Memoirs
of the Defense Academy
Japan, Vol. XVII. No. 3
pp. 95–105, 1977

On the Fundamental Singularity in the Theory of Ship Motions in a Seaway

(*Dedicated to Professor Koya Makiyama*)

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(Received April 26, 1977)

Abstract

The fundamental singularity which is a velocity potential of water motion by a pulsating singularity in a uniform stream with free surface must be evaluated to study the theory of ship motions in a sea way.

It is represented by a double Fourier integral in usual, but may be reduced to a single integral form which is an aim of the present paper, so that its analytical property may be facilitated to discuss or compute its numerical value.

1. Introduction

It is necessary to study the theory of oscillation of ships in a sea-way to compute a fundamental singularity which is usually represented by double integral.

This is only inconvenient for numerical computations but difficult to estimate its analytical properties in various cases.

The present paper aims to convert this double integral into a single integral so that such study may become easy.

The method is similar to the one used in the case of steady and non-oscillating problem[2] and somewhat sophisticated one but it is justified by the differentiation after obtaining the result and agrees with the latter in its limit.

2. Derivation and Definition

The velocity potential at $P(x, y, z)$ of water motion by a unit source pulsating with circular frequency ω situated at $Q(x', y', z')$ in a uniform stream with unit velocity is a harmonic function and satisfies at the free surface the condition;

$$\left[\left(i\omega-\frac{\partial}{\partial x}\right)^2+g\frac{\partial}{\partial z}\right]G(P, Q)|_{z=0}=0 , \tag{2.1}$$

where the co-ordinate system are taken as in Fig. 1 and g stands for gravity constant,

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and, as well known, is represented except time factor $\exp i\omega t$, as follows:

$$G(P,Q)=\frac{1}{4\pi}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)+gT(x-x',y-y',z+z')\ , \quad (2.2)$$

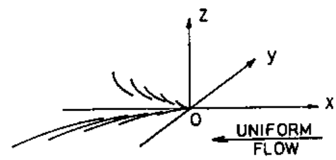

Fig. 1. Co-ordinate system.

where

$$T(x,y,z)=-\lim_{\varepsilon\to+0}\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_0^\infty\frac{\exp[kz+ik\tilde{\omega}(\theta)]kdkd\theta}{A(k,\theta)}\ , \qquad z<0\ , \tag{2.3}$$

$$A(k,\theta)=(k\cos\theta-\omega)^2-gk+i\varepsilon(k\cos\theta-\omega)\ , \tag{2.4}$$

$\tilde{\omega}(\theta)=x\cos\theta+y\sin\theta$, $r_1=\overline{PQ}$ and $r_2=\overline{P\bar{Q}}$ where $\bar{Q}$ is an image point of Q with regard to the free surface.

Hence, it is sufficient to consider the function T and additionally its following derivatives.

$$\left(i\omega-\frac{\partial}{\partial x}\right)T(x,y,z)=S(x,y,z)\ , \tag{2.5}$$

$$\frac{\partial}{\partial z}T(x,y,z)=-S_2(x,y,z)\ , \tag{2.6}$$

In another point of view, the function S is a velocity potential of a point pressure on water surface and S_2 the surface elevation[1].

The function T satisfies the following differential equation throughout the water.

$$\left[\left(i\omega-\frac{\partial}{\partial x}\right)^2+g\frac{\partial}{\partial z}\right]T(x,y,z)=-\frac{z}{2\pi r^3}\ , \tag{2.7}$$

where $r=\sqrt{x^2+y^2+z^2}$, because we have an integral representation:

$$\frac{z}{r^3}=-\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^\infty\exp[kz+ik\tilde{\omega}(\theta)]kdkd\theta\ , \qquad z<0\ .$$

These functions are all even with regard to y, that is,

$$\left.\begin{aligned}T(x,-y,z)&=T(x,y,z)\ ,\\ S(x,-y,z)&=S(x,y,z)\ ,\\ S_2(x,-y,z)&=S_2(x,y,z)\ ,\end{aligned}\right\} \tag{2.8}$$

Therefore, it is sufficient to consider only for positive y.

On the contrary, for negative x they have radiating terms and if we define the following function as:

$$T_w(x,y,z)=\bar{T}(x,y,z)-T(-x,y,z)\ , \tag{2.9}$$

where the bar over the letter stands for complex conjugate to be taken, we may calculate the function T for negative x as,

$$T(-x, y, z)=\bar{T}(x, y, z)-T_w(x, y, z)\ , \tag{2.10}$$

so that it may sufficient to consider only for positive x in the following.

Now, by its definition (2.9), T_w can be written as,

$$T_w(x, y, z)=\frac{1}{4\pi^2}\int_{-\pi}^{\pi} d\theta\int_0^{\infty} \exp\,[kz-ik\tilde{\omega}(\theta)]\left[\frac{1}{A(k, \theta)}-\frac{1}{\bar{A}(k, \theta)}\right]kdk\ , \tag{2.11}$$

and since A can be decomposed as

$$\left.\begin{aligned} &A(k, \theta)=\cos^2\theta\{k-\kappa(\theta)\}\{k-\kappa'(\theta)\}\ ,\\ &\left.\begin{matrix}\kappa(\theta)\\ \kappa'(\theta)\end{matrix}\right\}=\left[\left(\omega-\frac{i\varepsilon}{2}\right)\cos\theta+\frac{g}{2}\left(1\mp\sqrt{1+\frac{4\omega}{g}\cos\theta}\right)\right]\sec^2\theta\ , \end{aligned}\right\} \tag{2.12}$$

the integration in k of (2.11) becomes only the ones around its poles in the k-plane as shown in Fig. 2.

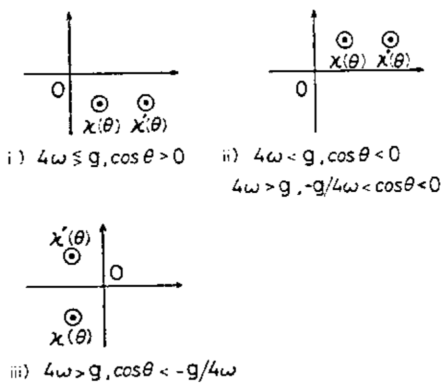

Fig. 2. Poles in k-plate.

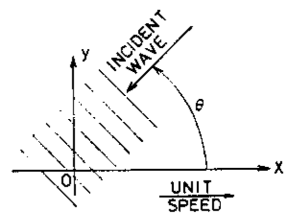

Fig. 3. Direction of incident wave.

Calculating residuces, we have

$$T_w(x, y, z)=\frac{i}{2\pi g}\int_{-\pi}^{\pi}\Big[\bar{\phi}_0(x, y, z; \kappa(\theta), \theta)\kappa(\theta) - \bar{\phi}_0(x, y, z; \kappa'(\theta), \theta)\kappa'(\theta)\,\mathrm{sgn}\,(\cos\theta)\Big]\frac{d\theta}{\sqrt{1+\frac{4\omega}{g}\cos\theta}}\ , \tag{2.13}$$

where

$$\phi_0(x, y, z; k, \theta)=\exp\,[kz+ik(x\cos\theta+y\sin\theta)]\ , \tag{2.14}$$

In the latter integral of the right hand side, the range where $\cos u>g/4\omega$, if exists, must be excluded.

To obtain more clear image of these trailing or radiatiating waves, let us consider an incident wave coming from the direction θ with the encounter circular frequency ω and wave number k as in Fig. 3.

$$\exp[ik\tilde{\omega}(\theta)+i\omega t]\,, \tag{2.15}$$

where t means time.

The relation between the encounter frequency and the one of the wave in fixed co-ordinate is

$$\omega_0=\omega-k\cos\theta\,, \tag{2.16}$$

and since $\omega_0=\sqrt{gk}$, this gives the relation:

$$\frac{\omega_0}{\sqrt{g}}=\sqrt{k}=-\frac{\sqrt{g}}{2\cos\theta}\left[1\mp\sqrt{1+\frac{4\omega}{g}\cos\theta}\,\right], \tag{2.17}$$

Comparing this with (2.12), it is easy to see that

$$\omega_0=\sqrt{g\kappa(\theta)}\,,\qquad \omega_0'=\sqrt{g\kappa'(\theta)}\,, \tag{2.18}$$

where ω_0' is the other root of (2.15).

Therefore, all the waves of (2.13) are the waves having the encounter circular frequency ω.

3. Integral Representation

In the integral representation of T (2.3), we may also write it by the substitution of integration variables:

$$k\cos\theta=p\,,\quad k\sin\theta=q\,,\quad kdkd\theta=dpdq\,, \tag{3.1}$$

like as

$$T(x,y,z)=-\lim_{\varepsilon\to+0}\frac{1}{4\pi^2}\times\iint_{-\infty}^{+\infty}\frac{\exp[z\sqrt{p^2+q^2}+i(px+qy)]dpdq}{(p-\omega)^2-g\sqrt{p^2+q^2}+i\varepsilon(p-\omega)}\,, \tag{3.2}$$

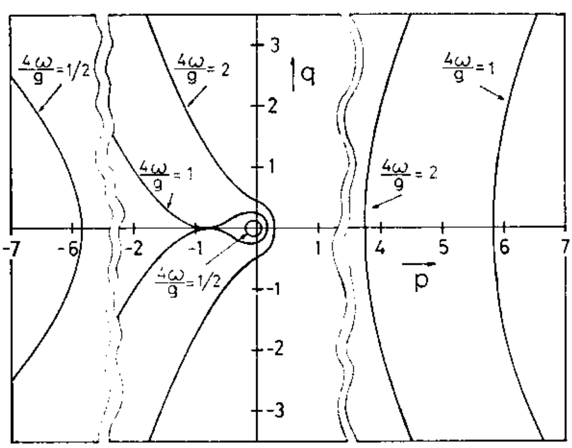

Fig. 4. Curves of singularity in p-q plane.

Then, their poles lie on lines in $p-q$ plane like as in Fig. 4.

Moreover, substituting q to

$$q=|p|\sinh v\,, \tag{3.3}$$

we may write (3.2), as follows:

$$T(x,y,z)=-\frac{1}{4\pi^2}\iint_{-\infty}^{+\infty}\frac{\exp[|p|(z\cosh v+iy\sinh v)+ipx]|p|\cosh v dvdp}{(p-\omega)^2-g|p|\cosh v+i\varepsilon(p-\omega)}\,, \tag{3.4}$$

abbreviating the letters to take the limit.

Putting

that is,

$$\left.\begin{aligned} &z=-\rho\sin\delta\ , \qquad y=\rho\cos\delta\ , \\ &z\cosh v+iy\sinh v=i\rho\sinh(v+i\delta)\ , \end{aligned}\right\} \tag{3.5}$$

and since the poles in the complex v-plane are given by

$$\cosh v=\frac{(p-\omega)^2}{g|p|}+i\varepsilon\frac{(p-\omega)}{|p|}\ , \tag{3.6}$$

and lie as in Fig. 6, referring to Fig. 5, we may shift the path of integration in v to the horizontal line $(-i\delta)$ or $(\pi-i\delta)$ according to p positive or negative in the complex v-plane with residue at poles in this domain surrounded by both lines, and we have,

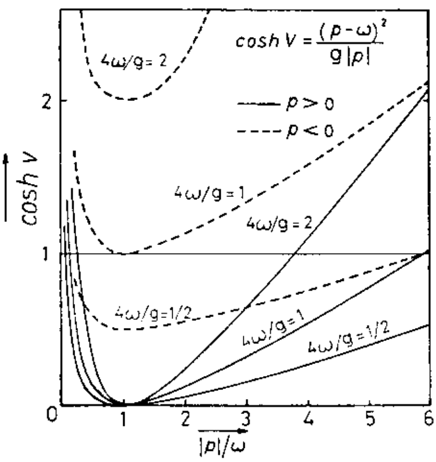

Fig. 5. Relation Between v and p.

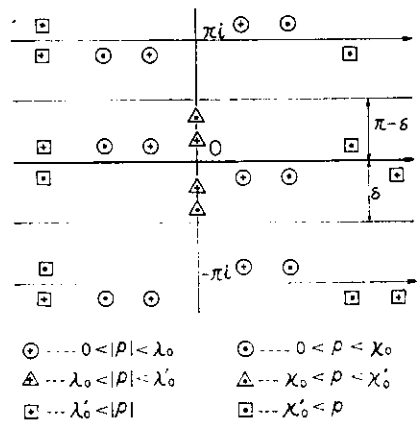

Fig. 6. Poles in v-plane.

$$T=T_S+T_R\ , \tag{3.7}$$

$$T_S=-\frac{1}{4\pi^2}\iint_{-\infty}^{\infty}\frac{\exp[(x+\rho\sinh u)]p\cosh(u-i\delta)dpdu}{(\rho-\omega)^2-gp\cosh(u-i\delta)}\ , \tag{3.8}$$

$$\begin{aligned} T_R=&\frac{1}{2\pi ig}\left\{\int_{-\infty}^{-\lambda_0'}+\int_{-\lambda_0}^{0}\right\}\frac{\exp[ipx+(z/g)(p-\omega)^2-(iy/g)\sqrt{(p-\omega)^4-g^2p^2}](p-\omega)^2dp}{\sqrt{(p-\omega)^4-g^2p^2}} \\ &-\frac{1}{2\pi g}\int_{-\lambda_0'}^{\lambda_0}\frac{\exp[ipx+(z/g)(p-\omega)^2+(y/g)\sqrt{g^2p^2-(p-\omega)^4}](p-\omega)^2dp}{\sqrt{g^2p^2-(p-\omega)^4}} \\ &+\frac{1}{2\pi ig}\left\{\int_{0}^{\kappa_0}-\int_{\kappa_0'}^{\infty}\right\}\frac{\exp[ipx+(z/g)(p-\omega)^2+(iy/g)\sqrt{(p-\omega)^4-g^2p^2}](p-\omega)^2dp}{\sqrt{(p-\omega)^4-g^2p^2}} \\ &+\frac{1}{2\pi g}\int_{\kappa_0}^{\kappa_0'}\frac{\exp[ipx+(z/g)(p-\omega)^2-(y/g)\sqrt{g^2p^2-(p-\omega)^4}](p-\omega)^2dp}{\sqrt{g^2p^2-(p-\omega)^4}}\ , \end{aligned} \tag{3.9}$$

where

$$\left.\begin{aligned} &\left.\begin{matrix}\kappa_0\\ \kappa_0'\end{matrix}\right\}=\omega+\frac{g}{2}\left(1\mp\sqrt{1+\frac{4\omega}{g}}\right),\\ &\left.\begin{matrix}\lambda_0\\ \lambda_0'\end{matrix}\right\}=-\omega+\frac{g}{2}\left(1\mp\sqrt{1-\frac{4\omega}{g}}\right).\end{aligned}\right\} \tag{3.10}$$

The last integral can be written also, changing the variables, like as

$$(p-\omega)^4=g^2p^2\cosh t\,, \tag{3.11}$$

or

$$\left.\begin{aligned} &p=\left\{\begin{matrix}\nu'(t)\\ \nu(t)\end{matrix}\right\}=-\omega+\frac{g}{2}\left(1\pm\sqrt{1-\frac{4\omega}{g}\operatorname{cosech}t}\right)\cosh t\,,\quad \text{for}\quad p\leq-\omega' \text{ or } -\omega<p<0\,,\\ &p=\left\{\begin{matrix}\mu(t)\\ \mu'(t)\end{matrix}\right\}=\omega+\frac{g}{2}\left(1\mp\sqrt{1+\frac{4\omega}{g}\operatorname{cosech}t}\right)\cosh t\,,\quad \text{for}\quad 0<p<\omega \text{ or } \omega<p\,.\end{aligned}\right\} \tag{3.12}$$

$$\left.\begin{aligned} &\frac{1}{\nu}\frac{d\nu}{dt}\ \text{ or }\ \frac{1}{\nu'}\frac{d\nu'}{dt}=\mp\frac{\sinh t}{\sqrt{1-\frac{4\omega}{g}\operatorname{cosech}t}}\,,\\ &\frac{1}{\mu}\frac{d\mu}{dt}\ \text{ or }\ \frac{1}{\mu'}\frac{d\mu'}{dt}=\mp\frac{\sinh t}{\sqrt{1+\frac{4\omega}{g}\operatorname{cosech}t}}\,,\end{aligned}\right\} \tag{3.13}$$

we have

$$\begin{aligned} T_R=&\frac{1}{2\pi ig}\int_{-i\delta}^{\infty}\frac{\exp[i\mu\{x+\rho\sinh(t+i\delta)\}]\mu(t)dt}{\sqrt{1+\frac{4\omega}{g}\operatorname{cosech}t}}\\ &+\frac{i}{2\pi g}\int_{i\delta}^{\infty}\frac{\exp[i\mu'\{x+\rho\sinh(t-i\delta)\}]\mu'(t)dt}{\sqrt{1+\frac{4\omega}{g}\operatorname{cosech}t}}\\ &+\frac{1}{2\pi ig}\int_{-\cosh^{-1}(4\omega/g)}^{\infty}[\exp[i\nu\{x-\rho\sinh(t-i\delta)\}]\nu(t)+\exp[-i\nu'\{x-\rho\sinh(t-i\delta)\}]\nu'(t)]\\ &\times\frac{dt}{\sqrt{1-\frac{4\omega}{g}\operatorname{cosech}t}}\,, \end{aligned} \tag{3.14}$$

On the other hand, since the poles of the integrand of (3.8) lie like as in Fig. 7, the integration in p gives

$$\begin{aligned} T_S=&\frac{i}{2\pi g}\left\{\int_{-i\delta}^{\infty-i\delta}-\int_{-\infty-i\delta}^{-u_0-i\delta}\right\}\frac{\exp[i\mu\{x+\rho\sinh(v+i\delta)\}]\mu(v)dv}{\sqrt{1+\frac{4\omega}{g}\operatorname{cosech}v}}\\ &+\frac{1}{2\pi ig}\int_{-u_0-i\delta}^{-i\delta}\frac{\exp[i\mu'\{x+\rho\sinh h(v+i\delta)\}]\mu'(v)dv}{\sqrt{1+\frac{4\omega}{g}\operatorname{cosech}v}}\,, \end{aligned} \tag{3.15}$$

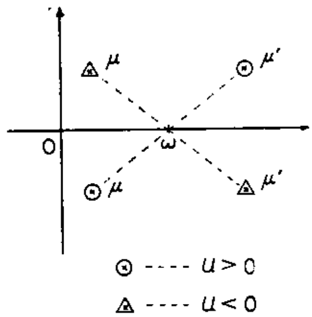

Fig. 7. Poles in p-plane.

where $u_0=\sinh^{-1} x/\rho$, deforming the path of integration appropriately to a semi-circle of inifinte radius according to $(x+\rho\sinh u)$ is positive or negative.

Thence, shifting the paths of integration of (3.15) and summing up (3.14) and (3.15), we have finally

$$T(x,y,z)=\frac{1}{2\pi ig}\int_{-\infty}^{-u_0-i\delta}\frac{dt}{\sqrt{1+\frac{4\omega}{g}\operatorname{cosech} t}}$$
$$\times[\exp[i\mu\{x+\rho\sinh(t+i\delta)\}]\mu(t)-\exp[i\mu'\{x+\rho\sinh(t+i\delta)\}]\mu'(t)]$$
$$+\frac{1}{2\pi ig}\int_{-\infty}^{-\cosh^{-1}(4\omega/g)}\frac{dt}{\sqrt{1-\frac{4\omega}{g}\operatorname{cosech} t}}$$
$$\times[\exp[-i\nu\{x-\rho\sinh(t+i\delta)\}]\nu(t)+\exp[-i\nu'\{x-\rho\sinh(t+i\delta)\}]\nu'(t)]\,, \qquad (3.16)$$

Differentiating this, we have by the definition

$$S(x,y,z)=\frac{1}{2\pi}\left(\frac{xz}{r\rho^2}+\frac{y}{\rho^2}\right)$$
$$+\frac{1}{2\pi g}\int_{-\infty}^{-u_0-i\delta}\frac{dt}{\sqrt{\prime\prime}}\{\exp[i\mu]\mu(\omega-\mu)-\exp[i\mu']\mu'(\omega-\mu')\}$$
$$+\frac{1}{2\pi g}\int_{-\infty}^{-\cosh^{-1}(4\omega/g)}\frac{dt}{\sqrt{\prime\prime}}\{\exp[-i\nu]\nu(\omega+\nu)]+\exp[-i\nu']\nu'(\omega+\nu')\}\,, \qquad (3.17)$$

$$S_2(x,y,z)=\frac{1}{2\pi r\rho^4}(ry-ixz)^2$$
$$+\frac{i}{2\pi g}\int_{-\infty}^{-u_0-i\delta}\frac{\cosh t\,dt}{\sqrt{\prime\prime}}\{\exp[i\mu]\mu^2-\exp[i\mu']\mu'^2\}$$
$$+\frac{i}{2\pi g}\int_{-\infty}^{-\cosh^{-1}(4\omega/g)}\frac{\cosh t\,dt}{\sqrt{\prime\prime}}\{\exp[-i\nu]\nu^2+\exp[-i\nu']\nu'^2\} \qquad (3.18)$$

where the exponential terms and the root terms in the integrands are the same as the expression (3.16) but abbreviated.

Differentiating once more and summing up them like the differential equation (2.7), we can verify that the obtained representation (3.16) satisfies it. This result justifies the preceding deformation of the integral.

Lastly, to compare the representation (3.16) with the wave term (2.13), changing the integration varible t to θ like as $\cos ht = \sec\theta$, we have

$$T(x, y, z) = \frac{1}{2\pi i g}\int_{\alpha-\pi}^{\varphi-\pi/2-i\varepsilon} \frac{d\theta}{\sqrt{1+\frac{4\omega}{g}\cos\theta}} \times[\phi_0(x, y, z; \kappa(\theta), \theta)\kappa(\theta) - \phi_0(x, y, z; \kappa'(\theta), \theta)\kappa'(\theta)\,\mathrm{sgn}(\cos\theta)]\,, \tag{3.19}$$

where $\varphi = \tan^{-1}(y/x)$, $\varepsilon = \sin h^{-1}(|z|/\sqrt{x^2+y^2})$ and $\alpha = \cos^{-1}(g/4\omega)$.

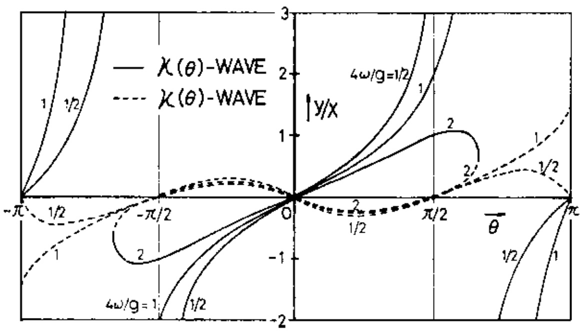

Fig. 8. Points of stationary phase.

4. Asymtotic Property in Far Field

It is well known that these function have four wave systems far from the origin, which are dominant terms.

Refering to the literature[1] with respect to their real form, we reproduce here only points of stationary phase of integrands of (2.13) and (3.19) in Fig. 8, that is, points

$$\left.\begin{aligned} \frac{y}{x} &= \frac{\sin\theta\cos\theta}{\sqrt{1+\frac{4\omega}{g}\cos\theta-\sin^2\theta}}\,, \quad \text{for } \kappa\text{-wave}\,, \\ \frac{y}{x} &= \frac{-\sin\theta\cos\theta}{\sqrt{1+\frac{4\omega}{g}\cos\theta+\sin^2\theta}}\,, \quad \text{for } \kappa'\text{-wave}\,, \end{aligned}\right\} \tag{4.1}$$

corresponding to, when $z=0$,

$$\frac{\partial}{\partial\theta}\left[\begin{matrix}\kappa(\theta)\\ \kappa'(\theta)\end{matrix}\right\}(x\cos\theta+y\sin\theta)\Bigg] = 0\,, \tag{4.2}$$

Hence, as is well known, the function T (3.19) has no wave up-stream side when $4\omega > g$, because there exists no stationary point, but the integral between $(-\pi/2)$ and $(\varphi-\pi/2-i\varepsilon)$ has a contribution and

$$T(x, y, z) \fallingdotseq \frac{1}{2\pi g r}\,, \qquad \text{for } x, y \gg 1\,. \tag{4.3}$$

For negative x, by the formula (2.10), we have

$$T(x, y, z) \fallingdotseq \frac{1}{2\pi g r} - T_w(-x, y, z)\,, \tag{4.4}$$

and T_w of (2.13) may be evaluated by the method of stationary phase.

5. Special Value

There is no difficulty to put x and z to zero in expressions (3.16), (3.19), for example, we have from (3.16),

$$T(0, y, 0)=\frac{1}{2\pi i g}\int_{-\infty}^{0}\{\exp[i\mu y\sinh t]\mu(t)-\exp[i\mu' y\sinh t]\mu'(t)\}\frac{dt}{\sqrt{1+\frac{4\omega}{g}\operatorname{cosech} t}}$$
$$+\frac{1}{2\pi i g}\int_{-\infty}^{\cosh^{-1}(4\omega/g)}\{\exp[i\nu y\sinh t]\nu(t)+\exp[i\nu' y\sinh t]\nu'(t)\frac{dt}{\sqrt{1-\frac{4\omega}{g}\operatorname{cosech} t}}, \quad (5.1)$$

Of course, there is no jump along the y-axis and the wave term becomes the imaginary part of T by (2.9):

$$T_w(0, y, 0)=\bar{T}(0, y, 0)-T(0, y, 0), \quad (5.2)$$

and from (2.13)

$$T_w(0, y, 0)=\frac{i}{2\pi g}\int_{-\pi}^{\pi}[\exp[-i\kappa y\sin\theta]\kappa(\theta)-\exp[-i\kappa' y\sin\theta]\kappa'(\theta)\operatorname{sgn}(\cos\theta)]\frac{d\theta}{\sqrt{1+\frac{4\omega}{g}\cos\theta}}, \quad (5.3)$$

On the other hand, if we put y and z to zero, in the expression (3.16), first integral tends to the negative infinity because $u_0=\sinh^{-1} x/\rho$ tends infinity and its limiting value becomes indeterminate.

Therefore, we may put z to 0, that is $\delta=0$ at first and then take the limiting value for $y=0$.

Since $\mu(t)$ tends to zero for greater t and

$$\lim_{x\to+0}\int_{-\infty}^{-u_0}\frac{\exp[i\mu'(t)(x+y\sinh t)]\mu'(t)dt}{\sqrt{1+\frac{4\omega}{g}\operatorname{cosech} t}}=\frac{1}{ix}, \quad (5.4)$$

we have

$$T(x, 0, 0)=\frac{1}{2\pi g x}$$
$$+\frac{1}{2\pi i g}\int_{-\infty}^{-\cosh^{-1}(4\omega/g)}\{\exp[-i\mu x]\nu(t)+\exp[-i\nu' x]\nu'(t)\}\frac{dt}{\sqrt{1-\frac{4\omega}{g}\operatorname{cosech} t}}. \quad (5.5)$$

For the wave term T_w, there is no difficulty for this limiting process.

Lastly, other expressions (3.17) and (3.19) will be discussed in the same manner as the above, but it may be aasier to obtain by the differentiation of the above expression by defenitions (2.5) and (2.7) like as

$$\left.\begin{aligned} S(x,0,0)&=\left(i\omega-\frac{\partial}{\partial x}\right)T(x,0,0)\,, \\ S_2(x,0,0)&=\frac{1}{g}\left(i\omega-\frac{\partial}{\partial x}\right)S(x,0,0)\,, \end{aligned}\right\} \tag{5.6}$$

6. The Function for Steady Case

When ω tends to zero, we have the case of steady and non-oscillating problem. Then, we see from (2.12) and (3.12)

$$\left.\begin{aligned} &\kappa(\theta)\fallingdotseq 0\,, && \kappa'(\theta)\fallingdotseq g\sec^2\theta-\frac{i\varepsilon}{2}\sec\theta\,, \\ &\mu(t)\fallingdotseq\nu(t)\fallingdotseq 0\,, && \mu'\fallingdotseq\nu'\fallingdotseq g\cosh t\,. \end{aligned}\right\} \tag{6.1}$$

Therefore, we have at first from the definition (2.3)

$$T(x,y,z)=-\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_0^{\infty}\frac{\exp\left[kz+ik\tilde{\omega}(\theta)\right]dkd\theta}{k\cos^2\theta-g+i\varepsilon\cos\theta}\,, \tag{6.2}$$

and from the expression (3.16)

$$\begin{aligned} T=&\frac{i}{2\pi}\int_{-\infty}^{-u_0-i\delta}\exp\left[ig(x+y\sinh t-iz\cosh t)\cosh t\right]\cosh t\,dt \\ &+\frac{1}{2\pi i}\int_{-\infty}^{\pi/2\cdot i}\exp\left[ig(-x+y\sinh t-iz\cosh t)\cosh t\right]\cosh t\,dt\,. \end{aligned} \tag{6.3}$$

The last expression agrees with the one obtained in the former paper of the author[2], who defined like as

$$T(x,y,z)=\frac{1}{\pi}O_{-2}^{(1)}(gx,gy,-gz)\,. \tag{6.4}$$

7. The Function for the Case without Advance Speed

When the advance speed becomes zero, it means that g tends infinity but ω^2/g, say K, stays at some finite value, we have from (2.12),

$$\kappa(\theta)\fallingdotseq\frac{\omega^2}{g}=K\,,\qquad \kappa'(\theta)\fallingdotseq g\sec^2\theta\fallingdotseq\infty\,. \tag{7.1}$$

Putting these values in the expression (3.20), we see the second term of the right hand side has no contribution and we have

$$\left[gT(x,y,z)\right]_{g=\infty}=\frac{K}{2\pi i}\int_{-\pi-i\infty}^{\varphi-\pi/2-i\varepsilon}\exp\left[ikR\cos(\theta-\varphi)+kz\right]d\theta\,, \tag{7.2}$$

where $R=\sqrt{x^2+y^2}$.

Deforming the path of integration in the complex θ-plane, we may integrate it respectively as the following portions,

$$\int_{-\pi-i\infty}^{\varphi-\pi/2-i\varepsilon}=\int_{-\pi+\varphi-i\infty}^{-\pi+\varphi}+\int_{\varphi-\pi}^{\varphi-\pi/2}+\int_{\varphi-\pi/2}^{\varphi-\pi/2-i\varepsilon}\ , \tag{7.3}$$

and we have

$$[gT]_{\varphi=\infty}=\frac{K}{2i}e^{Kz}H_0^{(2)}(KR)-\frac{K}{4}e^{Kz}\{H_0(KR)-Y_0(KR)\}$$
$$+\frac{K}{2\pi i}e^{Kz}\int_0^{-\sinh^{-1}z/R}\exp[-KR\sinh v]dx\ . \tag{7.4}$$

This agrees with the well known expression in this case[3].

8. Conclusion

As we see in the precedings, the fundamental singularity, which appears in the theory of ship motions in a sea way can be represented by a single integral, otherwise it is done by a double integral.

This is based on a simple reason. Namely, all such functions are harmonic so that they must be represented by a double Fourier integral like as

$$f(x,y,z)=\frac{1}{4\pi^2}\iint_{-\infty}^{\infty}\exp[z\sqrt{p^2+q^2}+ipx+iqy]F(p,q)dpdq\ , \tag{8.1}$$

but they must satisfy also the free surface condition (2.7), so that p and q must satisfy the relation:

$$(p-\omega)^2=g\sqrt{p^2+q^2}\ . \tag{8.2}$$

Therfore, the above double integral means substantially a simple integral, or we can say that all such functions can be represented by the summation of the elementary solution

$$\exp[z\sqrt{p^2+q^2}+ipx+iqy]\ , \tag{8.3}$$

in which p and q satisfies (8.2).

References

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