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A Contribution to The Theory of Free Surface Flow

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Synopsis

There is a variational principle equivalent to the boundary value problem of a flow with free surface making use of Lagrangean. In this paper, this principle is transformed to the one of the doublet or circulation strength. Then, for example, it is found that the circulation strength over the free surfrce without gravity must be constant. Such simple relation enables us to construct a new integral equation for the flow with free surface.

Introduction

There is a variational principle for the water flow with free surface that Lagrangean of the actual flow is minimum.[2] This variational principle can be transformed so that the variational problem could be equivalent to the boundary value problem.[3)5] This method is very attractive but seems not to be practical for numerical analysis.[5]
In this paper, we try to construct a variational principle for the doublet or circulation distribution over every surfaces equivalent to this boundary value problem. Then, we will find that the singularity density over free surface is very simple and given by the coordinate of that surface. This knowledge will introduce us to a new integral equation of this boundary value problem but it will be similar as Trefftz's[2] and Zwick's.[4]

1. Variational Principle

Let us consider a functional J with respect to a flow outside a body and cavity as in Fig. 1:[3)5]

$$J=\frac{1}{2}\iint_{B+F}\mu(\phi_\nu+2x_\nu)dS-\frac{c^2}{2}V-\frac{g}{2}\iint_{B+F_U-F_L}\eta^2dxdz\ , \tag{1.1}$$

where ϕ means a velocity potential with doublet distribution μ over B and F, that is, the the body surface and free surface as follows:

$$\phi(P)=\frac{1}{4\pi}\iint_{B+F}\mu(Q)\frac{\partial}{\partial\nu}\frac{1}{r(P,Q)}dS(Q)\ , \tag{1.2}$$

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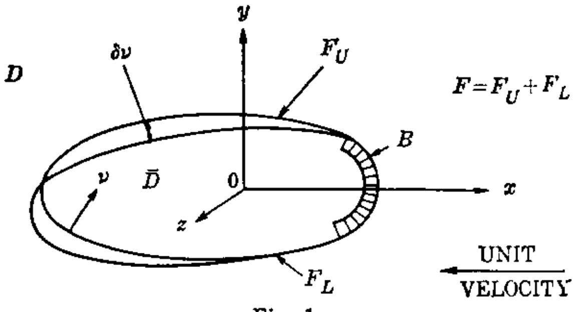

Fig. 1.

V the volume of the domain $\bar{D}$, that is, the body plus cavity:

$$V=\iiint_{\bar{D}} d\tau \,, \tag{1.3}$$

g the gravity constant, η the vertical location of the free surface and C a constant velocity velocity defined as

$$\frac{p}{\rho}+\frac{1}{2}q^2+g\eta=\frac{c^2}{2} \,, \tag{1.4}$$

where p means the pressure, ρ the density of the fluid and

$$q^2=(\nabla\Phi)^2 \,, \tag{1.5}$$

$$\Phi=x+\phi \,. \tag{1.6}$$

Now, the velocity potential ϕ is regular in D and $\bar{D}$ by definition and that has a continuous normal derivative on B and F but jumps its value on crossing B and F as follows:

$$\phi_\nu^{+}=\phi_\nu^{-} \quad \text{on} \quad B \text{ and } F \,, \tag{1.7}$$

$$\mu=\phi^{-}-\phi^{+} \quad \text{on} \quad B \text{ and } F \,, \tag{1.8}$$

where $\pm$ sign on the shoulder means that its value stands for the one of the outward domain D (+) or the inner one $\bar{D}$ (−).

Putting (1.7) and (1.8) into (1.1), we have

$$\begin{aligned} J&=\frac{1}{2}\iint_{B+F}\phi^{-}(\phi_\nu^{-}+2x_\nu)\,dS-\frac{1}{2}\iint_{B+F}\phi^{+}(\phi_\nu^{+}+2x_\nu)dS-\frac{c^2}{2}V-\frac{g}{2}\iint_{B+F}\eta^2 dxdz \\ &=-\iiint_{\bar{D}}\frac{1}{2}[(\nabla\phi^{-})^2+2\phi_x^{-}]\,d\tau-\frac{1}{2}\iiint_{D}[(\nabla\phi^{+})^2+2\phi_x^{+}]d\tau-\frac{c^2}{2}\iiint_{\bar{D}}d\tau-g\iiint_{\bar{D}}y\,d\tau \\ &=-\frac{1}{2}\iiint_{D}[(\nabla\Phi^{+})^2-1]d\tau-\frac{1}{2}\iiint_{\bar{D}}[(\nabla\Phi^{-})^2-1+c^2-2gy]d\tau \,. \end{aligned} \tag{1.9}$$

When ϕ changes slightly by $\delta\phi$ and free surface deflects by $\delta\nu$ the variation of J becomes

$$\delta J=-\iiint_D \nabla\delta\phi^+\nabla\Phi^+ d\tau-\iiint_{\bar{D}}\nabla\delta\phi^-\nabla\Phi^- d\tau-\frac{1}{2}\iint_F[(\nabla\Phi^+)^2-(\nabla\Phi^-)^2-c^2+2gy]\delta\nu dS$$

$$=-\iint_{B+F}\delta\mu(\phi_\nu+x_\nu)dS-\frac{1}{2}\iint_F[(\nabla\Phi^+)^2-(\nabla\Phi^-)^2-c^2+2gy]\delta\nu dS\,, \tag{1.10}$$

where

$$\delta\mu=\delta\phi^- - \delta\phi^+\,. \tag{1.11}$$

Then the stationary conditions for J are[3)5)]

$$\phi_\nu+x_\nu=0 \quad \text{on} \quad B \text{ and } F\,, \tag{1.12}$$

and

$$\frac{1}{2}\{(\nabla\Phi^+)^2-(\nabla\Phi^-)^2\}+gy=\frac{c^2}{2}\,. \tag{1.13}$$

If the former condition is satisfied, the inner potential Φ^- must be zero;

$$\Phi^-=x+\phi^-=0 \quad \text{in} \quad \bar{D}\,, \tag{1.14}$$

because its normal derivative vanishes on B and F and it is regular throughout the finite domain $\bar{D}$.
Hence, by (1.8), we have

$$\mu=\phi^- - \phi^+=-(x+\phi^+)=-\Phi^+\,. \tag{1.15}$$

Then, the latter condition becomes as

$$\frac{1}{2}q^2+gy=\frac{c^2}{2} \quad \text{on} \quad F\,, \tag{1.16}$$

which means the constancy of the pressure.
Moreover, the doublet density becomes by (1.12), (1.15) and (1.16) as

$$\frac{1}{2}(\mu_s^2+\mu_t^2)=\frac{c^2}{2}-g\eta \quad \text{on} \quad F\,, \tag{1.17}$$

where s and t means the orthogonal curvilinear co-ordinates over the surface F.
Inversely, if we assume the doublet density on F as (1.17) beforehand, we have

$$\delta J=-\iint_B\delta\mu(\phi_\nu+x_\nu)dS-\frac{1}{2}\iint_F[\{(\Phi_\nu^+)^2-(\Phi_\nu^-)^2\}$$
$$+\{(\Phi_s^+)^2-(\Phi_s^-)^2-\mu_s^2\}+\{(\Phi_t^+)^2-(\Phi_t^-)^2-\mu_t^2\}]\delta\nu dS$$
$$=-\iint_B\delta\mu(\phi_\nu+x_\nu)dS+\iint_F(\mu_s\Phi_s^-+\mu_t\Phi_t^-)\delta\nu dS\,, \tag{1.18}$$

because it might be $\delta\mu=0$ on F, that is, μ is given by (1.17) and there is the continuity (1.7).

Then, the stationary conditions of J are

$$\phi_\nu + x_\nu = 0 \quad \text{on} \quad B\,, \tag{1.19}$$

$$\mu_s \Phi_s^- + \mu_t \Phi_t^- = 0 \quad \text{on} \quad F\,. \tag{1.20}$$

Since μ is not vanish identically, the latter condition means

$$\Phi^- = 0 \quad \text{on} \quad F\,, \tag{1.21}$$

but this means also that Φ^- must be identically zero in $\bar{D}$ and

$$\phi^- = \Phi^- - x = -x$$

that is,

$$\phi_\nu^- = -x_\nu = \phi_\nu^+ \quad \text{on} \quad F\,. \tag{1.22}$$

Thence, the constant pressure condition (1.16) is satisfied naturally. In any way, the stationary value of J is[3)5)]

$$[J] = \frac{1}{2}\iiint_D (\nabla\phi)^2 \, d\tau - \frac{c^2-1}{2} V - \frac{g}{2}\iint_{B+F_U-F_L} \eta^2 dx dz\,. \tag{1.23}$$

The doublet density over F of (1.17) is determined if the surface F is given and is very simple. Hence, it may be preferable to select the latter procedure for the practical analysis.

Moreover, from these observation, we may obtain the following integral equation:

$$\phi_\nu(P) = \frac{1}{4\pi}\iint_{B+F} \mu(Q) \frac{\partial^2}{\partial\nu_P \partial\nu_Q} \frac{1}{r(P,Q)} dS(Q) = -x_\nu(P)\,, \tag{1.24}$$

where μ on F is to be integrated from (1.17).

In this equation, the unknowns are μ on B and the form and location of F. If this equation is satisfied by some μ and F, the conditions (1.12) and (1.14) are fullfilled and then, by (1.15), (1.16) and (1.17), the pressure condition does also.

It will be seen in the later section that this equation is also consistent with our linear theory.

The integration of μ over F by (1.17) may not be so simple for a general surface but is very simple for a rotationally symmetric flow and a two-dimensional flow.

Finally it is noticed that the integral equation (1.18) is applicable for a infinite cavity although the variational problem is not for such case.[2)5)]

2. Two-Dimensional Problem

In two-dimensional case, it is preferable and simple to take up the circulation distribution instead of the doublet.

The functional becomes

$$J=\frac{1}{2}\int_{B+F}\gamma(\phi+2y)\,dS-\frac{c^2}{2}A-\frac{g}{2}\int_{B+F_U-F_L}\eta^2dx\,, \tag{2.1}$$

where

$$\phi(P)=\frac{1}{2\pi}\int_{B+F}\gamma(Q)\log r(P,Q)dS(Q)\,, \tag{2.2}$$

$$A=\iint_{\bar{D}}dxdy\,, \tag{2.3}$$

and c given also by (1.4) but

$$q^2=(\nabla\Psi)^2\,, \tag{2.4}$$

$$\Psi=\phi+y\,. \tag{2.5}$$

For the simplicity, let us assume

$$\int_{B+F}\gamma dS=0\,, \tag{2.6}$$

so that ϕ would be one valued.
From (2.2), we have

$$\phi_\nu^- - \phi_\nu^+=\gamma\,, \tag{2.7}$$

and

$$\phi^-=\phi^+\,. \tag{2.8}$$

In a similar way as the preceding, since we have

$$J=-\frac{1}{2}\iint_D[(\nabla\phi^+)^2+2\phi_y^+]dxdy-\frac{1}{2}\iint_{\bar{D}}[(\nabla\phi^-)^2+2\phi_y^-+c^2-2gy]dxdy\,, \tag{2.9}$$

taking the variation as in the preceding, we have

$$\delta J=\iint_{B+F}\delta\gamma(\phi+y)\,dS-\frac{1}{2}\int_F[(\nabla\phi^+)^2-(\nabla\phi^-)^2-c^2+2g\eta]\delta\nu dS\,, \tag{2.10}$$

where

$$\delta\gamma=\delta\phi_\nu^- - \delta\phi_\nu^+\,, \tag{2.11}$$

from which we can conclude that the stationary conditions of J are

$$\phi+y=0 \quad \text{on} \quad B+F\,, \tag{2.12}$$

then

$$\Psi^-=0\,,\quad \phi=-y \quad \text{in} \quad \bar{D}\,, \tag{2.13}$$

and

$$\frac{1}{2}q^2+g\eta=\frac{c^2}{2}\,. \tag{2.14}$$

The circulation strength, then, becomes

$$\gamma^2=c^2-2g\eta \ . \tag{2.15}$$

Inversely, if we assume that the circulation strength over F is given by (2.15), we have for the stationary conditions of J the boundary condition (2.12) and consequently (2.14) as in the preceding. The stationary value of J becomes

$$[J]=\frac{1}{2}\iint_D (\nabla\psi)^2 dxdy-\frac{c^2-1}{2}A-\frac{g}{2}\int_{B+F_U-F_L}\eta^2 dx \ . \tag{2.16}$$

Finally, if we put the circulation on F as (2.14), then we have the integral equation for this boundary value problem as follows;

$$\psi=\frac{1}{2\pi}\int_B \gamma\log rdS+\frac{1}{2\pi}\int_F[\pm\sqrt{c^2-2g\eta}\,]\log rdS=-y \ , \tag{2.17}$$

where the sign of γ on F is to be determined as the problem becomes appropriate.

3. Linearization

In general, the variational technique is not so practical that we have had not yet few examples in application.

The integral equations, however, introduced here are very simple and analogous to Trefftz's method[2] and may be practical in application.

Thence, in this section, let us ascertain their validity by comparing their linearized forms with the well-known ones in the two-dimensional case.

Firstly, let us assume that $g=0$ and the body is very thin and symmetric as Fig. 2.

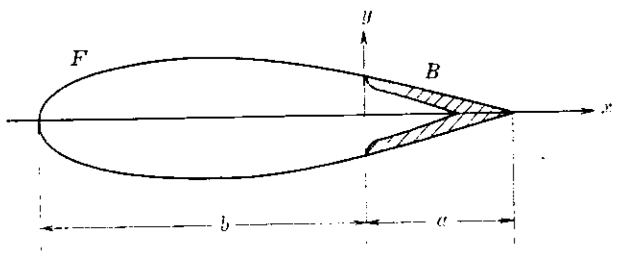

Fig. 2.

Then, the stream function in (2.17) can be approximated as follows;

$$\psi \doteqdot \frac{1}{2\pi}\int_B \gamma(\xi)\log\sqrt{\frac{(x-\xi)^2+(y-\eta)^2}{(x-\xi)^2+(y+\eta)^2}}\,d\xi+\frac{c}{2\pi}\int_F \log\sqrt{\frac{(x-\xi)^2+(y-\eta)^2}{(x-\xi)^2+(y+\eta)^2}}\,d\xi \ ,$$

$$\doteqdot -\frac{1}{\pi}\int_0^a \frac{\gamma(\xi)\eta(\xi)y}{(x-\xi)^2+y^2}d\xi-\frac{c}{\pi}\int_{-b}^0 \frac{\eta(\xi)y}{(x-\xi)^2+y^2}d\xi \ , \tag{3.1}$$

where η means a half breadth of the body or cavity.

The corresponding potential will be

$$\phi \doteqdot \frac{1}{\pi}\int_0^a \frac{\gamma(\xi)\eta(\xi)(x-\xi)}{(x-\xi)^2+y^2}d\xi+\frac{c}{\pi}\int_{-b}^0 \frac{\eta(\xi)(x-\xi)}{(x-\xi)^2+y^2}d\xi . \tag{3.2}$$

The integral equation (2.17) now becomes

$$\lim_{y\to+0}\psi=-\mu(x)\eta(x)-\frac{\eta(x)}{\pi}\int_0^a\frac{\gamma\eta}{(x-\xi)^2}d\xi-\frac{c\eta(x)}{\pi}\int_{-b}^0\frac{\eta(\xi)d\xi}{(x-\xi)^2}=-\eta(x) , \tag{3.3}$$

or

$$1=\mu(x)+\frac{1}{\pi}\int_0^a\frac{\gamma(\xi)\eta(\xi)}{(x-\xi)^2}d\xi+\frac{c}{\pi}\int_{-b}^0\frac{\eta(\xi)d\xi}{(x-\xi)^2} , \tag{3.4}$$

where

$$\left.\begin{aligned}\mu(x)&=c \quad \text{for} \quad -b<x<0\\ &=\gamma(x) \quad \text{for} \quad a>x>0\end{aligned}\right\} \tag{3.5}$$

and the improper integrals stand for following values

$$\int\frac{f(\xi)}{(x-\xi)^2}d\xi=-\frac{\partial}{\partial x}\left[P.V.\int\frac{fd\xi}{(x-\xi)}\right]=\lim_{y\to0}\int\frac{fd\xi}{(x-\xi)^2+y^2} .$$

On the other hand, the usual linearized boundary conditions for ϕ are[2)]

$$\left.\begin{aligned}\psi(x, 0)&=-\eta(x) \quad \text{for} \quad a>x>0\\ \phi_x(x, 0)&=c-1 \quad \text{for} \quad -b<x<0\end{aligned}\right\} \tag{3.6}$$

The integral equation (3.3) is equivalent to the former condition for $a>x>0$, and also to the latter one for $-b>x>0$, because we have, from (3.2) and (3.4),

$$\phi_x(x, 0)\doteqdot-\frac{1}{\pi}\int_0^a\frac{\gamma\eta d\xi}{(x-\xi)^2}-\frac{c}{\pi}\int_{-b}^0\frac{\eta d\xi}{(x-\xi)^2}=c-1 \quad \text{for} \quad -b<x<0 . \tag{3.7}$$

Thus, although the expression seems different with usual one, the solution of (3.4) must give the same one as the usual method except that we treat as doublet distribution

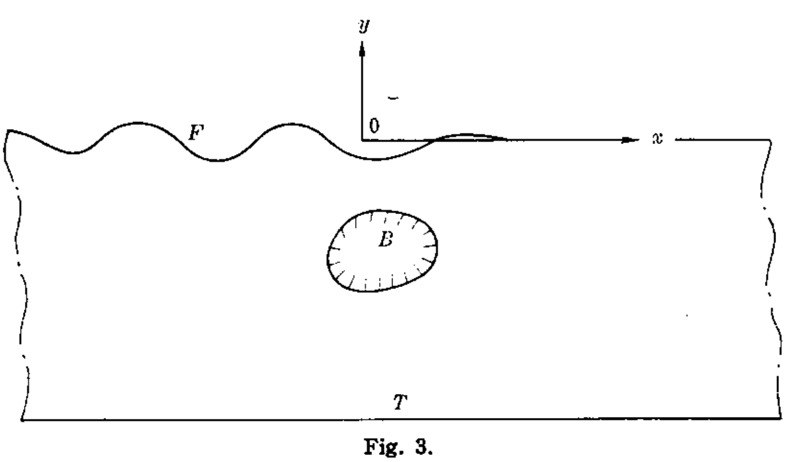

Fig. 3.

instead of a source-sink one.

Secondly, let us imagine a submerged body in a uniform stream of infinite depth as Fig. 3. Thence, the stream function becomes from (2.17) as

$$\psi(x, y)=\frac{1}{2\pi}\int_B \gamma \log r dS-\frac{1}{2\pi}\int_F \sqrt{1-2g\eta}\log r dS+\frac{1}{2\pi}\int_T \gamma \log r dS\,. \tag{3.8}$$

The circulation at the bottom T will be uniform and the unit because we assume it lies at infinitely deep place, and since we could have

$$\int_{-\infty}^{\infty}\log\sqrt{(x-\xi)^2+(T+y)^2}d\xi-\int_{-\infty}^{\infty}\log\sqrt{(x-\xi)^2+y^2}d\xi=0\,, \tag{3.9}$$

subtracting this from (3.8) and neglecting higher order terms, we have a linearized form

$$\psi(x, y)=\frac{1}{2\pi}\int_B \gamma \log r dS+\frac{g}{2\pi}\int_{-\infty}^{\infty}\eta(\xi)\log\sqrt{(x-\xi)^2+y^2}d\xi+\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{y\eta(\xi)d\xi}{(x-\xi)^2+y^2}\,, \tag{3.10}$$

where it is to be assumed that

$$\int_{-\infty}^{\infty}\eta(\xi)d\xi=0 \quad \text{and} \quad \int_B \gamma dS=0\,. \tag{3.11}$$

The integral equation (2.17) at the free surface gets the from

$$-\eta(x)=\frac{1}{\pi}\int_B \gamma\log\sqrt{(x-\xi)^2+\eta^2}dS+\frac{g}{\pi}\int_{-\infty}^{\infty}\eta(\xi)\log|x-\xi|d\xi\,. \tag{3.12}$$

Differentiating by x, mulplying by e^{ikx} and integrating both hands, we have

$$A(k)=\frac{H(k)}{k-g}\,,\qquad \overline{A(k)}=\frac{\overline{H(k)}}{k-g}\,, \tag{3.13}$$

where

$$A(k)=\int_{-\infty}^{\infty}\eta(x)e^{ikx}dx\,, \tag{3.14}$$

$$H(k)=\int_B \gamma e^{ky+ikx}dx\,. \tag{3.15}$$

Putting this result into (3.10), we have

$$\psi(x, y)=\frac{1}{2\pi}\int_B \gamma\log r dS-\frac{1}{\pi}\int_B \gamma dS\int_0^{\infty}\frac{k+g}{k-g}e^{k(y+\eta)}\cos k(x-\xi)\frac{dk}{k}\,. \tag{3.16}$$

This is the same expression as the usual linearized stream function.[1]

Conclusion

We have considered the boundary value problem of a water flow with free surface and introduced the equivalent variational principle with respect to the doublet or circula-

tion density over the every boundary surfaces.
Then, we find that its density on a free surface is determined solely by a location of its surface, especially the circulation must be constant on the cavity surface without gravity. This fact is deduced naturally if we assume that there is still water outside the water flow considered, then we must put a singularity density appropriate to cancell out the dynamical effect of the water flow so that the water outside the domain considered would be rest.

From these results, we have proposed a new integral equation to solve our boudary value problem, which is similar as of Trefftz's and Zwick's but simpler than them and is verified that it is equivalent to the theory which we have had when linearized. This integral equation may fit well to the numerical analysis and may be applied to an infinite flow, although the variational one could not be.

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