Memoirs of the Defense Academy, Japan, Vol. VIII, No. 1, pp. 183-200, 1968

"On Boundary Value Problems of an Oscillating Body Floating on Water"

(To Professor H. Kato this paper is dedicated)

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(Recived March 30, 1968)

Abstract

The author deals with the boundary value problem of a floating ship when it oscillates or diffracts the incident wave. At first, he defines introductly the velocity potentials, forces and moments for the oscillations corresponding to six degrees of freedom and the diffraction and shows the reciprocity between the forces and damping integrals. Secondly, he derives the asymptotic expansion of Neumann function which solves these boundary value problems and finds those coefficients of expansion relates closely to the forces and moments when the ship oscillates and diffracts the wave. Considering the variation of Neumann function, he also shows the variation of the forces and moments by a slight change of the ship form. Thirdly, he converts the boundary value problem to the integral equation for Kotchin's function and derives similar relations between the force coefficients and Kotchin's function to Kramers-Kronig's. Lastly, he introduces some integrals of which extremum problems are equivalent to satisfy the boundary condition and gives two examples of solutions at high frequency.

Introduction

Recent progress of the theory of ship motion is built up soundly on the theoretical ground of the water wave and many works have been down on the boundary value problems of oscillating ships theoretically and numerically[1]. In this paper, the author is interested in the mutual relations of various force coefficients, asymptotic character of the velocity potential, the variation of forces by a slight change of the ship form, Kotchin's function and the extremum property of the boundary condition.

He gives the formulation of problems and definitions in the first chapter, discusses the asymptotic property and variation of Neumann function in the second chapter, the integral equation for Kotchin's function in the third chapter and the boundary value problem as the extremum in the last chapter. In each chapter, the discussion for two-dimensional problem is added in the sub-section as far as it differs from the three dimensional one.

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1. Velocity Potentials, Forces and Reciprocity[1,5,6,7,14,15]

1.1 *Three-dimensional Problem*

Let us consider a ship floating on the water surface and take the origin at the center of the water plane of the ship, the x-axis longitudinally forwards, y-axis horizontally and z-axis vertically downwards. The velocity potential of the water motion by the harmonic oscillation of the ship or the diffraction of the incident regular wave can be represented as

$$\mathcal{R}_e\{\varphi(x, y, z)e^{i\omega t}\}, \tag{1.1.1}$$

where ω means the circular frequency of the periodic motion, and the water pressure

$$\mathcal{R}_e\{p(x, y, z)e^{i\omega t}\}, \qquad p(x, y, z)=\rho i\omega\varphi(x, y, z), \tag{1.1.2}$$

under the restriction of the linearized theory, where ρ means the water density. The velocity potential must satisfy the condition

$$\left(\frac{\partial}{\partial z}+K\right)\varphi(x, y, 0)=0, \qquad K=g/\omega^2, \tag{1.1.3}$$

on the water surface and the conditions on the ship surface, which can be satisfied by superposing the velocity potentials of the following motions.

i)	The oscillation parallel to x-axis (surging)	suffix 1
ii)	〃 y-axis (swaying)	2
iii)	〃 z-axis (heaving)	3
iv)	the rotational oscillation around x-axis (rolling)	4
v)	〃 y-axis (pitching)	5
vi)	〃 z-axis (yawing)	6
vii)	the diffraction	d
viii)	the incident wave	o

In the following we use these suffixes to denote each motions. These potentials are all linear with respect to their amplitude of each motion so that we may normalize them dividing by their amplitude as follows;

$$\varphi_j=i\omega X_j\phi_j, \quad j=1, 2, \cdots, 6; \qquad \varphi_j=\frac{iga}{\omega}\phi_j, \quad j=0, d, \tag{1.1.4}$$

where a means the amplitude of the incident wave, and X_j the amplitude of the oscillation of each motion. The boundary conditions on the ship surface are

$$\frac{\partial}{\partial n}\phi_j=-\frac{\partial x_j}{\partial n}, \quad j=1, 2, \cdots, 6, \qquad \frac{\partial}{\partial n}\phi_d=-\frac{\partial}{\partial n}\phi_0, \tag{1.1.5}$$

where

$$\left.\begin{aligned} &x_1\equiv x, \quad x_2\equiv y, \quad x_3\equiv z\,; \qquad \frac{\partial}{\partial n}x_4=y\frac{\partial z}{\partial n}-z\frac{\partial y}{\partial n} \\ &\frac{\partial}{\partial n}x_5=z\frac{\partial x}{\partial n}-x\frac{\partial z}{\partial n}\,, \quad \frac{\partial}{\partial n}x_6=x\frac{\partial y}{\partial n}-y\frac{\partial x}{\partial n} \end{aligned}\right\} \tag{1.1.6}$$

and n means the normal to the ship surface into the water. The force or moment acting on the ship may be also normalized as follows:

$$F_{ij}=-\iint_s p_i\frac{\partial x_j}{\partial n}ds=\rho gKX_i f_{ij}\,, \tag{1.1.7}$$

$$f_{ij}=\iint_s \phi_i\frac{\partial x_j}{\partial n}ds=-\iint_s \phi_i\frac{\partial \phi_j}{\partial n}ds\,, \tag{1.1.8}$$

for the force by the i-th motion to j-th direction and for the exciting force or moment by the regular plane wave coming from the direction of α radian with respect to the x-axis

$$E_j=-\iint_s (p_0+p_d)\frac{\partial x_j}{\partial n}ds=\rho gaH_j(K,\alpha)\,, \tag{1.1.9}$$

$$H_j(K,\alpha)=\iint_s (\phi_0+\phi_d)\frac{\partial x_j}{\partial n}ds=-\iint_s (\phi_0+\phi_d)\frac{\partial \phi_j}{\partial n}ds\,, \tag{1.1.10}$$

where

$$\phi_0(x,y,z;\alpha)=\exp\left[-Kz+iK(x\cos\alpha+y\sin\alpha)\right]. \tag{1.1.11}$$

Now, by Green's theorem, we may easily verify the reciprocity relation for the radiation and diffraction potentials

$$\iint_s\left(\phi_i\frac{\partial}{\partial n}\phi_j-\phi_j\frac{\partial}{\partial n}\phi_i\right)ds=0\,, \qquad i,j=1,2,\cdots,6,d\,, \tag{1.1.12}$$

then we have

$$f_{ij}=f_{ji}\,, \qquad i,j=1,2,\cdots,6\,, \tag{1.1.13}$$

and

$$H_j(K,\alpha)=\iint_s\left(\phi_j\frac{\partial}{\partial n}\phi_0-\phi_0\frac{\partial}{\partial n}\phi_j\right)ds\,, \qquad j=1,2,\cdots,6 \tag{1.1.14}$$

which is called Haskind's formula[7]. Another application of (1.1.12) is the following relation for the diffracted wave

$$H_d(K,\theta;\alpha)=H_d(K,\alpha;\theta)\,, \tag{1.1.15}$$

by making use of the equality

$$\iint_s\left\{\phi_0(\alpha)\frac{\partial}{\partial n}\phi_0(\beta)-\phi_0(\beta)\frac{\partial}{\partial n}\phi_0(\alpha)\right\}ds=0\,, \tag{1.1.16}$$

that is, the diffraction of the wave has a reversible property. In the same way as the above, we have by Green's theorem, making use of the asymptotic expansion (2.1.14),

$$\iint_s \left(\phi_i \frac{\partial}{\partial n}\bar{\phi}_j - \bar{\phi}_j \frac{\partial}{\partial n}\phi_i\right) ds = \frac{iK}{2\pi}\int_0^{2\pi} H_i(K, \theta)\bar{H}_j(K, \theta) d\theta\,, \qquad i, j=1, 2, \cdots, 6, d\,, \quad (1.1.17)$$

where the bar over the letter means the complex conjugate to be taken. Since the boundary condition is real, we have

$$\begin{aligned} \bar{f}_{ji} - f_{ij} &= \bar{f}_{ij} - f_{ji} = \bar{f}_{ij} - f_{ij} \\ &= \frac{iK}{2\pi}\int_0^{2\pi} H_i(K, \theta)\bar{H}_j(K, \theta) d\theta = \frac{iK}{2\pi}\int_2^{2\pi} H_j(K, \theta)\bar{H}_i(K, \theta) d\theta\,, \end{aligned} \quad (1.1.18)$$

when i equals j, this is the energy dissipated by the wave. Putting d for i and $1, 2, \cdots$, 6 for j, since

$$\phi_d \frac{\partial}{\partial n}\bar{\phi}_j - \bar{\phi}_j \frac{\partial}{\partial n}\phi_d = (\phi_0 + \phi_d)\frac{\partial}{\partial n}\bar{\phi}_j + \left(\bar{\phi}_j \frac{\partial}{\partial n}\phi_0 - \phi_0 \frac{\partial}{\partial n}\bar{\phi}_j\right),$$

we have

$$\bar{H}_j(K, \alpha+\pi) - H_j(K, \alpha) = \frac{iK}{2\pi}\int_0^{2\pi} H_d(K, \theta, \alpha)\bar{H}_j(K, \theta) d\theta\,, \qquad j=1, 2, \cdots, 6\,. \quad (1.1.19)$$

If both i and j are the diffraction potential, (1.1.17) gives the relation

$$\bar{H}_d(K, \alpha+\pi, \beta) - H_d(K, \beta+\pi, \alpha) = \frac{iK}{2\pi}\int_0^{2\pi} H_d(K, \theta, \alpha)\bar{H}_d(K, \theta, \beta) d\theta\,. \quad (1.1.20)$$

The reciprocity (1.1.15), the relation (1.1.19) and (1.1.20) give us much knowledge of the diffracted wave and its relation to the radiation potentials[15].

1.2. *Two-dimensional Problem*

If the ship is very long so that the water motion is limited in the $y-z$ plane, we have two-dimensional problems corresponding to the preceding. The definition and notations are preserved but the motion is limited to the followings.

i)	swaying oscillation	(y-axis)	suffix 2
ii)	heaving 〃	(z-axis)	3
iii)	rolling		4
iv)	diffraction		d
v)	incident wave		o

Thus, many of the formulas are the same form as in the former case, except that the integration is performed on the periphery curve C not on the surface.

In the following let us pick up the difference. The regular wave potential may be defined as

$$\phi_0(y, z; K)=\exp(-|K|z+iKy), \tag{1.2.1}$$

and Kotchin's function

$$\begin{aligned} H_j(K) &= -\int_C \{\phi_0(K)+\phi_d(K)\}\frac{\partial}{\partial n}\phi_j ds \\ &= \int_C \left\{\phi_j\frac{\partial}{\partial n}\phi_0(K)-\phi_0(K)\frac{\partial}{\partial n}\phi_j\right\}ds , \end{aligned} \tag{1.2.2}$$

$$H_d(K, \pm K)=H_d(\pm K, K)=\int_C \{\phi_0(K)+\phi_d(K)\}\frac{\partial}{\partial n}\phi_0(\pm K)ds . \tag{1.2.3}$$

The formula (1.1.17) can be deformed as

$$\int_C \left(\phi_i\frac{\partial}{\partial n}\bar{\phi}_j-\bar{\phi}_j\frac{\partial}{\partial n}\phi_i\right)ds=i\{H_i(K)\bar{H}_j(K)+H_i(-K)\bar{H}_j(-K)\} , \tag{1.2.4}$$

then

$$\begin{aligned} \bar{f}_{ji}-f_{ij} &= \bar{f}_{ij}-f_{ji}=\bar{f}_{ij}-f_{ij} \\ &= i\{H_i(K)\bar{H}_j(K)+H_i(-K)\bar{H}_j(-K)\} \\ &= i\{\bar{H}_i(K)H_j(K)+\bar{H}_i(-K)H_j(-K\} . \end{aligned} \tag{1.2.5}$$

For example, if the ship is symmetric with respct to the $x-z$ plane, we have clearly

$$H_3(K)=H_3(-K) , \qquad H_2(K)=-H_2(-K) , \qquad H_4(K)=-H_4(-K) , \tag{1.2.6}$$

then by the above formula we have

$$H_2(K)/\bar{H}_2(K)=H_4(K)/\bar{H}_4(K) \tag{1.2.7}$$

which shows that the exciting moment and sway force of the wave have the same phase.

Putting d for i in (1.2.4), we have

$$\bar{H}_j(-K)-H_j(K)=i\{H_d(K, K)\bar{H}_j(K)+H_d(K, -K)\bar{H}_j(-K)\} , \tag{1.2.8}$$

and that

$$\bar{H}_d(K, -K)-H_d(K, -K)=i\{|H_d(K, K)|^2+|H_d(K, -K)|^2\} . \tag{1.2.9}$$

Thus, if we know H_2 and H_3, we may calculate $H_d(K, \pm K)$ from (1.2.8)[14].

2. Neumann function[2,13,14]

2.1. *Three-dimensional Problem*

The velocity potential of the unit source at Q satisfying the water surface condition is well-known[1] and let us call it the fundamental singularity, namely,

$$S(P, Q)=\frac{1}{4\pi}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)+\frac{1}{4\pi^2}\lim_{\mu\to+0}\int_{-\pi}^{\pi}\int_0^\infty\frac{\exp\{-k(z+z')+ik(\tilde{\omega}-\tilde{\omega}')\}}{k-K+\mu i}kdkdu , \tag{2.1.1}$$

where

$$P\equiv(x, y, z)\,, \quad Q\equiv(x', y', z'), \quad \bar{Q}\equiv(x', y', -z')\,, \quad r_1=\overline{PQ}, \quad r_2=\overline{P\bar{Q}}\,,$$
$$\tilde{\omega}=x\cos u+y\sin u, \quad \tilde{\omega}'=x'\cos u+y'\sin u\,.$$

If there exists Neumann function of the present problems, which has the following property:

$$\frac{\partial}{\partial n}N(P, Q)=0 \qquad \text{on } S\,, \tag{2.1.2}$$

$$N(P, Q)=S(P, Q)+A(P, Q)\,, \tag{2.1.3}$$

where A is a regular function, then we may solve our boundary value problem as

$$\phi(P)=-\iint_S \frac{\partial\phi}{\partial n}N(P, Q)dS_Q\,. \tag{2.1.4}$$

Since the function A is regular and must satisfy the condition

$$\frac{\partial}{\partial n}A(P, Q)=-\frac{\partial}{\partial n}S(P, Q) \qquad \text{on } S\,, \tag{2.1.5}$$

it may be written by (2.1.4) as follows:

$$A(P, Q)=\iint_S N(Q, R)\frac{\partial}{\partial n}S(P, R)dS_R\,. \tag{2.1.6}$$

Lastly, it is natural to assume that Neumann function has the reciprocity, namely,

$$N(P, Q)=N(Q, P)\,, \tag{2.1.7}$$

because $S(P, Q)$ has the same relation.

Now, let us consider its asymptotic expansion. At first $S(P, Q)$ can be expanded asymptotically as follows:

$$\begin{aligned} S(P, Q)\underset{P\to\infty}{\longrightarrow}&\sqrt{\frac{K}{2\pi\rho i}}\,e^{-Kz-iK\rho}\phi_0(Q, \varphi)+\cdots \\ &+\left(z'-\frac{1}{K}\right)\{u_1(P)+x'u_2(P)+y'u_3(P)\}+\cdots\,, \end{aligned} \tag{2.1.8}$$

where $\qquad x=\rho\cos\varphi, \quad y=\rho\sin\varphi, \quad x'=\rho'\cos\varphi', \quad y'=\rho'\sin\varphi'$

and

$$\left.\begin{aligned} u_1(P)&=-\frac{1}{2\pi}\left(\frac{\partial}{\partial z}-\frac{1}{K}\frac{\partial^2}{\partial z^2}\right)\frac{1}{r}\,, \qquad r=\sqrt{x^2+y^2+z^2}\,, \\ u_2(P)&=-\frac{\partial}{\partial x}u_1(P)\,, \qquad u_3(P)=-\frac{\partial}{\partial y}u_1(P)\,. \end{aligned}\right\} \tag{2.1.9}$$

Putting this expansion into (2.1.6), we have also

$$A(P,Q)\underset{P\to\infty}{\longrightarrow}\sqrt{\frac{K}{2\pi\rho i}}\,e^{-Kz-iK\rho}\phi_d(Q,\varphi)+\cdots$$
$$+\phi_3(Q)u_1(P)+\phi_{\rm II}(Q)u_2(P)+\phi_{\rm III}(Q)u_3(P)+\cdots, \tag{2.1.10}$$

where

$$\left.\begin{aligned}\phi_d(Q,\varphi)&=\iint_S N(Q,P)\frac{\partial}{\partial n}\phi_0(P,\varphi)dS,\\ \phi_{\rm II}(Q)&=\iint_S N(Q,P)\frac{\partial}{\partial n}\left\{x\left(z-\frac{1}{K}\right)\right\}dS,\\ \phi_{\rm III}(Q)&=\iint_S N(Q,P)\frac{\partial}{\partial n}\left\{y\left(z-\frac{1}{K}\right)\right\}dS.\end{aligned}\right\} \tag{2.1.11}$$

Adding (2.1.8) and (2.1.10), we have finally

$$N(P,Q)\underset{P\to\infty}{\longrightarrow}\sqrt{\frac{K}{2\pi\rho i}}\,e^{-Kz-iK\rho}\Phi_d(Q,\varphi)+\cdots$$
$$+\Phi_3'(Q)u_1(P)+\Phi_{\rm II}(Q)u_2(P)+\Phi_{\rm III}(Q)u_3(P)+\cdots, \tag{2.1.12}$$

where

$$\left.\begin{aligned}&\Phi_d(Q,\varphi)=\phi_0(Q,\varphi)+\phi_d(Q,\varphi),\\ &\Phi_3'(Q)=\Phi_3(Q)-\frac{1}{K},\qquad \Phi_3(Q)=\phi_3(Q)+z',\\ &\Phi_{\rm II}(Q)=\phi_{\rm II}(Q)+x'\left(z'-\frac{1}{K}\right),\\ &\Phi_{\rm III}(Q)=\phi_{\rm III}(Q)+y'\left(z'-\frac{1}{K}\right).\end{aligned}\right\} \tag{2.1.13}$$

Then, the velocity potential becomes approximately at infinity,

$$\phi(P)\underset{P\to\infty}{\longrightarrow}\sqrt{\frac{K}{2\pi\rho i}}\,e^{-Kz-iK\rho}H(K,\varphi)+\cdots$$
$$-u_1(P)\iint_S \Phi_3'\frac{\partial\phi}{\partial n}dS-u_2(P)\iint_S \Phi_{\rm II}\frac{\partial\phi}{\partial n}dS-u_3(P)\iint_S \Phi_{\rm III}\frac{\partial\phi}{\partial n}ds+\cdots, \tag{2.1.14}$$

where

$$H(K,\varphi)=-\iint_S \Phi_d(P,\psi)\frac{\partial\phi}{\partial n}dS, \tag{2.1.15}$$

which is the same as the exciting force or moment by (1.1.10). Thus its first term is the wave diverging outwards but this term vanishes exponentially into the water depth, where the second term dominates over other terms and this is the doublet with the z-axis as its axis and its strength for heaving motion is

$$\iint_S \Phi_3' \frac{\partial \phi_3}{\partial n} dS = -\iint_S \Phi_3' \frac{\partial z}{\partial n} dS = \frac{1}{K}[A_w - K(\nabla + f_{33})], \tag{2.1.16}$$

where ∇ stands for the displacement volume and A_w the water plane area, namely, this is equal to the sum of the statical and dynamical vertical force divided by ρgK. Conversely, we may verify that this term vanishes if there acts no vertical force upon the the ship, and then the influence of the oscillation of the ship is very small in deeper region of the water. When the free surface effects nothing, that is, the body is fully submerged deep, these terms as in (2.1.16) are symmetric in every direction of motion and moreover they have extremum property which results from their positive definite character[2], but we may not verify anything in the present case, because the kinetic energy of the present system becomes infinite. The last example of the utility of Neumann function is to estimate the variation of various coefficients resulting from a slight change of the ship surface. According to the reference[2], we have the variation of Neumann function when the ship surface swells outwards by the amount $\delta\nu$ measured along the normal as follows:

$$\delta N(P, Q) = \iint_S \operatorname{grad} N(P, R) \operatorname{grad} N(R, Q)(\delta\nu dS)_R. \tag{2.1.17}$$

In the same way, the variation of the velocity potential is

$$\delta\Phi(P) = \iint_S \operatorname{grad} \Phi(R) \operatorname{grad} N(P, R)\delta\nu dS, \tag{2.1.18}$$

where

$$\frac{\partial}{\partial n}\Phi = 0 \qquad \text{on } S, \tag{2.1.19}$$

Then, putting the asymptotic expansion (2.1.12) into the above formulas and comparing both sides, we have

$$\delta H(K, \varphi) = \iint_S \operatorname{grad} \Phi \operatorname{grad} \Phi_a \delta\nu dS. \tag{2.1.20}$$

Although the variation of heaving force is also deduced in the same manner as the above, other ones can not be, so that we must calculate them by the same manner as the one used to deduce (2.1.17) or (2.1.18).

The result is

$$\delta \iint_S \Phi_i \frac{\partial x_j}{\partial n} dS = \iint_S \operatorname{grad} \Phi_i \operatorname{grad} \Phi_j \delta\nu dS, \tag{2.1.21}$$

or

$$-\delta\iint_S \phi_i \frac{\partial x_j}{\partial n} dS=\iint_S (\text{grad}\, x_i\, \text{grad}\, x_j - \text{grad}\, \Phi_i\, \text{grad}\, \Phi_j)\delta\nu dS\,, \quad i, j=1, 2, 3\,. \qquad (2.1.22)$$

These formula permits us to estimate the change of various forces by a slight change of the ship form, for example, we see by (2.1.22) that added mass increases as the ship swells at the part where the velocity is greater than the body velocity[2].

2.2. *Two-dimensional Problem*[1,2,13,14]

The fundamental singularity is given as

$$S(P, Q)=\frac{1}{2\pi}\log\frac{r_2}{r_1}+\frac{1}{\pi}\lim_{\mu\to+0}\int_0^\infty \frac{e^{-k(z+z')}\cos k(x-x')}{k-K+\mu i}dk\,, \qquad (2.2.1)$$

where

$$P\equiv(y, z)\,, \qquad Q\equiv(y', z'), \quad \bar{Q}\equiv(y', z')\,, \qquad r_1=\overline{PQ}, \quad r_2=\overline{P\bar{Q}}\,.$$

Neumann function may be written also as (2.1.3) and the boundary condition of A is (2.1.5) and it is represented as (2.1.6). The asymptotic expansion of S becomes as

$$S(P, Q)\xrightarrow[P\to\infty]{} -ie^{-K(z+z')-iK|y-y'|}+\left(z'-\frac{1}{K}\right)\{u_1(P)+y'u_3(P)\}+\cdots\,, \qquad (2.2.2)$$

where

$$\left.\begin{aligned} u_1(P)&=\frac{1}{\pi}\left(\frac{z}{r^2}-\frac{y^2-z^2}{Kr^4}\right), \qquad r=\sqrt{y^2+z^2}\,, \\ u_3(P)&=-\frac{\partial}{\partial y}u_1(P)\,. \end{aligned}\right\} \qquad (2.2.3)$$

Putting this expansion into (2.1.6), (2.1.3) and (2.1.4), we have the following expansions.

$$A(P, Q)\xrightarrow[P\to\infty]{} -ie^{-Kz-iK|y|}\phi_d(Q, (\text{sgn}\, y)K)+\phi_3(Q)u_1(P)+\phi_{\text{III}}(Q)u_3(P)+\cdots\,, \qquad (2.2.4)$$

$$N(P, Q)\xrightarrow[P\to\infty]{} -ie^{-Kz-iK|y|}\Phi_d(Q, (\text{sgn}\, y)K) +\Phi_3'(Q)u_1(P)+\Phi_{\text{III}}(Q)u_3(P)+\cdots\,, \qquad (2.2.5)$$

$$\phi(P)\xrightarrow[P\to\infty]{} ie^{-Kz-iK|y|}H\{(\text{sgn}\, y)K\} -u_1(P)\int_C \Phi_3'\frac{\partial\phi}{\partial n}dS-u_3(P)\int_C \Phi_{\text{III}}\frac{\partial\phi}{\partial n}dS+\cdots\,. \qquad (2.2.6)$$

The meaning of notations is like as the preceding. The last discussions of the variational formula are the same as in the three-dimensional case.

3. Integral equation for Kotchin's function[1,4,5,6,12]

3.1. *Three-dimensional Problem*

The velocity potential may be represented by making use of $S(P, Q)$ as follows:

$$\phi(P)=\iint_S \left(\phi\frac{\partial}{\partial n}-\frac{\partial}{\partial n}\phi\right)S(P, Q)dS_Q\,, \tag{3.1.1}$$

Changing the order of integration, it may also be written as

$$\phi(P)=\phi_1(P)+\phi_2(P)\,, \tag{3.1.2}$$

$$\phi_1(P)=\frac{1}{4\pi}\iint_S \left(\phi\frac{\partial}{\partial n}-\frac{\partial}{\partial n}\phi\right)\left(\frac{1}{r_1}-\frac{1}{r_2}\right)dS\,, \tag{3.1.3}$$

$$\phi_2(P)=\frac{1}{4\pi^2}\lim_{\mu\to+0}\int_{-\pi}^{\pi}\int_0^{\infty}\frac{H(k, u+\pi)\exp\{-kz+ik\tilde{\omega}\}}{k-K+\mu i}kdkdu\,, \tag{3.1.4}$$

$$H(k, u)=\iint_S \left(\phi\frac{\partial}{\partial n}-\frac{\partial}{\partial n}\phi\right)\exp\{-kz+ik\tilde{\omega}\}dS\,. \tag{3.1.5}$$

Now Neumann function of the domain outside the surface S with its mirror image with respect to the water surface is often known if its water surface is considered as rigid. Let it be $N(P, Q)$, then

$$\phi_1(P)=-\iint_S M(P, Q)\frac{\partial}{\partial n}\phi_1 dS\,, \tag{3.1.6}$$

where

$$M(P, Q)=N(P, Q)-N(P, \bar{Q})\,,$$

$$\frac{\partial}{\partial n}\phi_1=\frac{\partial}{\partial n}(\phi-\phi_2)\,.$$

Then, we may write down the velocity potential as follows;

$$\phi(P)=-\iint_S M(P, Q)\frac{\partial\phi}{\partial n}dS+\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_0^{\infty}\frac{H(k, u+\pi)\mathfrak{N}(P;k, u)}{k-K+\mu i}kdkdu\,, \tag{3.1.7}$$

$$\mathfrak{N}(P;k, u)=\exp\{-kz+ik\tilde{\omega}\}+\mathfrak{n}(P;k, u)\,, \tag{3.1.8}$$

$$\mathfrak{n}(P;k, u)=\iint_S M(P, Q)\frac{\partial}{\partial n}\exp\{-kz+ik\tilde{\omega}\}dS\,, \tag{3.1.9}$$

$$\frac{\partial}{\partial n}\mathfrak{N}(P;k, u)=0 \qquad \text{on } S\,. \tag{3.1.10}$$

Putting (3.1.7) into (3.1.5), we have the integral equation to determine Kotchin's function, that is,

$$H(p,\theta)=H_0(p,\theta)+\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_0^{\infty}\frac{H(k,u+\pi)L(p,\theta;k,u)}{k-K+\mu i}kdkdu\,, \tag{3.1.11}$$

where

$$H_0(p,\theta)=-\iint_S \mathfrak{N}(P;p,\theta)\frac{\partial\phi}{\partial n}dS\,, \tag{3.1.12}$$

$$L(p,\theta;k,u)=\iint_S \mathfrak{N}(P;k,u)\frac{\partial}{\partial n}\exp\{-pz+ip\tilde{\omega}(\theta)\}dS\,. \tag{3.1.13}$$

This is of the second type of Fredholm's equation so that it may have a unique solution. In general, to solve this equation numerically is difficult and is not the present problem. Here we will proceed to obtain some interesting relations by applying (3.1.7) and (3.1.11).

Firstly, multiplying boundary value to (3.1.7) and integrating over S, we have

$$f_{ij}=-\iint_S \phi_j\frac{\partial\phi_i}{\partial n}dS=-\iint_S \phi_{i0}\frac{\partial\phi_j}{\partial n}dS+\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_0^{\infty}\frac{H_j(k,u+\pi)H_{i0}(k,u)}{k-K+\mu i}kdkdu\,, \tag{3.1.14}$$

where

$$\phi_{i0}(P)=-\iint_S M(P,Q)\frac{\partial\phi_i}{\partial n}dS\,, \tag{3.1.15}$$

$$H_{i0}(k,u)=-\iint_S \mathfrak{N}(P;k,u)\frac{\partial\phi_i}{\partial n}dS\,. \tag{3.1.16}$$

Secondly, putting d for j in (3.1.14) and adding a simple term, we have

$$H_i(K,\alpha)=H_{i0}(K,\alpha)+\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_0^{\infty}\frac{H_d(K,\alpha;k,u+\pi)H_{i0}(k,u)}{k-K+\mu i}kdkdu\,, \tag{3.1.17}$$

and putting K,α for p,θ in (3.1.11), we have also very similar formula as the above, that is

$$H(K,\alpha)=H_0(K,\alpha)+\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_0^{\infty}\frac{H(k,u+\pi)L(K,\alpha;k,u)}{k-K+\mu i}kdkdu\,, \tag{3.1.18}$$

where it is to be noticed that

$$H_{d0}(K,\alpha;k,u)=-\iint_S \mathfrak{N}(P;k,u)\frac{\partial\phi_d(K,\alpha)}{\partial n}dS=L(K,\alpha;k,u)\,. \tag{3.1.19}$$

These are similar to Kramers-Kronig relations[9] and usefull to estimate the second approximation to the first one[7].

3.2. *Two-dimensional Problem*

The velocity potential is to be

$$\phi(P)=\phi_1(P)+\phi_2(P)\,, \tag{3.2.1}$$

$$\phi_1(P)=\frac{1}{2\pi}\int_C\left(\phi\frac{\partial}{\partial n}-\frac{\partial}{\partial n}\phi\right)\log\frac{r_2}{r_1}dS\,, \tag{3.2.2}$$

$$\phi_2(P)=\frac{1}{2\pi}\lim_{\mu\to+0}\int_{-\infty}^{\infty}\frac{H(-k)e^{-|k|x+iky}}{|k|-K+\mu i}dk\,, \tag{3.2.3}$$

$$H(k)=\int_C\left(\phi\frac{\partial}{\partial n}-\frac{\partial}{\partial n}\phi\right)e^{-k|x|+iky}dS\,. \tag{3.2.4}$$

If there is Neumann function stated as the preceding section, we may write as (3.1.6), namely,

$$\phi_1(P)=-\int_C M(P,Q)\frac{\partial}{\partial n}(\phi-\phi_2)dS\,, \tag{3.2.5}$$

then putting this into (3.2.1) with (3.2.3) we have finally

$$\phi(P)=-\int_C M(P,Q)\frac{\partial\phi}{\partial n}dS+\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{H(-k)\mathcal{N}(P;k)}{|k|-K+\mu i}dk\,, \tag{3.2.6}$$

where

$$\mathcal{N}(P,k)=e^{-|k|x+iky}+\mathfrak{N}(P,k)\,, \tag{3.2.7}$$

$$\mathfrak{N}(Q,k)=\int_C M(Q,P)\frac{\partial}{\partial n}e^{-|k|x+iky}dS\,, \tag{3.2.8}$$

and

$$\frac{\partial}{\partial n}\mathcal{N}(P,k)=0 \qquad \text{on } C\,\cdot \tag{3.2.9}$$

The integral equation for $H(k)$ is obtained by putting (3.2.6) into (3.2.4) and we get

$$H(p)=H_0(p)+\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{H(-k)L(p,k)}{|k|-K+\mu i}dk\,, \tag{3.2.10}$$

where

$$H_0(k)=-\int_C\mathcal{N}(P,k)\frac{\partial\phi}{\partial n}dS=\int_C\left(\phi_0\frac{\partial}{\partial n}-\frac{\partial}{\partial n}\phi_0\right)e^{-|k|x+iky}dS\,, \tag{3.2.11}$$

$$\phi_0(P)=-\int_C M(P,Q)\frac{\partial\phi}{\partial n}dS\,, \tag{3.2.12}$$

and

$$L(p,k)=\int_C\mathcal{N}(P,k)\frac{\partial}{\partial n}e^{-|p|x+ipy}dS\,. \tag{3.2.13}$$

In the same way as the preceding the force coefficients becomes

$$f_{ij}=-\int_C\phi_{i0}\frac{\partial}{\partial n}\phi_j dS+\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{H_j(-k)H_{i0}(k)}{|k|-K+\mu i}dk\,, \tag{3.2.14}$$

and

$$H(K)=H_0(K)+\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{H_d(K,-k)H_0(k)}{|k|-K+\mu i}dk, \tag{3.2.15}$$

$$H(K)=H_0(K)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{H(-k)H_{d0}(K,k)}{|k|-K+\mu i}dk, \tag{3.2.16}$$

where

$$H_{d0}(K,k)=L(K,k). \tag{3.2.17}$$

4. Boundary Value Problem as Extremum[8,10)]

4.1. *Pressure distribution*

A. H. Flax introduces the variational principle into the theory of thin wing to solve its boundary value problem by making use of Rayleight-Ritz's method[10)]. His principle is directly applicable to the present problem for the pressure distribution pulsating on the water surface, and it is easier than his case because of the reciprocity of the velocity potential and no use of the reverse flow potential.

Now, let us consider the pressure distribution $p(x, y)\exp(i\omega t)$ over a part S of the water surface, and the resulting velocity potential $i\omega\phi\exp(i\omega t)$, then they must satisfy the following equation at the water surface

$$\left(K+\frac{\partial}{\partial z}\right)\phi(x,y,0)=\begin{cases}\dfrac{1}{\rho g}p(x,y) & \text{on } S\\ 0 & \text{outside of } S\end{cases}. \tag{4.1.1}$$

Hence, using the fundamental singularity (2.1.1), we have the representation

$$\phi(P)=-\frac{1}{\rho g}\iint_S p(Q)S(P,Q)dSQ, \qquad z'=0. \tag{4.1.2}$$

The boundary value problem is to determine p so as to satisfy the given vertical velocity on S.

It is easy to see the reciprocity

$$\iint_S p_i\frac{\partial}{\partial z}\phi_j dS=\iint_S p_j\frac{\partial}{\partial z}\phi_i dS. \tag{4.1.3}$$

Then, let us consider the integral

$$I=\iint_S p\left(\frac{\partial\phi}{\partial z}-2\frac{\partial f}{\partial z}\right)dS, \tag{4.1.4}$$

where $\partial f/\partial z$ is the given boundary value, ϕ is supposed to be any potential resulting from the pressure p, and f is the accurate potential of the pressure π. Taking the variation of I and making use of (4.1.3), we have

$$\delta I=2\iint_S \delta p\left(\frac{\partial\phi}{\partial z}-\frac{\partial f}{\partial z}\right)dS\,, \tag{4.1.5}$$

then if $\partial\phi/\partial z$ equals $\partial f/\partial z$, I equals zero, and inversely if δI equals zero under arbitrary variation of p, $\partial\phi/\partial z$ equals $\partial f/\partial z$. Namely, the boundary value problem is equivalent to the extremum problem of I. Thus we may use Rayleigh-Ritz's procedure to solve the present problem.

Although this numerical process is practically the same as the usual one in which we approximate the boundary value by making use of any set of orthogonal functions, but this is basically rational especially as the method to obtain approximate force coefficients.

4.2. *Gauss's integral*

The preceding method is unfortunately not applicable to the displacement ship but we may find Gauss's integral which is made use of to solve Dirichlet Problem in the text book of the potential theory[3]. It is the integral as

$$G=\int_C \sigma(2f-\phi)dS\,, \tag{4.2.1}$$

where ϕ is any comparable potential of the source distribution σ over C and f is the given boundary value. It is easily seen that the extremum of this integral is equivalent to Dirchlet problem. In the same way, we may consider for Neumann Problem the following integral

$$N=\int_C \mu\left(\frac{\partial\phi}{\partial n}-2\frac{\partial f}{\partial n}\right)dS\,, \tag{4.2.2}$$

where ϕ is any potential of the doublet distribution to be determined and $\partial f/\partial n$ is the given boundary value, but this integral is usually improper and more difficult to evaluate numerically than the former. Fortunately, since Neumann problem is converted to Dirichlet Problem in the two-dimensional problem by introducing the stream function, we may make use of Gauss's integral in the present case. The integral to be extremized is

$$G'=\int_C \gamma(2f-\psi)dS\,, \tag{4.2.3}$$

where ψ is any stream function by the circulation distribution γ and f is the given boundary value.

These two integrals may be equally usefull to the present problem but there is an inconvenient point. That is, what the extremum of these integral gives is not always the force coefficient as it does in the preceding section especially in the case of the floating body. In fact, if we define

$$\text{Extrem.}\,(N)=N_0=-\int_C \mu\frac{\partial\phi}{\partial n}dS\,, \tag{4.2.5}$$

$$\text{Extrem.}\,(G')=G_0'=-\int_C \gamma\phi dS\,, \tag{4.2.6}$$

considering μ, ϕ and γ, ϕ is the correct value, since

$$\left.\begin{aligned} &\mu=\phi_+-\phi_-\,, \\ &\gamma=\frac{\partial}{\partial S}(\phi_+-\phi_-)\,, \end{aligned}\right\} \tag{4.2.7}$$

where suffix + − means the value at just outside and inside of C, we may calculate them as follows;

$$N_0=G_0'=-\int_C \phi_+\frac{\partial\phi}{\partial n}dS+\int_C \phi_-\frac{\partial\phi}{\partial n}dS\,, \tag{4.2.8}$$

The first term of the right hand side is the force and the second term the force from the inside of C when we assume that C is a very thin shell bowl immersed in the water, so that the force acting on the floating ship may not be estimated directly from N and G' except the special case. The heaving motion is one of such special cases, and we have

$$\begin{aligned} \int_C \phi_-\frac{\partial\phi}{\partial n}dS &= -\int_C \left\{\phi_-\frac{\partial}{\partial n}\left(z-\frac{1}{K}\right)-\left(z-\frac{1}{K}\right)\frac{\partial}{\partial n}\phi_-\right\}dS+\int_C\left(z-\frac{1}{K}\right)\frac{\partial z}{\partial n}dS \\ &= \int_C\left(z-\frac{1}{K}\right)\frac{\partial z}{\partial n}dS\,, \end{aligned} \tag{4.2.10}$$

because both $(z-1/K)$ and ϕ_- are regular inside C and satisfy the water surface condition. Other cases are when the ship is fully submerged and the motion is translational and then this term becomes simply the displacement volume.

4.3. *Examples of Application*

There may be proposed another type of such integral, namely,

$$J=\iint_S \phi\left(\frac{\partial\phi}{\partial n}-2\frac{\partial f}{\partial n}\right)dS\,. \tag{4.3.1}$$

of which extremum problem is also equivalent to the boundary value problem, and its extremum equals the force.

As an example of its utility, let us consider a semi-submerged circular cylinder of unit radius in heaving motion with very high frequency. Assume the potential take the following form,

$$\left.\begin{aligned} &\phi=\phi_1+A\phi_2\,, \qquad y=r\cos\theta\,, \qquad z=r\sin\theta\,, \\ &\phi_1=\frac{\sin\theta}{r}-\frac{\cos 2\theta}{Kr^2}\,, \qquad \frac{\partial f}{\partial n}=\frac{\partial f}{\partial r}=-\sin\theta\,, \end{aligned}\right\} \tag{4.3.2}$$

$$\phi_2 = \lim_{\mu \to +0} \int_0^\infty \frac{e^{-kx}\cos ky}{k-K+\mu i} dk ,$$

where A is assumed as a small parameter to be determined and ϕ_1 is considered to be the predominant part of ϕ at high frequency[8], then

$$J = 2(\phi_1, f) - (\phi_1, \phi_1) + 2A(\phi_2, f-\phi_1) - A^2(\phi_2, \phi_2) , \tag{4.3.3}$$

where

$$(\phi_i, \phi_j) = (\phi_j, \phi_i) = -\int_C \phi_i \frac{\partial}{\partial n}\phi_j dS , \tag{4.3.4}$$

then the extremum condition becomes

$$\frac{1}{2}\frac{\delta I}{\delta A} = (\phi_2, f-\phi_1) - A(\phi_2, \phi_2) = 0 , \tag{4.3.5}$$

namely

$$A = (\phi_2, f-\phi_1)/(\phi_2, \phi_2)) . \tag{4.3.6}$$

Putting this value into (4.3.3), we have

$$\text{Extrem.}\,(J) = f_{3,3} = 2(\phi_1, f) - (\phi_1, \phi_1) + (\phi_2, f-\phi_1)^2/(\phi_2, \phi_2) . \tag{4.3.7}$$

Calculating these terms, they are

$$\left.\begin{aligned} &(\phi_1, \phi_1) = \frac{\pi}{2} + \frac{2}{K} + \frac{\pi}{K^2} , \qquad (\phi_1, f) = \frac{\pi}{2} + \frac{2}{3K} , \\ &2(\phi_1, f) - (\phi_1, \phi_1) = \frac{\pi}{2}\left(1 - \frac{4}{3\pi K} - \frac{1}{K^2}\right) , \end{aligned}\right\} \tag{4.3.8}$$

$$\left.\begin{aligned} &\phi_2 \xrightarrow[K\to\infty]{} -\frac{1}{K}\phi_1 - \pi i e^{-Kx-iK|y|} , \qquad (\phi_2, \phi_2) \fallingdotseq -i\pi^2 e^{-2iK} , \\ &(\phi_2, f-\phi_1) \fallingdotseq -\frac{4\pi i}{K^2} e^{-iK}\left(1 + \frac{i}{3\pi}e^{iK}\right) , \\ &A \fallingdotseq \frac{4}{\pi K^2} e^{iK}\left(1 + \frac{i}{3\pi}e^{iK}\right) , \\ &\phi \xrightarrow[K\to\infty]{} \phi_1 - \frac{4i}{K^2}\left(1 + \frac{i}{3\pi}e^{iK}\right) e^{-Kx+iK(1-|y|)} . \end{aligned}\right\} \tag{4.3.9}$$

Accordingly, we have

$$f_{3,3} \fallingdotseq \frac{\pi}{2}\left(1 - \frac{4}{3\pi K} - \frac{4}{K^2}\right) + \frac{16}{\pi^2 K^4} e^{2iK}\left(1 + \frac{i}{3\pi}e^{iK}\right)^2 + \cdots , \tag{4.3.10}$$

so that the coefficient of the added mass k may be

$$k \doteqdot 1-\frac{4}{3\pi K}+\cdots . \tag{4.3.11}$$

The results (4.3.10) and (4.3.12) and the similar ones of the following example are the same as ones except smaller terms[8,11] obtained by F. Ursell after very long arguments.

For the other example, let us consider a heaving semi-submerged sphere at high frequency, and assume the potential as

$$\left.\begin{aligned}&\phi=\phi_1+A\phi_2\,, \qquad \partial f/\partial r=-\cos\theta\\ &\phi_1=\frac{\cos\theta}{2r^2}+\frac{3\cos^2\theta-1}{2Kr^3}\,, \qquad z=r\cos\theta\,, \qquad \sqrt{x^2+y^2}=r\sin\theta=\rho\\ &\phi_2=\int_0^\infty\frac{e^{-kz}J_0(kr\sin\theta)}{k-K+\mu i}kdk\xrightarrow[K\to\infty]{}-\frac{2}{K}\phi_1+\sqrt{\frac{2\pi K}{i\rho}}\,e^{-Kz-iK\rho}\end{aligned}\right\} \tag{4.3.13}$$

$$(\phi_1,\phi_1)=\frac{\pi}{3}\left(1+\frac{15}{8K}+\frac{18}{5K^2}\right), \qquad 2(\phi,f)-(\phi_1,\phi_1)=\frac{\pi}{3}\left(1-\frac{3}{8K}-\frac{18}{5K^2}\right), \tag{4.3.14}$$

$$\left.\begin{aligned}&(\phi_2,\phi_2)\doteqdot 2\pi^2Ke^{-2iK}\,, \qquad (\phi_2,f-\phi_1)\doteqdot 3\sqrt{\frac{2\pi^3}{iK^3}}\,e^{-iK}+\frac{3\pi}{4K^2}\\ &A\doteqdot\frac{3e^{iK}}{\sqrt{2\pi iK^5}}+\frac{3e^{2iK}}{8\pi K^3}\,,\end{aligned}\right\} \tag{4.3.15}$$

and

$$\left.\begin{aligned}&\phi\xrightarrow[K\to\infty]{}\phi_1+\frac{3}{iK^2\sqrt{\rho}}\,e^{-Kz+iK(1-\rho)}\,,\\ &k\doteqdot\frac{1}{2}\left(1-\frac{3}{8K}-\cdots\right).\end{aligned}\right\} \tag{4.3.16}$$

Conclusion

Considering the force acting on and the boundary value problem of the floating ship without advance speed, we have the conclusion as follows:

There are the mutual relations between forces, dampings and the diffracted wave is completely determined from the wave of the radiation potentials especially in two-dimensional problem. The first term of the asymptotic expansion of Neumann function which solves the present boundary value problem is of course the radiated wave of which amplitude is the diffraction potential and the second term is the doublet potential with vertical axis and its strength proportional to the heaving potential. The variation of forces acting on the ship by a slight change of its form is estimated if we know the velocity around the ship surface and for example the added mass coefficient increases as the surface swells slightly at the part where the local velocity is greater than the oscillating velocity.

The boundary value problem is converted to the problem to solve the integral equation for Kotchin's function and the relation between the force and this function similar to Kramers-Kronig's one is obtained and will be usefull to the approximate evaluation of forces and waves.

The method to solve the boundary value problem as the extremum of some integral which is proposed by A. H. Flax in the wing theory and Gauss in the potential theory is applicable to the present problem and proposes a sound basis of the approximate evaluation of the potential.

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