpha)$ は図のように入射波束の断面積である。



$\frac{U}{2}\cos\theta$ は群速度.

∴ $\sec^3\theta \sin(\theta+\alpha) = a^2 \sec^3\theta' \sin(\theta'-\alpha)$, (2)

(1)と(2)から $a = \cos\theta \sec\theta'$, . . . (3)

$$\sec^3\theta \sin(\theta+\alpha) = \sec^3\theta' \sin(\theta'-\alpha), \quad \text{. . (4)}$$

(4)式より与えられた θ, α に対し $0<\theta'<\frac{\pi}{2}$ が求められる。

特に $\alpha=0$ ならば $\theta'=\theta$ ///

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3. 動揺しつつ前進する場合

$$\phi_I = \frac{g}{i\omega_0} e^{Kz + iK(x\cos\theta + y\sin\theta)},$$

$$\zeta_I = \left(i\omega - U\frac{\partial}{\partial x}\right)\phi_I = e^{iK(x\cos\theta + y\sin\theta)}.$$

ω ; 出会円周波数, ω_0 ; 波の 〃 $\}$ $\omega = \omega_0 + KU\cos\theta$,

$$\phi_R = \frac{a g}{i\omega_0(\theta_1)} e^{\kappa z + i\kappa(x\cos\theta_1 + y\sin\theta_1)} + \frac{b g}{i\omega_0(\theta_2)} e^{\kappa' z + i\kappa'(x\cos\theta_2 - y\sin\theta_2)},$$

$$\left.\begin{matrix}\kappa(\theta) \\ \kappa'(\theta)\end{matrix}\right\} = \frac{\omega}{U}\sec\theta + \frac{\gamma}{2}\sec^2\theta \pm \sqrt{\frac{\gamma^2}{4}\sec^4\theta + \frac{\omega\gamma}{U}\sec^3\theta}.$$

$$\left.\begin{matrix}\omega_0(\theta) \\ \omega_0'\end{matrix}\right\} = \omega - \left|\begin{matrix}\kappa(\theta) \\ \kappa'(\theta)\end{matrix}\right| U\cos\theta, \quad C(\theta) = \frac{\omega_0}{\kappa} \text{ ; 波速}, \quad C'(\theta) = \frac{\omega_0'}{\kappa'}$$

エネルギー保存より

$$\frac{\omega_0}{K}\sin(\alpha+\theta) = a^2 \frac{\omega_0(\theta_1)}{\kappa(\theta_1)}\sin(\theta_1-\alpha) + b^2 \frac{\omega_0'(\theta_2)}{\kappa'(\theta_2)}\sin(\theta_2-\alpha), \qquad (1)$$

法線速度 0 の条件から

$$\frac{K}{\omega_0}\sin(\theta+\alpha) = \frac{a\kappa(\theta_1)}{\omega_0(\theta_1)}\sin(\theta_1-\alpha) + b\frac{\kappa'(\theta_2)}{\omega_0'(\theta_2)}\sin(\theta_2-\alpha), \quad (2)$$

これでは不定である。

斯くで考えて見ると x'は進波成分で波長が入射波より
ずっと長く完全反射の時は現われそうにない。

それ故(1),(2)は.

$$\frac{\omega_0}{K}\sin(\theta+\alpha) = a^2\frac{\omega_0(\theta')}{\kappa(\theta')}\sin(\theta'-\alpha) \quad , \quad (3)$$

$$\frac{K}{\omega_0}\sin(\theta+\alpha) = a\frac{\kappa(\theta')}{\omega_0(\theta')}\sin(\theta'-\alpha), \quad (4)$$

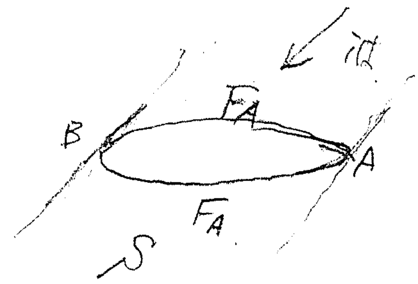
と考えられ

$$a\left\{\frac{\omega_0(\theta')}{\kappa(\theta')}\right\}^2 = \left(\frac{\omega_0}{K}\right)^2, \quad (5)$$

$$\left(\frac{K}{\omega_0}\right)^3\sin(\theta+\alpha) = \left\{\frac{\kappa(\theta')}{\omega_0(\theta')}\right\}^3\sin(\theta'-\alpha), \quad (6)$$

4. 幾何光学的近似. *

前節までの反射法則は波長が物体に比して充分小さい場合に成立つと考えられ、またその時は図のように影の領域 S が出来ると考えられる。



さて散乱ポテンシャル ϕ は

$$\phi_d(P) = \int_{F_A+F_B}\left[\phi_d(Q)\frac{\partial}{\partial n}S(P,Q) - \frac{\partial\phi_d}{\partial n}S(P,Q)\right]dF, \quad (1)$$

となり、境界上で

$$\frac{\partial\phi_d}{\partial n} = -\frac{\partial\phi_I}{\partial n}, \quad (2)$$

またPが物体の外とすると

$$\int_{F_A+F_B}\left[\phi_I(Q)\frac{\partial}{\partial n}S(P,Q) - \frac{\partial\phi_I}{\partial n}S\right]dF = 0, \quad (3)$$

であるから (1) と (3) を加え合せて

$$\phi_d(P) = \int_{F_A+F_B}[\phi_I(Q) + \phi_d(Q)]\frac{\partial S}{\partial n}\,dF, \quad (4)$$

となるが影の部分では波はないのであるから

$$\phi_I + \phi_d = 0 \quad \text{on } F_B, \quad (5)$$

それ故

$$\phi_d(P) = \int_{F_A}\Phi\frac{\partial S}{\partial n}\,dF, \quad (6)$$

$\Phi = \phi_I + \phi_R$, (ϕ_R は前節までの反射波), (7)

となり，F_A 上の Φ は前節迄のように求められるから，
散乱ポテンシャルは求まった事になる。

同様にしてコッチン関数を

$$H_d(K,\theta) = \int_{F_A} \Phi \frac{\partial}{\partial n} e^{Kz + iK(x\cos\theta + y\sin\theta)} dF, \quad \cdots \quad (8)$$

勿論問題によって exp. 項の形は少し変わる。

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あるいは又書きかえて、

$$f_{ij}-\overline{f_{ji}}=\iint_S(\varphi_i+\overline{\tilde{\varphi}_i})\frac{\partial}{\partial n}\tilde{\varphi}_j\,dS,$$

$$h_{ij}+\overline{h_{ji}}=\iint_S(\varphi_i-\overline{\tilde{\varphi}_i})\frac{\partial}{\partial n}\tilde{\varphi}_j\,dS.$$

(6).

となるから§3 (16)の式を代入して. (19)を考えると.

$$f_{ij}-\overline{f_{ji}}=\frac{i}{2\pi}\int_{-\pi}^{\pi}\overline{H_j(\kappa,u)}H_i(\kappa,u)\frac{\kappa^2\,du}{\sqrt{\gamma^2+\frac{4\omega\gamma\omega}{U}\cos u}}$$

$$+\frac{1}{2\pi i}\int_{-\pi}^{\pi}\overline{H_j(\kappa',u)}H_i(\kappa',u)\frac{\kappa'^2\,\mathrm{sgn}(\cos u)\,du}{\sqrt{\gamma^2+\frac{4\gamma\omega}{U}\cos u}},\quad(7)$$

又§3 (16)の両辺に$(i\omega-U\frac{\partial}{\partial x})$の演算を施こすと

$$h_{ij}+\overline{h_{ji}}=-\frac{1}{2\pi}\int_{-\pi}^{\pi}\overline{H_j(\kappa,u)}H_i(\kappa,u)\frac{\omega_0\kappa^2\,du}{\sqrt{\gamma^2+\frac{4\gamma\omega}{U}\cos u}}$$

$$+\frac{1}{2\pi}\int_{-\pi}^{\pi}\overline{H_j(\kappa',u)}H_i(\kappa',u)\frac{\omega_0'\kappa'^2\,du}{\sqrt{\gamma^2+\frac{4\gamma\omega}{U}\cos u}},\quad(8)$$

よって(3)により、$\omega_0=\omega-\kappa U\cos u$なる事から

$$r_{ij}+\overline{r_{ji}}=-\frac{1}{2\pi}\int_{-\pi}^{\pi}\overline{H_j(\kappa,u)}H_i(\kappa,u)\frac{\kappa^3\cos u\,du}{\sqrt{\gamma^2+\frac{4\gamma\omega}{U}\cos u}}$$

$$+\frac{1}{2\pi}\int_{-\pi}^{\pi}\overline{H_j(\kappa',u)}H_i(\kappa',u)\frac{\kappa'^3|\cos u|\,du}{\sqrt{\gamma^2+\frac{4\gamma\omega}{U}\cos u}},\quad(9)$$

を得る。

$f_{ij}-\overline{f_{ji}}\ \ \frac{i}{2\pi}\ \ \overline{H_i}H_j$

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(6) において $i=d$ とおくと、$\overline{\tilde{\varphi}_E} = -\varphi_E$ であるから

$$f_{dj} - \overline{f_{jd}} = \iint_S \left(\varphi_d + \overline{\tilde{\varphi}_d}\right)\frac{\partial}{\partial n}\tilde{\varphi}_j \, dS = \iint_S \left(\Phi_d + \overline{\tilde{\Phi}_d}\right)\frac{\partial}{\partial n}\tilde{\varphi}_j \, dS$$

$$= -\tilde{H}_j(k,\alpha) + \overline{H_j}(k,\alpha)$$

$$h_{dj} + \overline{h_{jd}} = -i\omega\left[\tilde{H}_j(k,\alpha) - \overline{H_j}(k,\alpha)\right],$$

又

$$r_{dj} + \overline{r_{jd}} = ik\cos\alpha\left[\tilde{H}_j(k,\alpha) - \overline{H}_j(k,\alpha)\right], \qquad (10)$$

つまり

$$-\left[\tilde{H}_j(k,\alpha) - \overline{H_j}(k,\alpha)\right] = \frac{i}{2\pi}\int_{-\pi}^{\pi} \overline{H_j}(k,u)\, H_d(x,u;\,k,\alpha)\frac{x^2\,du}{\sqrt{r^2 + \frac{4\omega}{g}\cos u}}$$

$$+ \frac{1}{2\pi i}\int_{-\pi}^{\pi} \overline{H_j}(x',u)\, H_d(x',u;\,k,\alpha)\frac{x'^2\,\mathrm{sgn}(\cos u)\,du}{\sqrt{r^2 + \frac{4\omega\nu}{g}\cos u}} \qquad (11)$$

h, r についても同様な式が与られる。

又

$$f_{ji} - \overline{f_{ij}} = -\left[\overline{f_{ij} - \overline{f_{ji}}}\right]$$

$$h_{ji} + \overline{h_{ij}} = \left[\overline{h_{ij} + \overline{h_{ji}}}\right] \qquad (12)$$

$$r_{ji} + \overline{r_{ij}} = \left[\overline{r_{ij} + \overline{r_{ji}}}\right]$$

であるから (11) において左辺で j と d を とりかえると 共役値の 符号がかわったものとなる。

又 h, r に対する式では j と d をかえると 共役値をとる。

又さらに (11) において $j=d$ とおくと

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$$\overline{H_d}(k,\alpha;\lambda,\beta)-\widetilde{H_d}(k,\alpha;\lambda,\beta)$$

$$=\frac{\nu}{2\pi}\int_{-\pi}^{\pi}\overline{H_d(x,u;\lambda,\beta)}\,H_d(x,u;k,\alpha)\frac{x^2\,du}{\sqrt{\gamma^2+\frac{4\gamma\omega}{U}\cos u}}$$

$$+\frac{1}{2\pi i}\int_{-\pi}^{\pi}\overline{H_d}(x',u;\lambda,\beta)H_d(x',u;k,\alpha)\frac{x'^2\,\mathrm{sgn}(\cos u)\,du}{\sqrt{\gamma^2+\frac{4\gamma\omega}{U}\cos u}},\qquad(13)$$

$\widetilde{H_d}$ と H_d については次の相互性が成立つ。

$$\widetilde{H_d}(k,\alpha;\lambda,\beta)=-\iint_S\Phi_d(k,\alpha)\frac{\partial}{\partial n}\widetilde{\Phi_d}(\lambda,\beta)\,dS$$

$$=-\iint_S\left[-\phi_I(k,\alpha)\frac{\partial}{\partial n}\widetilde{\Phi_I}(\lambda,\beta)+\phi_d(k,\alpha)\frac{\partial}{\partial n}\widetilde{\Phi_d}(\lambda,\beta)\right]dS.$$

$$=-\iint_S\left[\widetilde{\Phi_d}(\lambda,\beta)\frac{\partial}{\partial n}\phi_d(k,\alpha)\right]dS$$

$$+\iint_S\left[\phi_I(k,\alpha)\frac{\partial}{\partial n}\widetilde{\phi_I}(\lambda,\beta)-\widetilde{\phi_I}(\lambda,\beta)\frac{\partial}{\partial n}\phi_I(k,\alpha)\right]dS$$

$$=H_d(\lambda,\beta;k,\alpha),\qquad\cdots\cdots(14)$$

何故なら上の ϕ_I に関する積分において積分領域 S は水線面をつけ足して閉領域とする事が出来るが そうすると この積分は任意の表面の上にとる事が出来 極限において 0 となるからである。

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又次の等式もある。

$$\iint_S \left[\Phi_d(\lambda,\beta)\frac{\partial}{\partial n}\overline{\Phi_d(k,\alpha)} - \overline{\Phi_d(k,\alpha)}\frac{\partial}{\partial n}\Phi_d(\lambda,\beta) \right] dS = 0$$

であるから

$$0 = \iint_S \Big[+ \phi_I(\beta)\frac{\partial}{\partial n}\overline{\phi_I(\alpha)} - \overline{\phi_I(\alpha)}\frac{\partial}{\partial n}\phi_I(\beta)$$

$$+ \phi_d(\beta)\frac{\partial}{\partial n}\overline{\phi_I(\alpha)} - \overline{\phi_d(\alpha)}\frac{\partial}{\partial n}\phi_I(\beta)$$

$$+ \phi_d(\beta)\frac{\partial}{\partial n}\overline{\phi_d(\alpha)} - \overline{\phi_d(\alpha)}\frac{\partial}{\partial n}\phi_d(\beta)$$

$$+ \phi_I(\beta)\frac{\partial}{\partial n}\overline{\phi_d(\alpha)} - \overline{\phi_I(\alpha)}\frac{\partial}{\partial n}\phi_d(\beta) \Big] dS$$

$$= \iint_S \Big[\tilde{\overline{\phi_I}}(\alpha)\frac{\partial}{\partial n}\phi_d(\beta) - \phi_d(\beta)\frac{\partial}{\partial n}\tilde{\overline{\phi_I}}(\alpha)$$

$$+ \tilde{\overline{\phi_d}}(\alpha)\frac{\partial}{\partial n}\phi_I(\beta) - \phi_I(\beta)\frac{\partial}{\partial n}\tilde{\overline{\phi_d}}(\alpha)$$

$$- \left\{\tilde{\overline{\phi_d}}(\alpha) + \overline{\phi_d}(\alpha)\right\}\frac{\partial}{\partial n}\phi_d(\beta) \Big] dS$$

$$= \tilde{H}_d(\lambda,\beta;k,\alpha) - H_d(k,\alpha;\lambda,\beta)$$

$$- \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\kappa,u;\lambda,\beta)\overline{H_d}(\kappa,u;k,\alpha)\frac{\kappa^2 du}{\sqrt{}}$$

$$- \frac{1}{2\pi i}\int_{-\pi}^{\pi} H_d(\kappa;u;\lambda,\beta)\overline{H_d}(\kappa;u;k,\alpha)\frac{\kappa^2\,\mathrm{sgn}(\cos u)\,du}{\sqrt{}}$$

$$= 0$$

G. 仕事積分

i 方向の運動方程式はその慣性を m_i、外力を g_i とすると（$F_i^{(s)}$ は静水力学的力とする）

$$-\omega^2 m_i X_i = g_i - \rho\iint_S [p_I + p_d + p]\frac{\partial}{\partial n}x_i\, dS + F_i^{(s)},$$

$$-\omega^2 m_i X_i - F_i^{(s)} = g_i + \rho\tilde{H}_i(k,\alpha) + \rho\sum_{j=1}^{6} X_j f_{ji}, \quad \cdots (1) \quad i=1\sim 6$$

$$p = \sum_{j=1}^{6} X_j p_j$$

この共役値は

$$-\omega^2 m_i \overline{X_i} - \overline{F_i^{(s)}} = \overline{g_i} + \rho\overline{\tilde{H}_i}(k,\alpha) + \rho\sum_{j=1}^{6}\overline{X_j}\,\overline{f_{ji}}, \quad (1')$$

であるから夫々に $\overline{X_i}$, X_i をかけて辺々相引きについて1から6まで辺々加え合せると、m_i および $F_i^{(s)}$ の係数は実である故消えて

$$\frac{1}{\rho}\sum_{i=1}^{6}(g_i\overline{X_i} - \overline{g_i}X_i) = \sum_{i=1}^{6}\left[X_i\overline{\tilde{H}_i}(k,\alpha) - \overline{X_i}\tilde{H}_i(k,\alpha)\right]$$

$$+\sum_{i=1}^{6}\sum_{j=1}^{6}\left[X_i\overline{X_j}\,\overline{f_{ji}} - X_j\overline{X_i}f_{ji}\right],$$

$$= \overline{\tilde{H}}(k,\alpha) - \tilde{H}(k,\alpha) + \iint_S\left(\phi\frac{\partial\overline{\phi}}{\partial n} - \overline{\phi}\frac{\partial\phi}{\partial n}\right)dS, \quad (2)$$

この式の左辺は水が外力に対してなす仕事であり右辺第1項は波の強制力が船など外力に対してなす仕事であり右辺第2項は船が波を放散して失うエネルギーである。

今左辺を $+iD$ とおくと F. (4), (7) により

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$$\overline{\tilde{H}}(k,\alpha) - \tilde{H}(k,\alpha) = iD - \frac{i}{2\pi}\int_{-\pi}^{\pi} |H(x,u)|^2 \frac{x^2\,du}{\sqrt{\gamma^2 + \frac{4gU}{D}\cos u}}$$

$$+ \frac{i}{2\pi}\int_{-\pi}^{\pi} |H(x',u)|^2 \frac{x'^2\,\mathrm{sgn}(\cos u)\,du}{\sqrt{\gamma^2 + \frac{4gU}{D}\cos u}}, \quad (3)$$

となるが F (11) により

$$\left.\begin{aligned} -\tilde{H} &= \overline{H} + i\left[\overline{H}\,H_d\right], \\ -\overline{\tilde{H}} &= H - i\left[H\,\overline{H_d}\right], \end{aligned}\right\}$$

[ ]は積分を意味するものとする。

これを上式に代入すると

$$\overline{H} - H = iD - i\left[|H|^2 + \overline{H}H_d + \overline{H_d}H\right], \quad (4)$$

を得る。

これにさらに F (13) に F (14) を代入して上式に加えると

$$\left\{\overline{H_d(k_0,\alpha_0;k,\alpha)} + \overline{H(k,\alpha)}\right\} - \left\{H_d(k_0,\alpha_0;k,\alpha) + H(k,\alpha)\right\}$$

$$= iD - i\left[|H_d + H|^2\right], \quad \cdots\cdots (5).$$

を得る。*

これらの式の左辺は波がその強制力によって船になす仕事でそれが右辺の外力に対してなす仕事と発散波をだす仕事の和に等しい事を示す。

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一方船が水に対してなす仕事は

$$iW = \iint_S \left[(p_s+p_a+p)\frac{\partial\bar{\phi}}{\partial n} + \overline{(p_s+p_a+p)}\frac{\partial\phi}{\partial n}\right] dS$$

$$= i\omega_0\left[\tilde{H}(k,\alpha) - \bar{\tilde{H}}(k,\alpha)\right] + \iint_S (p-\bar{p})\frac{\partial\bar{\phi}}{\partial n} dS, \quad (6)$$

となって右辺第2項はF(8)によって求められる。
そうすると抵抗は

$$RU = W - \mathcal{D} \quad ,\quad \text{- - -} \quad (7)$$

の形で与えられるがこれで得られるものは花岡の言う非定常造波抵抗であって、波の漂流力は出て来ない。
この事は線型化に伴う矛盾であると考えられ、滑走板でも同様である。

Journal of
SHIP RESEARCH

The Coupled Damping Coefficients of a Symmetric Ship

By R. Timman[1] and J. N. Newman[2]

A study is made of a floating or submerged body with longitudinal and transverse symmetry, which is moving with constant forward speed and performing small oscillations. The analysis is quite general in the sense that the shape of the body and the nature of the oscillations are unspecified, but it is assumed that the linearized free-surface condition holds. With this assumption the oscillatory velocity potential is found in terms of an unknown Green's function, the existence of which is also assumed. This potential is then used to show the symmetry properties of the cross-coupling damping coefficients.

A CONTROVERSY has arisen in ship-motion theory regarding the cross-coupling damping coefficients of a pitching and heaving ship. If the ship is symmetrical fore and aft and if it has no forward speed, then from symmetry the cross-coupling moment due to heave and the force due to pitch must both be zero, at least in the linearized solution of the problem. However, if the ship is moving with forward speed, an asymmetry is introduced and cross-coupling results. Haskind [1][3] has employed thin-ship theory to show that for a symmetric ship with constant forward speed, the two cross-coupling damping coefficients for pitch and heave are equal in magnitude and opposite in sign. However, this conclusion has received criticism in several papers and a dispute has arisen, which is reviewed in the survey of Vossers [2].

An analysis of particular relevance to this discussion is that of Havelock [3], which considers the case of a floating spheroid with a rigid free-surface condition. Havelock assumes that the spheroid is pitching and heaving and that there is a constant forward speed or, equivalently, that there is a uniform flow of the stream. With this model it is found that the two cross-coupling coefficients are of unequal magnitude, and Havelock concludes that equality of the cross-coupling is a consequence of the thin-ship approximation. However, the sum of the two coefficients can be expressed [4] in terms of energy radiated in outgoing surface waves and, if a rigid free surface is assumed, there can be no waves and therefore no energy radiation. This reasoning leads to a contradiction with Havelock's result, for if the sum of the two cross-coupling coefficients is zero, then they must be equal and opposite.

The source of this discrepancy lies in the fact that in

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Nomenclature

- B_{ij} = damping coefficients
- c = forward velocity
- $F(\mathbf{x})$ = equation of the body surface
- $f_1(\mathbf{x})$ = $\cos(n, x)$ = horizontal direction cosine
- $f_2(\mathbf{x})$ = $\cos(n, y)$ = transverse direction cosine
- $f_3(\mathbf{x})$ = $\cos(n, z)$ = vertical direction cosine
- $f_4(\mathbf{x})$ = $y\cos(n, z) - z\cos(n, y)$
- $f_5(\mathbf{x})$ = $z\cos(n, x) - x\cos(n, z)$
- $f_6(\mathbf{x})$ = $x\cos(n, y) - y\cos(n, x)$
- $G(\mathbf{x}, \boldsymbol{\xi})$ = Green's function
- g = gravitational acceleration
- $\mathbf{i}, \mathbf{j}, \mathbf{k}$, = unit vectors
- $\mathbf{n}$ = unit normal into body surface
- p = fluid pressure
- t = time
- $\mathbf{v}(\mathbf{x})$ = velocity vector of steady flow field
- (x, y, z) = Cartesian co-ordinates fixed in space
- (x', y', z') = Cartesian co-ordinates fixed in body
- $\boldsymbol{\alpha}$ = displacement vector of a point on body
- ζ_i = oscillation amplitudes of body
- ξ, η, ζ = dummy co-ordinates corresponding to (x, y, z)
- $\phi(\mathbf{x})$ = velocity potential of oscillatory flow
- ω = circular frequency of oscillations

Havelock's analysis the boundary condition is satisfied by taking the oscillatory normal velocity on the spheroid and equating this to the normal velocity of the fluid *on the mean position of the spheroid in space.* In fact, the oscillatory disturbance is a small perturbation of the steady flow field, and the boundary condition on the spheroid must be satisfied on the exact oscillating surface of the body, or else expanded to the mean surface in a systematic manner so as to include the oscillatory flow induced on the body surface by its change of position in the steady-state field. That is, the oscillations of a ship in a moving stream give rise to a small disturbance of the steady flow field, and various second-order effects enter into the unsteady problem as a result of the lower-order steady field. It can be shown by an extension of Havelock's analysis that if the boundary condition is satisfied on the exact surface of the spheroid, then the cross-coupling damping coefficients between pitch and heave are in fact equal in magnitude and opposite in sign. It thus seems plausible that this equivalence holds for any symmetrical ship or body, irrespective of the thin-ship assumption.

In order to study this question more generally, the present paper treats the problem of an arbitrary floating or submerged body with longitudinal and transverse symmetry, which is moving with constant forward speed and oscillating sinusoidally in any of the six degrees of freedom. The only significant assumptions are that the problem is linear, in the sense that the oscillations are small and that the disturbance of the free surface due to the forward motion is also small. To be physically realistic, the latter assumption implies that the body is thin, slender, or deeply submerged, or a combination of these, but the analysis and conclusions are equally valid for the case of a nonslender body with a rigid free-surface condition.[4] ("Shallow" ships, with small draft and finite beam, are not included in the present work.) With these two basic assumptions we show that the sum of the two complementary cross-coupling damping coefficients is zero for all pairs of modes of oscillation except for the coupling of surge with pitch and roll with sway. Furthermore, fifteen of the thirty cross-coupling coefficients are shown to be zero. The same conclusions have been obtained for a thin ship by Hanaoka [6].

As usual we assume irrotational incompressible flow and formulate the problem in terms of the velocity potential. The potential problem is solved in terms of a Green's function which is not explicitly known, but the existence of this function seems physically plausible and can probably be proved by recourse to the theory of Fredholm integral equations. The reciprocity properties of this Green's function are then established and the symmetry properties of the damping coefficients follow directly. It should be emphasized that we do not explicitly solve either the steady or unsteady potential problems.

The proper representation of the oscillating potential, equation (9), is particularly interesting, since it demonstrates the effect of satisfying the boundary condition on the exact (oscillating) surface of the body. The final results for the damping coefficients are shown in a matrix, Table 1.

Both the analysis and final results are analogous to reciprocity studies in aerodynamics [7, 8]. In fact this analogy was the original motivation for suspecting that the equivalence of the pitch and heave cross-coupling coefficients did not depend on the thin-ship assumption.

The Boundary-Value Problem

Let (x, y, z) be a Cartesian co-ordinate system, moving through the fluid with constant velocity c, with z vertically upward and x in the direction of forward motion. In addition we shall employ an oscillatory co-ordinate system $\mathbf{x}' = \mathbf{x} - \boldsymbol{\alpha} e^{i\omega t}$ where $\boldsymbol{\alpha}$ is an infinitesimal vector, which may depend on $\mathbf{x}'$, and the real part is to be taken in expressions involving $e^{i\omega t}$. The $\mathbf{x}'$ co-ordinates are fixed with respect to a body which is defined by the equation $F(x', y', z') = 0$. The velocity of the fluid is represented by the vector

$$\mathbf{v}(\mathbf{x}) + e^{i\omega t}\nabla\phi(\mathbf{x})$$

Thus $\mathbf{v}$ is the steady velocity field due to the forward motion of the body, in the presence of the free surface, and $\phi(\mathbf{x})$ is the potential of the oscillating velocity vector. The function ϕ must satisfy Laplace's equation, the linearized free-surface condition [4]

$$-g\frac{\partial\phi}{\partial z} + \omega^2\phi + 2i\omega c\frac{\partial\phi}{\partial x} - c^2\frac{\partial^2\phi}{\partial x^2} = 0 \tag{1}$$

on the undisturbed free surface $z = 0$, and a suitable radiation condition at infinity.

The boundary condition on the body is

$$\begin{aligned}
0 = \frac{DF}{Dt} &= \frac{\partial F}{\partial t} + (\mathbf{v} + e^{i\omega t}\nabla\phi)\cdot\nabla F \\
&= \frac{\partial F}{\partial x'}\frac{\partial x'}{\partial t} + \frac{\partial F}{\partial y'}\frac{\partial y'}{\partial t} + \frac{\partial F}{\partial z'}\frac{\partial z'}{\partial t} \\
&\quad + (\mathbf{v} + e^{i\omega t}\nabla\phi)\cdot\left[\mathbf{i}\left(\frac{\partial F}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial F}{\partial y'}\frac{\partial y'}{\partial x} + \frac{\partial F}{\partial z'}\frac{\partial z'}{\partial x}\right)\right. \\
&\quad + \mathbf{j}\left(\frac{\partial F}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial F}{\partial y'}\frac{\partial y'}{\partial y} + \frac{\partial F}{\partial z'}\frac{\partial z'}{\partial y}\right) \\
&\quad \left. + \mathbf{k}\left(\frac{\partial F}{\partial x'}\frac{\partial x'}{\partial z} + \frac{\partial F}{\partial y'}\frac{\partial y'}{\partial z} + \frac{\partial F}{\partial z'}\frac{\partial z'}{\partial z}\right)\right] \\
&= -i\omega e^{i\omega t}\boldsymbol{\alpha}\cdot\nabla_{x'}F + (\mathbf{v} + e^{i\omega t}\nabla\phi)\cdot \\
&\quad \left[\nabla_{x'}F - \mathbf{i}e^{i\omega t}\left(\frac{\partial\boldsymbol{\alpha}}{\partial x}\cdot\nabla_{x'}F\right) - \mathbf{j}e^{i\omega t}\left(\frac{\partial\boldsymbol{\alpha}}{\partial y}\cdot\nabla_{x'}F\right)\right. \\
&\quad \left. - \mathbf{k}e^{i\omega t}\left(\frac{\partial\boldsymbol{\alpha}}{\partial z}\cdot\nabla_{x'}F\right)\right]
\end{aligned}$$

[4] The distinction between *thin* and *slender* bodies is important. In both cases the beam is small compared to the length but the thin ship has a small beam-draft ratio as well, whereas the slender ship has beam and draft of the same order of magnitude. By *deeply submerged* we imply that the depth of submergence is sufficiently large that the waves will be small, but not so large that the free surface may be neglected and the fluid considered as infinite.

This condition holds on the actual surface of the body where, from Taylor's theorem

$$\mathbf{v} = (\mathbf{v})_{\text{mean}} + [(\boldsymbol{\alpha}\cdot\nabla)\mathbf{v}]_{\text{mean}}e^{i\omega t} + 0(\alpha^2)$$

The subscript "mean" denotes that the function is to be evaluated on the mean position of the body, or with $\mathbf{x} = \mathbf{x}'$. Thus we obtain the boundary condition

$$0 = -i\omega e^{i\omega t}(\boldsymbol{\alpha}\cdot\nabla_{x'}F) + [\mathbf{v}(\mathbf{x}') + e^{i\omega t}\nabla\phi + e^{i\omega t}(\boldsymbol{\alpha}\cdot\nabla_{x'})\mathbf{v}(\mathbf{x}')]\cdot\left[\nabla_{x'}F - \mathbf{i}e^{i\omega t}\left(\frac{\partial\boldsymbol{\alpha}}{\partial x}\cdot\nabla_{x'}F\right) - \mathbf{j}e^{i\omega t}\left(\frac{\partial\boldsymbol{\alpha}}{\partial y}\cdot\nabla_{x'}F\right) - \mathbf{k}e^{i\omega t}\left(\frac{\partial\boldsymbol{\alpha}}{\partial z}\cdot\nabla_{x'}F\right)\right] + 0(\alpha^2)$$

The steady-state term gives the boundary condition for $\mathbf{v}$,

$$\mathbf{v}(\mathbf{x}')\cdot\nabla_{x'}F = 0, \tag{2}$$

and the terms of first order in the small oscillatory functions give the boundary condition for ϕ,

$$\nabla\phi\cdot\nabla_{x'}F = i\omega\boldsymbol{\alpha}\cdot\nabla_{x'}F - [(\boldsymbol{\alpha}\cdot\nabla)\mathbf{v}]_{\text{mean}}\cdot\nabla_{x'}F + [\mathbf{v}]_{\text{mean}}\cdot\left[\mathbf{i}\left(\frac{\partial\boldsymbol{\alpha}}{\partial x}\cdot\nabla_{x'}F\right) + \mathbf{j}\left(\frac{\partial\boldsymbol{\alpha}}{\partial y}\cdot\nabla_{x'}F\right) + \mathbf{k}\left(\frac{\partial\boldsymbol{\alpha}}{\partial z}\cdot\nabla_{x'}F\right)\right]$$

or

$$\nabla\phi\cdot\nabla_{x'}F = i\omega\boldsymbol{\alpha}\cdot\nabla_{x'}F + [(\mathbf{v}\cdot\nabla)\boldsymbol{\alpha} - (\boldsymbol{\alpha}\cdot\nabla)\mathbf{v}]_{\text{mean}}\cdot\nabla_{x'}F \tag{3}$$

In equation (3) and hereafter, all of the terms are small, of the same order as $\boldsymbol{\alpha}$ or ϕ. Thus to this order of approximation it is no longer necessary to distinguish between the actual position of the body and its mean position, or between the co-ordinates $\mathbf{x}$ and $\mathbf{x}'$.

We now use the vector identity[5]

$$(\mathbf{v}\cdot\nabla)\boldsymbol{\alpha} - (\boldsymbol{\alpha}\cdot\nabla)\mathbf{v} = \nabla\times(\boldsymbol{\alpha}\times\mathbf{v}) - \boldsymbol{\alpha}\nabla\cdot\mathbf{v} + \mathbf{v}\nabla\cdot\boldsymbol{\alpha}$$

and since $\nabla\cdot\mathbf{v} = 0$ (from incompressibility) and $\mathbf{v}\cdot\nabla F = 0$ [from equation (2)], we find that

$$\nabla\phi\cdot\nabla F = i\omega\boldsymbol{\alpha}\cdot\nabla F + [\nabla\times(\boldsymbol{\alpha}\times\mathbf{v})]\cdot\nabla F \quad \text{on} \quad F = 0$$

Since ∇F is a vector normal to the body surface, it follows that the boundary condition for ϕ on the body may be written as

$$\frac{\partial\phi}{\partial n} = [i\omega\boldsymbol{\alpha} + \nabla\times(\boldsymbol{\alpha}\times\mathbf{v})]\cdot\mathbf{n} \tag{4}$$

Thus the effect of the steady flow is to increase the normal oscillating velocity by $\nabla\times(\boldsymbol{\alpha}\times\mathbf{v})\cdot\mathbf{n}e^{i\omega t}$.

Green's Theorem and the Green's Function

Now we employ Green's theorem

$$\phi(\mathbf{x}) = \frac{1}{2\pi}\iint\left[G(\mathbf{x},\boldsymbol{\xi})\frac{\partial}{\partial n_\xi}\phi(\boldsymbol{\xi}) - \phi(\boldsymbol{\xi})\frac{\partial}{\partial n_\xi}G(\mathbf{x},\boldsymbol{\xi})\right]ds_\xi \tag{5}$$

where $\mathbf{x}$ is a point on the mean body surface and the integration is over this surface, the undisturbed free surface $z = 0$, and a closing surface at infinity. The Green's function is any harmonic function of $\mathbf{x}$ and $\boldsymbol{\xi}$ which is singular like the potential for a source at the point $\mathbf{x} = \boldsymbol{\xi}$. We shall assume the existence of a Green's function which satisfies the free-surface condition, the same radiation condition is ϕ, and the condition

$$\frac{\partial G}{\partial n_\xi} = 0 \text{ on the body } F(\boldsymbol{\xi}) = 0$$

As stated in the introduction, the existence of this Green's function can probably be proven from the theory of Fredholm integral equations, and furthermore seems physically plausible since this function can be visualized as the potential due to a realistic fluid flow; i.e., the disturbance caused by a small pulsating sphere in the presence of the body and the free surface.

Before proceeding further we must establish the reciprocal properties of this Green's function. Assume that G^+ and G^- are two Green's functions satisfying the condition $\partial G/\partial n_\xi = 0$ on the body and satisfying the free-surface conditions

$$-g\frac{\partial G^\pm}{\partial\zeta} + \omega^2G^\pm \mp 2i\omega c\frac{\partial G^\pm}{\partial\xi} - c^2\frac{\partial^2G^\pm}{\partial\xi^2} = 0 \quad \text{on} \quad \zeta = 0. \tag{6}$$

These two functions correspond physically to the velocity potentials of an oscillating source at the point $\mathbf{x} = \boldsymbol{\xi}$, in the presence of the body and the free surface. The function G^+ corresponds to the case where there is a free-stream velocity c in the $-x$-direction and the function G^- to a flow with velocity c in the $+x$-direction. Because of this difference the sign of the third term in (6) must differ for the two cases.

From Green's theorem, with $\mathbf{x}$ and $\mathbf{y}$ two different vectors,

$$G^+(\mathbf{x},\mathbf{y}) - G^-(\mathbf{y},\mathbf{x}) = \frac{1}{2\pi}\iiint\nabla\cdot[G^+(\mathbf{x},\boldsymbol{\xi})\nabla G^-(\mathbf{y},\boldsymbol{\xi}) - G^-(\mathbf{y},\boldsymbol{\xi})\nabla G^+(\mathbf{x},\boldsymbol{\xi})]dV_\xi$$

$$= \frac{1}{2\pi}\iint\left[G^+(\mathbf{x},\boldsymbol{\xi})\frac{\partial}{\partial n_\xi}G^-(\mathbf{y},\boldsymbol{\xi}) - G^-(\mathbf{y},\boldsymbol{\xi})\frac{\partial}{\partial n_\xi}G^+(\mathbf{x},\boldsymbol{\xi})\right]dS_\xi$$

where the surface integral is over the body, the undisturbed free surface, and a closure at infinity. The inte-

[5] Cf. [9], equation (1.4.13).

$\dot{x} = \dot{x} + \dot{\theta}\times r$

gral on the body vanishes since $\partial G^{\pm}/\partial n_\xi = 0$ and the closure at infinity vanishes from the radiation condition.[6] The integral over the free surface is equal to

$$\frac{1}{2\pi}\iint \left[G^+ \frac{\partial G^-}{\partial \zeta} - G^- \frac{\partial G^+}{\partial \zeta}\right] d\xi d\eta$$

$$= \frac{1}{2\pi g}\iint \left[G^+\left(\omega^2 G^- + 2i\omega c \frac{\partial G^-}{\partial \xi} - c^2 \frac{\partial^2 G^-}{\partial \xi^2}\right) - G^-\left(\omega^2 G^+ - 2i\omega c \frac{\partial G^+}{\partial \xi} - c^2 \frac{\partial^2 G^+}{\partial \xi^2}\right)\right] d\xi d\eta$$

$$= \frac{1}{2\pi g}\iint \frac{\partial}{\partial \xi}\left[G^+\left(i\omega c G^- - c^2 \frac{\partial G^-}{\partial \xi}\right) + G^-\left(i\omega c G^+ + c^2 \frac{\partial G^+}{d\xi}\right)\right] d\xi d\eta$$

$$= \frac{1}{2\pi g}\oint \left[G^+\left(i\omega c G^- - c^2 \frac{\partial G^-}{\partial \xi}\right) + G^-\left(i\omega c G^+ + c^2 \frac{\partial G^+}{\partial \xi}\right)\right] d\eta$$

where the line integral is over the boundary or boundaries or the free surface, or the intersection of the free surface with the closure at infinity and (if any) with the body. From the radiation condition the integral over the boundary at infinity vanishes, and if the body is submerged there is no further boundary, with the result that

$$G^+(\mathbf{x}, \mathbf{y}) - G^-(\mathbf{y}, \mathbf{x}) = 0$$

or

$$G^+(\mathbf{x}, \xi) = G^-(\xi, \mathbf{x})$$

For a floating body, the beam must be small (i.e., the body is either thin or slender) and thus the line integral around the waterline is of order

$$\oint |d\eta|$$

which is of the same order as the beam. Thus it is consistent with the linearized free-surface condition that, in all cases,

$$G^+(\mathbf{x}, \xi) = G^-(\xi, \mathbf{x}) \tag{7}$$

That is, the Green's function is reciprocal if the direction of the streaming flow is reversed. This property is well known for the Green's function which does not satisfy a boundary condition on the body.

We now return to the construction of the velocity potential from equation (5) substituting the Green's function $G^+(\mathbf{x}, \xi)$. From the reciprocal property (7) it follows that

$$-g\frac{\partial G^+}{\partial z} + \omega^2 G^+ + 2i\omega c \frac{\partial G^+}{\partial x} - c^2 \frac{\partial^2 G^+}{\partial x^2} = 0$$

[6] This consequence of the radiation condition is not physically obvious. Some discussion of this point will be found in reference [10], page 458.

on the undisturbed free surface, and thus that ϕ satisfies the free-surface condition (1). Furthermore, the surface integral in (5) may be treated in exactly the same manner as we did in establishing the reciprocity relation between the Green's functions. It follows that

$$\phi(\mathbf{x}) = \frac{1}{2\pi}\iint G^+(\mathbf{x}, \xi)\frac{\partial}{\partial n_\xi}\phi(\xi) dS_\xi \tag{8}$$

where the integration is only over the body.

Substituting (4) in (8) we obtain

$$\phi = \frac{1}{2\pi}\iint G(\mathbf{x}, \xi)[i\omega\alpha + \nabla \times (\alpha \times \mathbf{v})]\cdot \mathbf{n} dS_\xi$$

This is equal to

$$\phi = \frac{1}{2\pi}\iint \{i\omega\alpha G + \nabla \times [(\alpha \times \mathbf{v})G] - (\nabla G) \times (\alpha \times \mathbf{v})\}\cdot \mathbf{n} dS$$

but from Stokes' theorem

$$\iint \nabla \times [(\alpha \times \mathbf{v})G]\cdot dS = \oint (\alpha \times \mathbf{v})G \cdot d\mathbf{l}$$

where the line integral is again over the intersection, if any, of the body with the undisturbed free surface. Once again we invoke the linearized free-surface condition; if the waves are small, then $\mathbf{v}$ on the free surface is, to first order, tangent to the undisturbed plane of the free surface. Since $\mathbf{v}$ is also tangent to the body surface, it is tangent to the intersection $\mathbf{l}$, and thus $(\alpha \times \mathbf{v})\cdot d\mathbf{l} = 0$, and the line integral vanishes. Thus we find that

$$\phi = \frac{1}{2\pi}\iint \{i\omega\alpha G - \nabla G \times (\alpha \times \mathbf{v})\}\cdot \mathbf{n} dS$$

or, since

$$[\nabla G \times (\alpha \times \mathbf{v})]\cdot \mathbf{n} = [(\nabla G\cdot \mathbf{v})\alpha - (\alpha \cdot \nabla G)\mathbf{v}]\cdot \mathbf{n} = (\mathbf{v}\cdot\nabla G)(\alpha\cdot \mathbf{n})$$

it follows that

$$\phi = \frac{1}{2\pi}\iint (i\omega G - \mathbf{v}\cdot\nabla_\xi G)(\alpha\cdot\mathbf{n})dS, \tag{9}$$

where the integral is over the body surface. *Thus the effect of the steady velocity field on the unsteady potential is expressed by the factor* $\mathbf{v}\cdot\nabla G$. Physically this can be thought of as a dipole distribution in the direction of $\mathbf{v}$ (and thus tangent to the body surface) and of strength equal to the normal displacement $(\alpha\cdot\mathbf{n})$ times the magnitude of $\mathbf{v}$. The same result has been derived for a thin ship [4].

The Forces and Moments

Equation (9) holds for any oscillatory displacement vector α, and is therefore not restricted to rigid body motions. We now assume that the body is rigid, with six degrees of freedom. It is convenient to introduce an indicial notation, where we denote the six oscillatory velocities by

$$i\omega\zeta_j e^{i\omega t} \qquad (j = 1, 2, \ldots 6)$$

These are, respectively, surge, sway, heave, roll, pitch, and yaw. We also define the six matrix elements

$$f_1(\mathbf{x}) = \cos(n, x)$$
$$f_2(\mathbf{x}) = \cos(n, y)$$
$$f_3(\mathbf{x}) = \cos(n, z)$$
$$f_4(\mathbf{x}) = y \cos(n, z) - z \cos(n, y)$$
$$f_5(\mathbf{x}) = z \cos(n, x) - x \cos(n, z)$$
$$f_6(\mathbf{x}) = x \cos(n, y) - y \cos(n, x)$$

Then the normal displacement $\alpha \cdot \mathbf{n}$ at the point x is given by

$$\alpha \cdot \mathbf{n} = \sum_{j=1}^{6} \zeta_j f_j(\mathbf{x}) \tag{10}$$

and if p is the hydrodynamic pressure, the hydrodynamic forces and moments are given by the six expressions

$$F_i = -\int\int p f_i(\mathbf{x}) dS_x \qquad (i = 1, 2, \ldots 6) \tag{11}$$

The pressure p is, from the linearized form of Bernoulli's equation,

$$p = -\rho[i\omega\phi e^{i\omega t} + (\mathbf{v} \cdot \nabla\phi)e^{i\omega t} + \tfrac{1}{2}(\mathbf{v} \cdot \mathbf{v})] \tag{12}$$

where terms of second order in the oscillatory potential ϕ are neglected.

The integral in (11) must be evaluated on the oscillating surface of the ship, and thus the zero-order term $\frac{1}{2}$ $\mathbf{x}(\mathbf{v} \cdot \mathbf{v})$ must be expanded to

$$\tfrac{1}{2}(\mathbf{v} \cdot \mathbf{v})_{\text{body}} = \tfrac{1}{2}(\mathbf{v} \cdot \mathbf{v})_{\text{mean}} + e^{i\omega t}(\alpha \cdot \nabla)(\tfrac{1}{2}\mathbf{v} \cdot \mathbf{v})_{\text{mean}} + 0(\alpha^2)$$

Also if the body intersects the free surface, the oscillatory change in the surface of integration must be included. However, both of these effects are in phase with the displacement $\alpha e^{i\omega t}$ and will not influence the damping coefficients.

We restrict ourselves then to the damping forces and moments, which may be represented by the imaginary part of the integral (11), taken over the mean surface of the body. Let the damping coefficients be represented by the matrix B_{ij}, where the first index denotes the direction of the force and the second index the velocity component involved. Thus, for example, B_{35} is the heave damping force due to pitching oscillations. The thirty-six coefficients are identified by the matrix in Table 1. Combining equations (9–12) and taking the imaginary part, we obtain the expressions

$$B_{ij} = \frac{\rho}{2\pi} \operatorname{Im} \iint_{S_x} f_i(\mathbf{x})(i\omega + \mathbf{v}(\mathbf{x}) \cdot \nabla_x) \iint_{S_\xi} f_j(\xi)(i\omega - \mathbf{v}(\xi) \cdot \nabla_\xi) G^+(\mathbf{x}, \xi) dS_\xi dS_x \tag{13}$$

In this form the possibility of symmetry, and the importance of the dipole distribution $-\mathbf{v} \cdot \nabla_\xi G$ are apparent.

In order to establish the symmetry properties of the coefficients we consider the same forces and moments with the direction of the forward velocity reversed. To distinguish between these two cases we shall denote the matrix appropriate to forward motion in the $+x$-direction as B_{ij}^+ and the matrix of the reverse flow by B_{ij}^-. Then

$$B_{ij}^\pm = \frac{\rho}{2\pi} \operatorname{Im} \iint_{S_x} f_i(\mathbf{x})(i\omega + \mathbf{v}^\pm(\mathbf{x}) \cdot \nabla_x) \iint f_j(\xi)(i\omega - \mathbf{v}^\pm(\xi) \cdot \nabla_\xi) G^\pm(\mathbf{x}, \xi) dS_\xi dS_x \tag{14}$$

where $G^\pm(\mathbf{x}, \xi)$ are the two Green's functions which we introduced before, and which possess the reciprocal property

$$G^+(\mathbf{x}, \xi) = G^-(\xi, \mathbf{x})$$

We need one further assumption based upon the linearized free-surface condition. This is that, to first order,

$$\mathbf{v}^+(\mathbf{x}) = -\mathbf{v}^-(\mathbf{x})$$

on the body. This seems consistent with the assumption of a small disturbance on the free surface, for if the waves are small, the effect of the free surface on the flow *at the body* is small, and thus to first order the steady velocity on the body is an odd function of the stream velocity c, as is the case for a body in a wave-free field.

Substituting $\mathbf{v}^- = -\mathbf{v}^+$ and

$$G^-(\mathbf{x}, \xi) = G^+(\xi, \mathbf{x}),$$

it follows that

$$B_{ij}^- = \frac{\rho}{2\pi} \operatorname{Im} \iint_{S_\xi} f_i(\mathbf{x})(i\omega - \mathbf{v}^+(\mathbf{x}) \cdot \nabla_x) \iint_{S_\xi} f_j(\xi)(i\omega + \mathbf{v}^+(\xi) \cdot \nabla_\xi) G^+(\xi, \mathbf{x}) dS_\xi dS_x$$

or, after interchanging the integrals and the variables of integration and comparing the resulting expression with (14),

$$B_{ij}^- = B_{ji}^+ \tag{15}$$

It should be noted that up to this point in the analysis, no assumption has been made regarding the symmetry of the body. Thus (15) holds for asymmetric bodies and, in particular, it follows that the six principal damping coefficients B_{jj} are independent of the direction of forward motion. Of greater practical importance is the fact that if there is no forward speed, B_{ij}^+ and B_{ij}^- must be the same, and therefore $B_{ij} = B_{ji}$ when $c = 0$. This result has also been obtained by Haskind [11] and confirms the frequent argument based upon strip theory.

Now we consider the physical relations between B_{ij}^+ and B_{ij}^-, assuming that the ship is symmetrical. For example, the pitch moment of a symmetrical body due to heave and heave force due to pitch are odd functions of the forward velocity, since changing the direction of the flow is equivalent to looking at the body from the opposite side and changing the sign of rotation about the pitch axis. Examination of all of the coefficients in this manner yields the following conclusions:

Table 1 Matrix of Symmetry Properties of Damping Cross-Coupling Coefficients

	Surge $j = 1$	Sway $j = 2$	Heave $j = 3$	Roll $j = 4$	Pitch $j = 5$	Yaw $j = 6$
Surge force $i = 1$	B_{11}	$B_{12} = 0$	$B_{13} = -B_{31}$	$B_{14} = 0$	$B_{15} = B_{51}$	$B_{16} = 0$
Sway force $i = 2$	$B_{21} = 0$	B_{22}	$B_{23} = 0$	$B_{24} = B_{42}$	$B_{25} = 0$	$B_{26} = -B_{62}$
Heave force $i = 3$	$B_{31} = -B_{13}$	$B_{32} = 0$	B_{33}	$B_{34} = 0$	$B_{35} = -B_{53}$	$B_{36} = 0$
Roll moment $i = 4$	$B_{41} = 0$	$B_{42} = B_{24}$	$B_{43} = 0$	B_{44}	$B_{45} = 0$	$B_{46} = -B_{64}$
Pitch moment $i = 5$	$B_{51} = B_{15}$	$B_{52} = 0$	$B_{53} = -B_{35}$	$B_{54} = 0$	B_{55}	$B_{56} = 0$
Yaw moment $i = 6$	$B_{61} = 0$	$B_{62} = -B_{26}$	$B_{63} = 0$	$B_{64} = -B_{46}$	$B_{65} = 0$	B_{66}

1 The six coefficients B_{jj} and the nine coefficients $B_{12}, B_{14}, B_{15}, B_{24}, B_{36}, B_{42}, B_{51}, B_{52}$, and B_{54} are even functions of the forward velocity c.

2 The twelve coefficients B_{13}, B_{16}, B_{26}, B_{31}, B_{32}, B_{34}, B_{35}, B_{46}, B_{53}, B_{56}, B_{62}, and B_{64} are odd functions of c.

3 The nine coefficients B_{21}, B_{23}, B_{25}, B_{41}, B_{43}, B_{45}, B_{61}, B_{63}, and B_{65} are all zero by symmetry; i.e., there are no transverse forces or moments due to longitudinal oscillations.

Combining these conclusions with equation (15) it follows that:

4 The sum $B_{ij} + B_{ji} = 0$ for all cross-coupling coefficients except $B_{15} + B_{51}$ (surge and pitch) and $B_{24} + B_{42}$ (roll and sway), where $B_{ij} - B_{ji} = 0$.

5 Half of the cross-coupling coefficients are zero (those listed under statement 3 plus their complementary members). Furthermore this conclusion holds for an asymmetric body as well, since statement 3 and equation (15) are valid without the assumption of longitudinal symmetry.

Discussion of Results

The symmetry properties of the damping coefficients for a symmetrical ship are shown in Table 1. For the cross-coupling coefficients it is seen that $B_{ij} + B_{ji} = 0$ except for coupling between pitch and surge, where $B_{16} = B_{61}$, and for coupling between roll and sway, where $B_{34} = B_{43}$. These results are consistent with the conclusions based upon thin-ship theory [1, 4, 5, 6], but the present derivation is valid for all thirty cross-coupling damping coefficients and does not require that the ship be thin. We have, however, assumed that the waves created by the forward motion are small, and this implies that the body is either thin, slender, or deeply submerged. Nevertheless the analysis and conclusions also hold for Havelock's [3] mathematical model of a nonslender body with a rigid free surface, which may correspond physically to very slow forward speed and a low frequency of oscillations.

There does not appear to be sufficient experimental evidence to support these conclusions completely, but the oscillator experiments of Gerritsma [12] and Golovato [13] are strongly suggestive of the equivalence between the pitch and heave coupling coefficients. Gerritsma has measured the coefficients B_{35} and B_{53} for a Series 60 model. Since this hull is not longitudinally symmetric, the present theory is not strictly valid. Following a suggestion of Vossers [2], however, we may separate the cross-coupling coefficients into two parts

$$B_{ij} = B_{ij}^{\mathrm{I}} + B_{ij}^{\mathrm{II}}$$

where B_{ij}^{I} is the value of B_{ij} at zero forward speed and B_{ij}^{II} is the difference due to the effects of forward speed. From equation (15) it follows that (even for an asymmetric body)

$$B_{ij}^{\mathrm{I}} = B_{ji}^{\mathrm{I}}$$

If the ship is only slightly asymmetric, the effects of the asymmetry on the coefficients B_{ij}^{II} will be small and thus, approximately,

$$B_{35}^{\mathrm{II}} = -B_{53}^{\mathrm{II}}$$

The experimental results in Fig. 5 of reference [12] do not confirm this relation exactly, but they do suggest, especially at the higher frequencies and speeds, that this is a meaningful approximation for a slightly asymmetric ship.

Experimental measurements of B_{35} and B_{53} have been made for a symmetrical hull by Golovato (the results of the pitch experiments are unpublished), who found that these two coefficients were approximately equal and opposite, with a maximum difference of about 10 per cent over a fairly wide range of speeds and frequencies.

With regard to the derivation of this theory an important result is the effect of the steady flow field on the oscillatory potential. In equation (9) this effect is seen to be a tangential dipole distribution, equal in strength to the product of the steady velocity and the normal oscillatory displacement. The consideration of this effect is vital to the present analysis as it is directly responsible for the symmetry properties of the cross-coupling coefficients. Physically this implies that the problem of an oscillating body in a moving fluid is not the same, even in the linearized sense, as the problem of a fixed body with the same distribution of normal velocity, for there is an additional effect from moving about in the steady flow field.

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