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A Contribution to the Theory of Two-Dimensional Hydro-Planing

(Dedicated to Professor T. Tanegashima, Professor M. Honda, Professor S. Murai and Assistant Professor E. Hattori)

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Abstract

The authors propose to solve the integral equation of a planing plate ordinate instead of the one of the plate inclination. This method enables us to treat wider class of the pressure distribution which does not always satisfy Kutta-Joukowski condition, for example, eigen-function of the integral equation of the plate inclination, which is verified to be the influence function of splash. The solutions are obtained and verified to agree well with the preceding results. This boundary value problem may be deduced from the variational principle of an integral which becomes Lagrangean of the system in its stationary point.

1. Introduction

In airplane wing theory, A. H. Flax introduced a variational method to solve the boundary value problem and showed that it was a systematic means of obtaining various approximate solutions.[12] However, F. Ursell and G. N. Ward showed that this method was applicable only for the linearized theory.[11] There is the other way to construct a variational method for such problems which the one of the authors proposed. This method may be applied to the problems of bodies having various shapes and not limited to the linear boundary value problem[14] and moreover permits us to obtain eigen-solutions which do not satisfy with Kutta-Joukowski condition at tail end. The authors try to apply this variational method to the two-dimensional hydroplaning problem and calculate solutions numerically. In way of these calculations and formulations, they find from the reverse flow relations that the eigen function so obtained is the influence function of splash.

2. Direct Flow[1)~8)]

Let us consider a water flow around a planing plate of infinite breadth on the surface

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of infinitely deep water flow as shown in Figure 1. As well known, this water flow is described approximately by the velocity potential of some pressure distribution over the water surface. The condition of water surface is

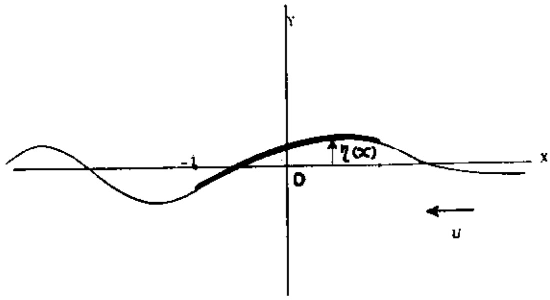

Fig. 1. Co-ordinate system.

$$\lim_{y\to -0} Re\left\{\left(\frac{d}{dz}+i\gamma\right)f(z)\right\}=\lim_{y\to -0}\left(\frac{\partial\phi}{\partial y}-\gamma\phi\right)$$
$$=\begin{cases}-p(x) & \text{for} \quad |x|<1\\ 0 & \text{for} \quad |x|>1\end{cases} \tag{2.1}$$

where the co-ordinate system is normalized so the half-length of the plate as to be unit and the velocity to be unit, then $\gamma=gl/U^2$ and the pressure to be $\rho U^2 p(x)$, (ρ the water density and g the gravity constant).

$f(z)$ means the complex velocity potential, φ the velocity potential, ϕ the stream function and $z=x+iy$.

They are expressed as follows:

$$f(z)=\varphi+i\phi=\frac{1}{\pi i}\int_{-1}^{1} p(\xi)S\{\gamma(z-\xi)\}d\xi \tag{2.2}$$

where

$$S(\gamma z)=\lim_{\mu\to +0}\int_0^\infty \frac{e^{-ikz}}{k-\gamma-\mu i}dk \tag{2.3}$$

The problem to be solved is to determine the pressure distribution $p(x)$ for given planing plate shape. This boundary condition can be written down in two ways, namely,

$$\left.\begin{aligned}&\phi(x, 0)=-\eta(x)\,,\\ \text{and}\quad &\left.\frac{\partial\varphi}{\partial y}\right|_{y=0}=-\left.\frac{\partial\phi}{\partial x}\right|_{y=0}=\frac{\partial\eta}{\partial x}\end{aligned}\right\} \quad \text{for} \quad |x|<1 \tag{2.5}$$

where $\eta(x)$ means the ordinate of planing plate.

Putting each condition into (2.2) and its derivative, we obtain two integral equations;

$$\eta(x)=\frac{1}{\pi}\int_{-1}^{1} p(\xi)S_c\{\gamma(x-\xi)\}d\xi \tag{2.5}$$

$$\frac{\partial\eta(x)}{\partial x}=\frac{1}{\pi}\int_{-1}^{1} p(\xi)\left[\frac{1}{\xi-x}+\gamma S_s\{\gamma(x-\xi)\}\right]d\xi \tag{2.6}$$

where

$$S_c(\gamma x)+iS_s(\gamma x)=S(\gamma x) \tag{2.7}$$

In the former equation, the kernel has logarithmic singularity at origin, so that it may have unique solution and the latter is a singular equation and not uniquely determined, so that the Kutta-Joukowski condition should be added to have a unique solution. From this consideration it may be clear that the eigen solution in the latter case will be the solution for $\eta(x)$=constant of the former case.

3. Reversed Flow[12,13]

Let us consider the water flow reversed the stream direction against the direct flow around the same planing plate and call it the reversed flow.

The water surface condition is the same as of the direct flow, namely,

$$\lim_{y\to -0} Re\left(\frac{d}{dz}+i\gamma\right)\tilde{f}(z)=\begin{cases}-\tilde{p}(x) & \text{for} \quad |x|<1 \\ 0 & \text{for} \quad |x|>1\end{cases} \tag{3.1}$$

and the complex velocity potential is written as

$$\tilde{f}(z)=\frac{1}{\pi i}\int_{-1}^{1}\tilde{p}(\xi)\tilde{S}\{\gamma(z-\xi)\}d\xi \tag{3.2}$$

where

$$\tilde{S}(\gamma z)=\lim_{\mu\to+0}\int_0^\infty \frac{e^{-ikz}}{k-\gamma+\mu i}dk \tag{3.3}$$

and sign ~ over the letter means the reversed flow quantity. The kernel $\tilde{S}$ has the wave train on the positive side of the x-axis and differs from the direct flow kernel S only by the wave term as follows;

$$\tilde{S}(\gamma z)=S(\gamma z)-2\pi i e^{-i\gamma z} \tag{3.4}$$

and that they are adjoint with each other in the following sense

$$S^*(\gamma z)=\tilde{S}(-\gamma \bar{z}) \tag{3.5}$$

where S^* means the complex conjugate value of S.

The boundary condition is now

$$\left.\begin{aligned}&\tilde{\phi}(x,0)=\eta(x)\\ \text{or}\quad &\left.\frac{\partial\tilde{\varphi}}{\partial y}\right|_{y=0}=-\left.\frac{\partial\tilde{\phi}}{\partial x}\right|_{y=0}=-\frac{\partial\eta}{\partial x}\end{aligned}\right\} \tag{3.6}$$

and the integral equation is

$$\eta(x)=-\frac{1}{\pi}\int_{-1}^{1}\tilde{p}(\xi)\tilde{S}_c\{\gamma(x-\xi)\}\,d\xi \tag{3.7}$$

or

$$\frac{\partial \eta}{\partial x}=\frac{1}{\pi}\int_{-1}^{1}\tilde{p}(\xi)\left[\frac{1}{x-\xi}+\gamma\tilde{S}_s\{\gamma(x-\xi)\}\right]d\xi \tag{3.8}$$

where

$$\tilde{S}=\tilde{S}_c+i\gamma\tilde{S}_s \tag{3.9}$$

Comparing (3.7) with (2.5) and remembering the adjoint relation (3.5) we have

$$\left.\begin{aligned} p(x)&=-\tilde{p}(-x) \quad \text{if} \quad \eta(x)=\eta(-x) \\ \text{and} \quad p(x)&=\tilde{p}(-x) \quad \text{if} \quad \eta(x)=-\eta(-x)\,. \end{aligned}\right\} \tag{3.10}$$

This duality is the fundamental property between the direct and reversed flow. The most important and basic properties between both flows are following integral identities called as the reversed flow theorem

$$\int_{-1}^{1}p_1(x)\eta_2(x)dx=-\int_{-1}^{1}\tilde{p}_2(x)\eta_1(x)dx \tag{3.11}$$

and

$$\int_{-1}^{1}p_1(x)\frac{\partial \eta_2}{\partial x}dx=\int_{-1}^{1}\tilde{p}_2(x)\frac{\partial \eta_1}{\partial x}dx \tag{3.12}$$

for p_1, $\tilde{p}_2$ corresponding boundary value η_1, η_2 respectively. p_1 and $\tilde{p}_2$ must vanish at tail end in the latter integral but not neccesary in the former.

4. Variational Principle[12),14)]

By the method of A. H. Flax, introducing the integral

$$J=\int_{-1}^{1}\{p+\tilde{p}\}\frac{\partial \eta}{\partial x}dx-\int_{-1}^{1}\int_{-1}^{1}\tilde{p}(x)p(\xi)S_s'\{\gamma(x-\xi)\}dxd\xi \tag{4.1}$$

where

$$S_s'(\gamma x)=-\frac{1}{x}+\gamma S_s(\gamma x)\,,$$

taking its variation with regard to $\tilde{p}$ and making use of the reciprocity (3.12), we have

$$\delta J=\int_{-1}^{1}\delta\tilde{p}dx\left(\frac{\partial \eta}{\partial x}-\int_{-1}^{1}pS_s'd\xi\right) \tag{4.2}$$

Then, the boundary value problem (2.6) is equivalent to the variational one of J. Here each of p and $\tilde{p}$ must vanish at each tail end because otherwise the integral does not exist.

The integral J is closely related to the linearized momentum integral and its stationary value is the pressure drag, so that this method is only applicable to very thin bodies.

The other way is to make use of the reciprocity (3.11). For this aim, introducing the integral

$$I=\int_{-1}^{1}\{p(x)-\tilde{p}(x)\}\eta(x)dx+\int_{-1}^{1}\int_{-1}^{1}\tilde{p}(x)p(\xi)S_c\{\gamma(x-\xi)\}dxd\xi \tag{4.3}$$

and taking the variation, we have

$$\delta I=-\int_{-1}^{1}\delta\tilde{p}dx\left\{\eta(x)-\int_{-1}^{1}pS_c d\xi\right\} \tag{4.4}$$

which is equivalent to the equation (2.5).

In this form there is no difficulty of the existence of the integral or non applicability to thick body. To understand the dynamical meaning of I, let us consider its stationary value, then by making use of the water surface conditions (2.1) and (3.1), we have

$$[I]=-\int_{-1}^{1}p\phi dx=\int_{-\infty}^{\infty}(\phi_y-\gamma\phi)\phi dx=\int_{-\infty}^{\infty}(\phi\phi_y-\gamma\eta^2)dx \tag{4.5}$$

and by Green's theorem it becomes

$$[I]=\int\int_D \nabla\phi\nabla\phi dxdy-\gamma\int_{-\infty}^{\infty}\eta^2 dx \tag{4.6}$$

The first term of the right hand side is the kinetic energy of the disturbed flow and the second term is the potential energy and both are infinite but the difference, Lagrangean or kinetic potential,[16] is finite because they are equal each other in the regular wave train.

Thus it becomes clear that the proposed method is founded on the stationary property of Lagrangean, so that it may be applied to the wider classes of the problem than Flax's method except non-linear problems of free surface.

5. Relations between Various Solutions

Before entering numerical calculations, let us obtain some mutual relations between solutions deduced from the formulas (3.10), (3.11) and (3.12).

Firstly, since all boundary conditions may be represented by Fourier series, let the solutions of (2.5) and (3.7) for $\eta_n=\cos n\theta$ be

$$\left.\begin{aligned} P_n(x)&=\frac{1}{\sin\theta}\sum_{m=0}^{\infty}A_m{}^n\cos m\theta \\ \tilde{P}_n(x)&=\frac{1}{\sin\theta}\sum_{m=0}^{\infty}\tilde{A}_m{}^n\cos m\theta \end{aligned}\right\}\quad x=-\cos\theta \tag{5.1}$$

then, by (3.11), we have adjoint relations

$$\left.\begin{array}{l} A_0{}^0=-\tilde{A}_0{}^0 \\ A_n{}^0=-2\tilde{A}_0{}^n\,, \quad 2A_0{}^n=-\tilde{A}_n{}^0\,, \quad \text{for} \quad n\neq 0 \\ A_m{}^n=-\tilde{A}_n{}^m \quad \text{for} \quad n,\, m\neq 0 \end{array}\right\} \tag{5.2}$$

and that by (3.10)

$$A_m{}^n=(-1)^{n+m+1}\tilde{A}_m{}^n\,, \tag{5.3}$$

so that

$$\left.\begin{array}{l} A_m{}^0=2(-1)^m A_0{}^m \quad \text{for} \quad m\neq 0 \\ A_m{}^n=(-1)^{n+m}A_n{}^m \quad \text{for} \quad m,\, n\neq 0 \end{array}\right\} \tag{5.4}$$

Secondly, let corresponding solutions satisfying Kutta-Joukowski condition to these be

$$\left.\begin{array}{l} p_n(x)=\dfrac{1}{\sin\theta}\sum\limits_{m=0}^{\infty} a_m{}^n \cos m\theta \\ \tilde{p}_n(x)=\dfrac{1}{\sin\theta}\sum\limits_{m=0}^{\infty} \tilde{a}_m{}^n \cos m\theta \end{array}\right\} \tag{5.5}$$

Their boundary conditions are

$$\frac{\partial}{\partial x}\eta_n(x)=v_n(x)=-n\frac{\sin n\theta}{\sin\theta} \tag{5.6}$$

and Kutta-Joukowski condition means

$$\sum_{m=0}^{\infty} a_m{}^n=0 \qquad \text{and} \qquad \sum_{m=0}^{\infty}(-1)^m\tilde{a}_m{}^n=0 \tag{5.7}$$

By the same way as the above, we have

$$a_m{}^n=(-1)^{n+m+1}\tilde{a}_m{}^n \tag{5.8}$$

and

$$\frac{1}{2n+1}\sum_{\mu=0}^{m} a_{2\mu}^{2n+1}=\frac{1}{2m+1}\sum_{\mu=0}^{n} a_{2\mu}^{2m+1}, \quad \text{etc.} \tag{5.9}$$

remembering the identity

$$\frac{\sin n\theta}{\sin\theta}=\sum_{\mu=0}^{n/2}\varepsilon_{n-2\mu-1}\cos(n-2\mu-1)$$

where

$$\varepsilon_0=1\,, \quad \varepsilon_n=2 \quad \text{for} \quad n>0$$

Thirdly, since p_n and P_n differs only with a constant in boundary values, we may write

$$\left.\begin{array}{l} p_n(x)=\alpha_n P_0(x)+P_n(x) \\ \tilde{p}_n(x)=\tilde{\alpha}_n\tilde{P}_0(x)+\tilde{P}_n(x) \end{array}\right\} \tag{5.10}$$

where α_n and $\tilde{\alpha}_n$ must be determined by the conditions (5.7), namely,

$$\left.\begin{aligned}\alpha_n&=-\sum_{m=0}^{\infty}A_m{}^n\Big/\sum_{m=0}^{\infty}A_m{}^0\\ \tilde{\alpha}_n&=-\sum_{m=0}^{\infty}(-1)^m\tilde{A}_m{}^n\Big/\sum_{m=0}^{\infty}(-1)^m\tilde{A}_m{}^0\end{aligned}\right\}\tag{5.11}$$

and that by the adjoint relation (5.3), we know

$$\tilde{\alpha}_n=(-1)^n\alpha_n\tag{5.12}$$

Making use of this value, when we know the former solution we may obtain the latter solution as follows;

$$a_m{}^n=\alpha_n A_m{}^0+A_m{}^n\tag{5.13}$$

Since the surface elevation corresponding $p_n(x)$ is $\eta=\alpha_n+\cos n\theta$, the coefficient α_n is the weighted mean elevation, that is,

$$\frac{1}{\pi}\int_{-1}^{1}(\alpha_n+\cos n\theta)\frac{dx}{\sqrt{1-x^2}}=\alpha_n$$

Another equality may be deduced by making use of both reciprocities (3.11) and (3.12) as

$$-\int_{-1}^{1}p_n(x)dx=\int_{-1}^{1}p_n v_1 dx=\int_{-1}^{1}\tilde{p}_1 v_n dx=\int_{-1}^{1}\tilde{P}_0(x)(\alpha_n+\cos n\theta)dx\tag{5.14}$$

from which we have

$$\alpha_n A_0{}^0=-\int_{-1}^{1}\tilde{p}_1 v_n dx+\int_{-1}^{1}\tilde{P}_0(x)\left(\int_{-1}^{x}v_n dx+1\right)dx\tag{5.14}$$

which permits us to calculate α_n if we know $\tilde{P}_0(x)$ and $\tilde{p}_1$.

Fourthly, let us obtain the representation of the following value

$$\left.\begin{aligned}\sigma_n&=\sum_{m=0}^{\infty}(-1)^m a_m{}^n\\ \tilde{\sigma}_n&=\sum_{m=0}^{\infty}\tilde{a}_m{}^n=(-1)^{n+1}\sigma_n\end{aligned}\right\}\tag{5.15}$$

which is necessary to estimate the splash resistance as seen in the following paragraph.

It is also written as

$$\sigma_n=2\sum_{m=0}^{\infty}a_{2m}^n=-2\sum_{m=0}^{\infty}a_{2m+1}^n\tag{5.16}$$

because of the condition (5.7).

Then, in the same way as the above, we obtain successively

$$\frac{\sigma_n}{2}=\lim_{m\to\infty}\sum_{m=0}^{m}a_{2m}^n=\lim_{m\to\infty}\frac{1}{\pi}\int_{-1}^{1}p_n(x)\frac{\sin(2m+1)\theta}{\sin\theta}d\theta$$

$$=-\frac{1}{\pi}\lim_{m\to\infty}\frac{1}{2m+1}\int_{-1}^{1}p_n v_{2m+1}dx$$
$$=-\frac{1}{\pi}\lim_{m\to\infty}\frac{1}{2m+1}\int_{-1}^{1}\tilde{p}_{2m+1}v_n dx$$
$$=-\frac{1}{\pi}\lim_{m\to\infty}\left(\frac{\tilde{\alpha}_{2m+1}}{2m+1}\int_{-1}^{1}\tilde{P}_0 v_n dx+\frac{1}{2m+1}\int_{-1}^{1}\tilde{P}_{2m+1}v_n dx\right).$$

The second term of the last line tends to zero, but the first term tends to

$$\lim_{m\to\infty}\frac{\tilde{\alpha}_{2n+1}}{2m+1}=-\lim_{m\to\infty}\frac{\alpha_{2m+1}}{2m+1}=-\frac{\sigma_1}{2A_0{}^0} \tag{5.17}$$

by making use of (5.14).

Thus we have finally

$$\sigma_n=\frac{\sigma_1}{\pi A_0{}^0}\int_{-1}^{1}\tilde{P}_0(x)v_n(x)dx \tag{5.18}$$

which permit us to calculate σ_n when we know $\tilde{P}_0$ and σ_1 which is the value for the flat plate.

Fifthly and lastly, let us consider the solutions for $\eta=\cos\gamma x$ and $\sin\gamma x$ and write

$$\left.\begin{aligned}P_c(x)&=\frac{1}{\sin\theta}\sum_{m=0}^{\infty}A_m{}^c\cos m\theta\\ P_s(x)&=\frac{1}{\sin\theta}\sum_{m=0}^{\infty}A_m{}^s\cos m\theta\end{aligned}\right\} \tag{5.19}$$

Corresponding results satisfying Kutta-Joukowski condition may be written as

$$p_{c,s}(x)=\frac{1}{\sin\theta}\sum_{m=0}^{\infty}a_m^{c,s}\cos m\theta=\alpha_{c,s}P_0(x)+P_{c,s}(x) \tag{5.20}$$

Since

$$e^{i\gamma x}=\sum_{n=0}^{\infty}\varepsilon_n i^n J_n(\gamma)\cos n\theta$$

where J_n means Bessel function, we have clearly

$$\left.\begin{aligned}A_m{}^c&=\sum_{n=0}^{\infty}\varepsilon_n(-1)^n A_m^{2n}J_{2n}(\gamma)\\ A_m{}^s&=-2\sum_{n=0}^{\infty}(-1)^n A_m^{2n+1}J_{2n+1}(\gamma)\end{aligned}\right\} \tag{5.21}$$

and corresponding equations for $a_m^{c,s}$.

This means that these solutions can be calculated from all p_n or P_n solutions. We call these diffraction potentials in the following.

6. Lift, Moment and Drag[8],[9]

The lift L, moment M and pressure drag D acting on the planing plate are given as follows;

$$\frac{L}{\rho U^2 l}=C_L=\int_{-1}^{1} p(x)dx \tag{6.1}$$

$$\frac{M}{\rho U^2 l^2}=C_M=\int_{-1}^{1} p(x)xdx \tag{6.2}$$

and

$$\frac{D_p}{\rho U^2 l}=C_{D_p}=\int_{-1}^{1} p(x)\frac{\partial \eta}{\partial x}dx \tag{6.3}$$

The drag is composed of two parts, that is, the wave-making and splash,

$$D_p=D_w+D_s \tag{6.4}$$

The wave drag is proportional to the square of the amplitude of the regular wave train and given as

$$\frac{D_w}{\rho U^2 l}=C_w=\gamma|F(\gamma)|^2 \tag{6.5}$$

where

$$F(\gamma)=\int_{-1}^{1} p(x)e^{i\gamma x}dx \tag{6.6}$$

On the other hand, the splash drag is given as

$$\frac{D_s}{\rho U^2 l}=C_s=\frac{\pi}{4}\sigma^2 \tag{6.7}$$

where

$$\sigma=\sum_{n=0}^{\infty}(-1)^n a_n \tag{6.8}$$

when

$$p(x)=\frac{1}{\sin\theta}\sum_{n=0}^{\infty} a_n \cos n\theta$$

Applying the reversed flow theorem (3.12), we have Munk's theorem for lift and moment as

$$C_L=-\int_{-1}^{1} \tilde{p}_1(x)\frac{\partial \eta}{\partial x}dx \tag{6.9}$$

$$C_M=\frac{1}{4}\int_{-1}^{1}\tilde{p}_2(x)\frac{\partial\eta}{\partial x}dx \tag{6.10}$$

In the same way, applying (3.11), we have similar results as

$$C_L=-\int_{-1}^{1}\tilde{P}_0(x)\eta(x)dx \tag{6.11}$$

$$C_M=\int_{-1}^{1}\tilde{P}_1(x)\eta(x)dx \tag{6.12}$$

These formulas enable us to calculate the lift and moment of the planing plate of arbitrary shape if we know only the pressure distribution of a flat plate.

Similarly, since we have for splash drag from (5.18)

$$\sigma=\frac{\sigma_1}{\pi A_0^0}\int_{-1}^{1}\tilde{P}_0\frac{\partial\eta}{\partial x}dx \tag{6.13}$$

we need only $\tilde{P}_0$. For wave drag, since we have

$$F(\gamma)=\frac{1}{i\gamma}\int_{-1}^{1}\{\tilde{p}_c(x)+i\tilde{p}_s(x)\}\frac{\partial\eta}{\partial x}dx=-\int_{-1}^{1}\{\tilde{P}_c+i\tilde{P}_s\}\eta(x)dx \tag{6.14}$$

we need only the diffraction potential.

7. Calculations

If we try to solve the boundary value problem by expanding η and p in Fourier series and applying the variational principle, the obtained equations are not other than the one of the usual Fourier expansion method. In another word, the method to solve by expanding into orthogonal series is justified by the variational principles in this case, so that the error in the drag or Lagrangean may be the least.

Preliminary calculations are carried out by Flax's method for a flat plate, putting $p(x)=a_0^1\tan\theta/2$ and making use of Miller's table.[10] The results are very good for small γ and given in Table 10 compared with other author's results.

Now, it may be convenient for calculation to decompose η into odd and even part, because numbers of the unknown of the system of linear equation reduce to half. Hence, standing c or s on the shoulder for the even or odd part respectively, we may write the equation (2.5) as

$$\left.\begin{aligned}\eta^c(x)&=\frac{1}{\pi}\int_{-1}^{1}p^c(\xi)\bar{S}_c\{\gamma(x-\xi)\}d\xi-F^s(\gamma)\cos\gamma x\\ \eta^s(x)&=\frac{1}{\pi}\int_{-1}^{1}p^s(\xi)\bar{S}_c\{\gamma(x-\xi)\}d\xi+F^c(\gamma)\sin\gamma x\end{aligned}\right\} \tag{7.1}$$

where

$$\bar{S}_c(\gamma x)=P.V.\int_0^\infty \frac{\cos kx}{k-\gamma}dk\ , \tag{7.2}$$

and

$$F^c(\gamma)+iF^s(\gamma)=\int_{-1}^{1}\{p^c(x)+ip^s(x)\}e^{i\gamma x}dx \tag{7.3}$$

These equations may be decomposed once more, if we put

$$\left.\begin{aligned} p^c(x)&=\bar{p}^c(x)+F^s(\gamma)\bar{\rho}^c(x)\\ p^s(x)&=\bar{p}^s(x)-F^c(\gamma)\bar{\rho}^s(x)\end{aligned}\right\} \tag{7.4}$$

we have four equations as follows;

$$\eta^j(x)=\frac{1}{\pi}\int_{-1}^{1}\bar{p}^j(\xi)\bar{S}_c\{\gamma(x-\xi)\}d\xi\ ,\quad j=c \text{ or } s \tag{7.5}$$

$$\left.\begin{matrix}\cos\\ \sin\end{matrix}\right\}(\gamma x)=\frac{1}{\pi}\int_{-1}^{1}\left\{\begin{matrix}\bar{\rho}^c\\ \bar{\rho}^s\end{matrix}\right\}\bar{S}_c d\xi \tag{7.6}$$

Solving these equations, determining F^c and F^s by (7.3) as

$$\left.\begin{aligned} F^c(\gamma)&=(\bar{F}^c+\bar{\varphi}^c\bar{F}^s)/(1+\bar{\varphi}^c\bar{\varphi}^s)\\ F^s(\gamma)&=(\bar{F}^s-\bar{\varphi}^s\bar{F}^c)/(1+\bar{\varphi}^c\bar{\varphi}^s)\end{aligned}\right\} \tag{7.7}$$

where

$$\left.\begin{aligned} \bar{F}^c+i\bar{F}^s&=\int_{-1}^{1}(\bar{p}^c+\bar{p}^s)e^{i\gamma x}dx\\ \bar{\varphi}^c+i\bar{\varphi}^s&=\int_{-1}^{1}(\bar{\rho}^c+\bar{\rho}^s)e^{i\gamma x}dx\end{aligned}\right\} \tag{7.8}$$

and adding them in (7.4), we have finally the solutions. The actual calculations are carried out by expanding the above equations into Fourier series as

$$\bar{\rho}^c(x)=\frac{1}{\sin\theta}\sum_{m=0}^{1}\bar{B}_{2m}\cos 2m\theta\ ,\quad \bar{\rho}^s(x)=\frac{1}{\sin\theta}\sum_{m=0}^{1}\bar{B}_{2m+1}\cos(2m+1)\theta \tag{7.9}$$

and

$$\bar{p}_n^c(x)=\frac{1}{\sin\theta}\sum_{m=0}^{1}\bar{A}_{2m}^n\cos 2m\theta\ ,\quad \bar{p}_n^s(x)=\frac{1}{\sin\theta}\sum_{m=0}^{1}\bar{A}_{2m+1}^n\cos(2m+1)\theta \tag{7.10}$$

for values

$$\eta=\cos n\theta\ ,\quad n=0,\,1,\,2,\,3.$$

The necessary coefficients

$$C_{n,m}=P.V.\int_0^\infty\frac{J_n(k)J_m(k)}{k-\gamma}dk \tag{7.11}$$

or

$$C_{2n,2m}=\frac{(-1)^{n+m}}{\pi^2}\int_0^\pi\int_0^\pi \bar{S}_c\{\gamma(\cos\theta-\cos\vartheta)\cos 2n\theta\cdot\cos 2m\vartheta d\theta d\vartheta$$

$$C_{2n+1,2m+1}=\frac{(-1)^{n+m}}{\pi^2}\int_0^\pi\int_0^\pi \bar{S}_c\{\gamma(\cos\theta-\cos\vartheta)\cos(2n+1)\theta\cos(2m+1)\vartheta d\theta d\vartheta \qquad (7.12)$$

are calculated numerically from Miller's table[10] and shown in Table 1. The number of the unknown in this case are two, and to solve them is very easy and the results are shown in Table 2 and 3.

After obtaining these coefficients, putting them into (7.8), we may obtain

$$\bar{\varphi}^c=\pi\sum_{m=0}^{1}(-1)^m\bar{B}_{2m}J_{2m}(\gamma)\ ,\quad \bar{\varphi}^s=-\pi\sum_{m=0}^{1}(-1)^m B_{2m+1}J_{2m+1}(\gamma) \qquad (7.13)$$

Table 1. Coefficient $C_{n,m}$

γ	$C_{0,0}$	$C_{0,2}=C_{2,0}$	$C_{2,2}$	$C_{1,1}$	$C_{1,3}=C_{3,1}$	$C_{3,3}$
0.05	3.0435	0.0080	0.2544	0.5233	0.0015	0.1685
0.1	2.2770	0.0170	0.2590	0.5489	0.0031	0.1704
0.2	1.4296	0.0357	0.2690	0.6026	0.0067	0.1744
0.5	0.0643	0.0759	0.3077	0.7450	0.0223	0.1886
1.0	−1.0471	0.0124	0.4048	0.7701	0.0644	0.2197
2.0	−0.8882	−0.5148	0.5264	−0.0426	0.0658	0.3376

Table 2. Coefficient $\bar{A}_m{}^n$

γ	$\bar{A}_0{}^0$	$\bar{A}_2{}^0$	$\bar{A}_0{}^2$	$\bar{A}_2{}^2$	$\bar{A}_1{}^1$	$\bar{A}_3{}^1$	$\bar{A}_1{}^3$	$\bar{A}_3{}^3$
0.05	0.3286	0.0104	0.0052	1.9660	0.9555	0.0083	0.0083	2.8537
0.1	0.4394	0.0289	0.0144	1.9318	0.9111	0.0164	0.0164	2.9343
0.2	0.7018	0.0931	0.0466	1.8653	0.8301	0.0318	0.0318	2.8683
0.5	21.9184	5.4043	2.7022	2.2911	0.6736	0.0797	0.0797	2.6603
1.0	−0.9547	−0.0293	−0.0146	1.2347	0.6656	0.1950	0.1950	2.3330
2.0	−0.7185	0.7027	0.3514	0.6062	−9.0251	−1.7579	−1.7579	1.1386

Table 3. Coefficient $\bar{B}_n$

γ	$\bar{B}_0$	$\bar{B}_1$	$\bar{B}_2$	$\bar{B}_3$
0.05	0.3284	−0.0478	0.0091	−0.0004
0.1	0.4383	−0.0910	0.0240	−0.0015
0.2	0.6943	−0.1652	0.0736	−0.0054
0.5	20.4044	−0.3260	4.9315	−0.0250
1.0	−0.7272	−0.5782	−0.2860	−0.0803
2.0	−0.4088	9.9566	−0.2704	2.3213

and

$$\left.\begin{aligned}&\bar{F}^c_{2n}=\pi\sum_{m=0}^{1}(-1)^m\bar{A}^{2n}_{2m}J_{2m}(\gamma)\,,\quad \bar{F}^s_{2n}=0\\&\bar{F}^c_{2n+1}=0\,,\quad \bar{F}^s_{2n+1}=-\pi\sum_{m=0}^{1}(-1)^m\bar{A}^{2n+1}_{2m+1}J_{2m+1}(\gamma)\end{aligned}\right\}\tag{7.14}$$

and they are shown in Table 4 and 5.

Putting these value into (7.7), we may obtain

$$\left.\begin{aligned}&F^c_{2n}=\frac{\bar{F}^c_{2n}}{1+\bar{\varphi}^c\bar{\varphi}^s}\,,\qquad F^s_{2n}=-\frac{\bar{\varphi}^s\bar{F}^c_{2n}}{1+\bar{\varphi}^c\bar{\varphi}^s}\\&F^c_{2n+1}=\frac{\bar{\varphi}^c\bar{F}^s_{2n+1}}{1+\bar{\varphi}^c\bar{\varphi}^s}\,,\qquad F^s_{2n+1}=\frac{\bar{F}^s_{2n+1}}{1+\bar{\varphi}^c\bar{\varphi}^s}\end{aligned}\right\}\tag{7.15}$$

shown in Table 6.

Lastly, we have the expansion coefficients of $P_n(x)$ in the form (5.1) by (7.4) as

$$\left.\begin{aligned}&A^{2n}_{2m}=\bar{A}^{2n}_{2m}+F^s_{2n}\bar{B}_{2m}\,,\qquad A^{2n}_{2m+1}=-F^c_{2n}\bar{B}_{2m+1}\\&A^{2n+1}_{2m}=F^s_{2n+1}\bar{B}_{2m}\,,\qquad A^{2n+1}_{2m+1}=\bar{A}^{2n+1}_{2m+1}-F^c_{2n+1}\bar{B}_{2m+1}\end{aligned}\right\}\tag{7.16}$$

which are shown in Table 7.

Then, calculating α_n by (5.11), we may obtain $a_m{}^n$ shown in Table 9 by (5.13). The results are compared with other author's results in Table 10 and Figure 2 and seen agreement to be good. Finally, Figure 3 shows the curves of $P_0(x)$ which is the eigen function of the equation (2.6) and the influence function of splash in the sense of (6.13). We may

Table 4. Coefficient $\bar{F}_n{}^c$ and $\bar{F}_n{}^s$

γ	$\bar{F}_0{}^c$	$\bar{F}_2{}^c$	$\bar{F}_1{}^s$	$\bar{F}_3{}^s$
0.05	1.0316	0.0144	−0.0750	−0.0006
0.1	1.3768	0.0376	−0.1429	−0.0024
0.2	2.1813	0.1156	−0.2595	−0.0084
0.5	64.1023	7.7466	−0.5120	−0.0393
1.0	−2.2844	−0.4809	−0.9082	−0.1262
2.0	−1.2843	−0.4248	15.6398	3.7466

Table 5. Coefficient $\bar{\varphi}^c$ and $\bar{\varphi}^s$

γ	$\bar{\varphi}^c$	$\bar{\varphi}^s$
0.05	1.0301	0.0038
0.1	1.3733	0.0143
0.2	2.1584	0.0516
0.5	59.6840	0.2479
1.0	−1.6448	0.7944
2.0	0.0122	−17.0995

Table 6. Coefficient $F_n{}^c$ and $F_n{}^s$

γ	$F_0{}^c$	$F_1{}^c$	$F_2{}^c$	$F_3{}^c$	$F_0{}^s$	$F_1{}^s$	$F_2{}^s$	$F_3{}^s$
0.05	1.0277	−0.0770	0.0143	−0.0006	−0.0039	−0.0747	−0.0001	−0.0006
0.1	1.3504	−0.1925	0.0369	−0.0032	−0.0193	−0.1402	−0.0005	−0.0023
0.2	1.9626	−0.5039	0.1040	−0.0164	−0.1013	−0.2335	−0.0054	−0.0075
0.5	4.0584	−1.9347	0.4904	−0.1484	−1.0060	−0.0324	−0.1216	−0.0025
1.0	7.4519	−4.8730	1.5687	−0.6769	−5.9194	2.9626	−1.2461	0.4115
2.0	−1.6233	0.2414	−0.5369	0.0578	−27.7570	19.7673	−9.1806	4.7354

Table 7. Coefficient A_n^m

γ	A_0^0	A_1^0	A_2^0	A_3^0	A_0^1	A_1^1	A_2^1	A_3^1
0.05	0.3273	0.0491	0.0103	0.0004	−0.0245	0.9519	−0.0007	0.0076
0.1	0.4309	0.1229	0.0284	0.0020	−0.0614	0.8936	−0.0034	0.0161
0.2	0.6314	0.3242	0.0857	0.0105	−0.1621	0.7469	−0.0172	0.0291
0.5	1.3910	1.3229	0.4430	0.1015	−0.6614	0.0429	−0.1599	0.0314
1.0	3.3497	4.3084	1.6636	0.5985	−2.1543	−2.1518	−0.8473	−0.1964
2.0	10.6289	16.1623	8.2090	3.7681	−8.0812	−11.4284	−5.3457	−2.3182

γ	A_0^2	A_1^2	A_2^2	A_3^2	A_0^3	A_1^3	A_2^3	A_3^3
0.05	0.0052	0.0007	1.9660	0.0000	−0.0002	0.0083	0.0000	2.8537
0.1	0.0142	0.0034	1.9318	0.0001	−0.0010	0.0161	−0.0001	2.9343
0.2	0.0428	0.0172	1.8649	0.0006	−0.0053	0.0291	−0.0006	2.8682
0.5	0.2215	0.1599	1.6916	0.0123	−0.0507	0.0314	−0.0123	2.6566
1.0	0.8915	0.9059	1.5911	0.1260	−0.2992	−0.1964	−0.1177	2.2787
2.0	4.1045	5.3457	3.0889	1.2463	−1.9359	−2.3336	−1.2806	1.0044

Table 8. Coefficient α_n

γ	α_0	α_1	α_2	α_3
0.05	−1.0000	−2.4132	−5.0931	−7.3918
0.1	−1.0000	−1.4460	−3.3366	−5.0481
0.2	−1.0000	−0.5673	−1.8305	−2.7490
0.5	−1.0000	0.2293	−0.6400	−0.8056
1.0	−1.0000	0.5393	−0.3543	−0.1679
2.0	−1.0000	0.7009	−0.3556	0.1173

Table 9. Coefficient a_m^n

γ	a_0^1	a_1^1	a_2^1	a_3^1	a_0^2	a_1^2	a_2^2	a_3^2
0.05	−0.8144	0.8334	−0.0256	0.0067	−1.6619	−0.2493	1.9133	−0.0021
0.1	−0.6846	0.7159	−0.0444	0.0131	−1.4237	−0.4066	1.8371	−0.0067
0.2	−0.5203	0.5630	−0.0658	0.0231	−1.1130	−0.5763	1.7081	−0.0187
0.5	−0.3425	0.3462	−0.0583	0.0536	−0.6687	−0.6867	1.4080	−0.0527
1.0	−0.3479	0.1717	0.0499	0.1264	−0.2952	−0.6024	1.0017	−0.0860
2.0	−0.6312	−0.0999	0.4082	0.3229	0.3251	−0.4014	0.1699	−0.0936

γ	a_0^3	a_1^3	a_2^3	a_3^3
0.05	−2.4197	−0.3545	−0.0764	2.8506
0.1	−2.1765	−0.6042	−0.1434	2.9241
0.2	−1.7411	−0.8621	−0.2360	2.8393
0.5	−1.1713	−1.0343	−0.3692	2.5748
1.0	−0.8616	−0.9196	−0.3970	2.1782
2.0	−0.6896	−4.2287	−0.3181	1.4462

Table 10. Comparison with other Methods

γ	By 1 term Flax's Method	By Squire			
	$-a_0^1=a_1^1$	$-a_0^1$	a_1^1	$-a_2^1$	a_3^1
0.05	0.8144	0.7840	0.8084	0.0370	0.0126
0.1	0.6840	0.6762	0.7152	0.0614	0.0224
0.2	0.5201	0.5112	0.5643	0.0912	0.0383
0.5	0.3375	0.3335	0.3513	0.1003	0.0825
1.0	0.4050	0.3428	0.1703	−0.0029	0.1695
2.0	−0.2604	0.6514	−0.1122	−0.3680	0.3956

γ	By Maruo			
	a_0^1	a_1^1	a_2^1	a_3^1
0.1	0.6829	−0.7163	0.0447	−0.0163
0.2	0.5185	−0.5640	0.0660	−0.0284
0.3	0.4248	−0.4683	0.0734	−0.0399
0.4	0.3704	−0.4005	0.0704	−0.0520
0.5	0.3392	−0.3479	0.0605	−0.0649

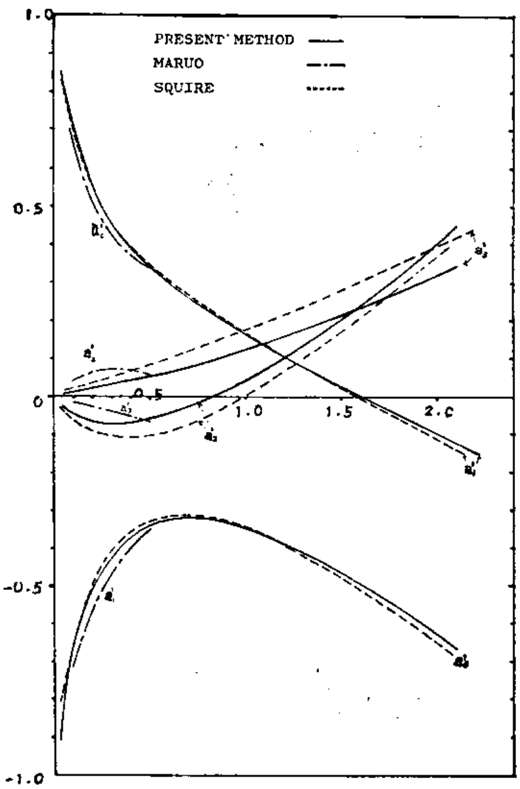

Fig. 2. Comparison with other author's results.

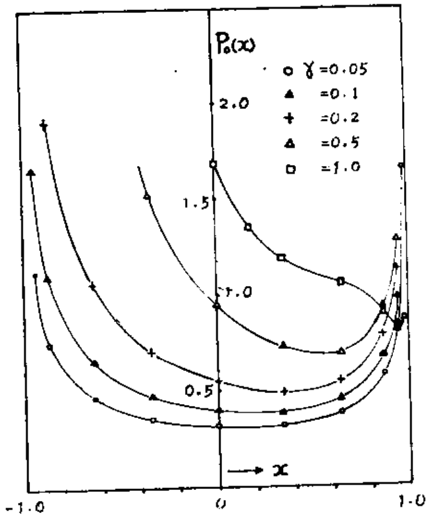

Fig. 3. Influence function for splash $P_0(x)$.

imagine easily the relation between the shape of plate and the splash from this figure.

8. Conclusion

Many works have been done about the problem of the hydro-planing and there is hardly anything left. We try to reconstruct here the boundary value problem on the basis of variational principles.

The conclusions are summerized as follows;

1) The integral equation may be written in two ways. The one represents the ordinate of planing plate and the other its inclination.

2) All preceding works treat with the equation of inclination. Flax's variational principle may be applied to it and the stationary value of the integral to be considered is the drag. The pressure distribution must satisfy Kutta-Joukowski condition.

3) The proposed method may be applied to the integral equation for the plate ordinate instead of the inclination and the integral to be extremized is Lagrangean. The pressure distribution may not always satisfy Kutta-Joukowski condition, namely, this method contains wider class of the pressure distribution than the former.

4) The eigen function of the integral equation for plate inclination is calculated by the latter method and it is found that this eigen function is also the influence function of splash.

5) The numerical calculations are carried out and found to be in good agreement with the preceding results.

References

1) J. V. Wehausen and E. V. Laitone; Surface Waves, Handbuch der Physik, Bd. 9, 1960.
2) L. I. Sedov; Two-Dimensional Problems in Hydrodynamics and Aerodynamics, Interscience Publ. New York, 1965.
3) H. Maruo; Journal of Zosen Kyokai, vol. 78, 1947.
4) H. Maruo; Journal of Zosen Kyokai, vol. 81, 1949.
5) H. Maruo; Journal of Zosen Kyokai, vol. 91, 1956.
6) H. Maruo; Journal of Zosen Kyokai, vol. 92, 1957.
7) H. Maruo; Journal of Zosen Kyokai, vol. 105, 1959.
8) H. Maruo; Memo. of Yokohama National University, Department of Engin. vol. 1, 1951.
9) H. B. Squire; Proc. Roy. Soc., A, vol. 243, 1958.
10) G. F. Miller; Proc. Roy. Soc., A, vol. 243, 1958.

11) F. Ursell and G. N. Ward; Q. J. Mech. and Appl. Mech., vol. 3, 1950.
12) A. H. Flax; J. of the Aeron. Science, vol. 19, 1952.
13) T. Hanaoka; Proc. of 9-th Japan Natio. Congress for Appl. Mech., 1959.
14) M. Bessho; Memo. of the Defense Academy, vol. 8, 1968.
15) J. C. Luke; J. of Fluid Mech., vol. 27, 1967.