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Solutions of Minimum problems of the wave-making resistance of the doublet distribution on the line and over the area perpendicular to the uniform flow

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Abstract

The numerical data of the solutions of the minimum problem of the wave-making resistance of the doublet distribution perpendicular to the uniform flow on the line at the water surface and over the strip extending vertically to the infinite depth are given.

Under the appropriate conditions, the solutions are determined uniquely in both cases, and the mimimum value of the wave-making resistance increases rapidly with the decreasing velocity.

Introduction

The general discussion of the minimum problem of the wave-making resistance was discussed in the previous paper[7] and the present problem was also formulated partly in the same work.

This paper gives the numerical details of the solution of the minimum propelm of the wave-making resistance of the doublet distribution on the line at the water surface and over the strip extending to the infinite depth.

The former is a mathematical model of the planing surface and H. Maruo discussed in detail[1)~5)] but here we consider it only in the present point of view.

The latter has no explicit practical application.

1.1 Area distribution

Let us consider the water motion due to the doublet distribution over the area $1 \geqq y \geqq -1$ and $0 > z > \infty$, where the water flows down to the negative x-axis with the unit velocity, its mean surface is taken to be the $x-y$ plane and the z-axis vertically upwards.

Then, the wave making resistance of this distribution is given by the formula

$$R = (\rho g^4)/\pi \int_0^{\pi/2} |F(g \sec^2 \theta, \theta)|^2 \sec^5 \theta d\theta , \tag{1.1.1}$$

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with

$$F(k, \theta)=\int_{-\infty}^{0}\int_{-1}^{1} h(y, z)e^{kz-iky\sin\theta}dy\, dz\,, \tag{1.2.2}$$

where ρ means the water density and g the gravity constant in this unit system and the distribution is assumed symmetric with respect to the origin.

Moreover, if we assume the distribution uniform in the z-direction, we have by integrarton

$$R=\rho\frac{g^2\bar{B}^2}{\pi}\int_{0}^{\pi/2}|f(g\sec^2\theta\sin\theta)|^2\sec\theta\, d\theta\,, \tag{1.1.3}$$

with

$$f(q)=\int_{-1}^{1} H(y)\exp.(-iqy)dy, \qquad q=g\sec^2\theta\sin\theta\,, \tag{1.1.4}$$

where

$$\bar{B}=(1/2)\int_{-1}^{1} h(y, z)dy\,, \tag{1.1.5}$$

and

$$H(y)=h(y, z)/\bar{B}\,,$$

then the condition (1.1.5) goes to

$$\int_{-1}^{1} H(y)dy=2\,. \tag{1.1.6}$$

Namely, $2\bar{B}$ means the total sum of the doublet distribution in any horizotal section.

Let us introduce the wave resistance coefficient and interchanging the order of the integration in (1.1.3), we have

$$R/(\rho\bar{B}^2/8)=c_w=32g^2\int_{-1}^{1} H(y)G^*(y)dy\,, \tag{1.1.7}$$

with

$$G^*(y)=(1/2\pi)\int_{-1}^{1} H(y')K_0(g|y-y'|/2)dy'\,. \tag{1.1.8}$$

Where K means the modified Bessel function.

Now, if we assume the next expansion in Mathieu functions,

$$\left.\begin{aligned} &H(\cos\theta)=\varphi(\theta)/\sin\theta\,,\\ &\varphi(\theta)=\sum_{n=0}^{\infty} a_{2n}ce_{2n}(\theta, -q), \quad q=g^2/16\,, \end{aligned}\right\} \tag{1.1.9}$$

then we have from (1.1.8) by integration

$$G^*(\cos\theta)=\sum_{n=0}^{\infty}\lambda_{2n}a_{2n}ce_{2n}(\theta, -q)\,, \tag{1.1.10}$$

and

$$C_w = 16g^2\pi \sum_{n=0}^{\infty} \lambda_{2n} a^2{}_{2n}\,, \tag{1.1.11}$$

If we ask the minimum wave-making resistance, we have a solution immediately putting G^* a constant, that is,

$$G^*(\cos\theta) = \lambda = 2\lambda \sum_{n=0}^{\infty} (-1)^n A^{(2n)} ce_{2n}(\theta, -q)\,. \tag{1.1,12}$$

Since this must be equal to (1.1.10), we have

$$a_{2n} = 2\lambda A_0^{(2n)}/\lambda_{2n} = a_{2n}^*\,, \tag{1.1.13}$$

and the constant is to be determined by (1.1.6) as

$$\lambda = 1/\pi A\,, \quad A = \sum_{n=0}^{\infty} (A^{(2n)})^2/\lambda_{2n}\,. \tag{1.1.14}$$

Putting these value in (1.1.7), we have finally

$$Cw = 64g^2\lambda = 64g^2/(\pi A)\,. \tag{1.1.15}$$

The above calculation is very simple, but we must use the table of Mathieu function. Thence, we write down them by the ordinary trigonometrical expansion.

Once more, if write (1.1.9) as

$$\varphi(\theta) = \sum_{n=0}^{\infty} \alpha_{2n} \cos 2n\theta\,, \tag{1.1.16}$$

then

$$G^*(\cos\theta) = (1/2) \sum_{m=0}^{\infty} \varepsilon_m \cos 2m\theta \sum_{n=0}^{\infty} B_{2n,2m} \alpha_{2n}\,, \tag{1.1.17}$$

where $\varepsilon_0 = 1$ and $\varepsilon_m = 2$ for $m \neq 0$ and

$$\left.\begin{aligned}
B_{2n,2m} &= (1/\pi^2) \int_0^{\pi}\!\!\int_0^{\pi} K_0(g|\cos\theta - \cos\theta'|2) \cos 2n\theta \cos 2m\theta' d\theta\, d\theta' \\
&= (1/2) \sum_{\mu=0}^{\infty} \varepsilon_\mu (-1)^{m-\mu} I_{\mu-n}(g/4) I_{\mu+n}(g/4) [K_{m+\mu}(g/4) I_{m-\mu}(g/4) \\
&\quad + K_{m-\mu}(g/4) I_{m+\mu}(g/4)]\,, \\
B_{0,0} &= 4 \sum_{r=0}^{\infty} \lambda_{2r} (A_0^{(2r)})^2\,, \\
B_{0,2m} &= B_{2m,0} = 2(-1)^m \sum_{r=0}^{\infty} \lambda_{2r} A_0^{(2r)} A_{2m}^{(2r)} \quad \text{for} \quad m > 0 \\
B_{2n,2m} &= (-1)^{n+m} \sum_{r=0}^{\infty} \lambda_{2r} A_{2n}^{(2n)} A_{2m}^{2r}\,. \quad \text{for} \quad \text{n}, m > 0
\end{aligned}\right\} \tag{1.1.18}$$

Especially, in the minimum solution, we have

$$\left.\begin{aligned} G^*(\cos\theta)=\lambda=(1/2)\sum_{n=0}^{\infty}\alpha_{2n}B_{2n,0}\,, \\ \text{and}\qquad 0=\sum_{n=0}^{\infty}\alpha_{2n}B_{2n,2m}\quad \text{for}\quad m\geqq 1\,, \end{aligned}\right\} \tag{1.1.19}$$

and the condition (1.1.6) goes to

$$\alpha_0=2/\pi\,. \tag{1.1.20}$$

The computations are carried out by the latter formula to $g=0.4\sim 2.0$ and by the former to $g=2\sim 8$ and shown in Tables 1, 2, 3 and Fig. 1.

Table 1

g	0.4	0.8	1.2	1.6	2.0	4	$\sqrt{40}$	8
F_r	1.118	0.7906	0.6455	0.5590	0.5000	0.3536	0.2812	0.2500
$2/\pi A$	1.5551	1.1431	0.9196	0.7732	0.6683	0.4000	0.2731	0.2222
Cw	7.962	23.41	42.38	63.34	85.54	204.8	349.5	455.1
Cwe	8.743	26.30	48.31	72.92	99.03	239.3	405.6	524.9
r_0^*	0.0995	0.5853	1.589	3.167	5.346	25.60	69.08	113.8
r_c^*	1.1051	2.570	4.696	7.052	10.25	35.74	85.61	135.0
a	11.10	4.392	2.955	2.227	1.919	1.396	1.239	1.187
b	10.10	3.392	1.955	1.227	0.919	0.396	0.239	0.187
α_0	0.6366	0.6366	0.6366	0.6366	0.6366	0.6366	0.6366	0.6366
α_2	−0.0124	−0.0334	−0.0565	−0.0788	−0.0997	−0.1810	−0.2389	−0.2667
α_4	0	0	0	−0.0003	−0.0007	−0.0040	−0.0098	−0.0144
α_6	0	0	0	0	0	0.0001	−0.0004	−0.0008
α_8	0	0	0	0	0	0	0	0
$\varphi_a(0)$	0.6242	0.6032	0.5801	0.5575	0.5362	0.4515	0.3874	0.3546
β_0	0.6366	0.6366	0.6366	0.6366	0.6366	0.6366	0.6366	0.6366
β_2	0.0492	0.1428	0.2337	0.3553	0.4522	0.7808	0.9509	1.0164
β_4	0.0002	0.0017	0.0065	0.0199	0.0301	0.1612	0.3489	0.4668
β_6	0	0	0.0001	0.0003	0.0008	0.0137	0.0632	0.1177
β_8	0	0	0	0	0	0.0006	0.0068	0.0175
β_{10}	0	0	0	0	0	0	0.0001	0.0003
$\varphi_b(0)$	0.6860	0.7811	0.8769	1.0120	1.1196	1.5923	2.0065	2.2553

Table 2

g	0.4	0.8	1.2	1.6	2	4	$\sqrt{40}$	8
F_r	1.118	0.7906	0.6455	0.5590	0.5000	0.3536	0.2812	0.2500
$B_{0,0}$	2.44286	1.79628	1.44648	1.21824	1.05549	0.64512	0.45400	0.37744
$B_{0,2}$	0.00487	0.01295	0.02159	0.02961	0.03672	0.05874	0.06660	0.06724
$B_{0,4}$	0	0.00003	0.00011	0.00027	0.00050	0.00278	0.00648	0.00904
$B_{0,6}$	0	0	0	0	0	0.00006	0.00036	0.00074
$B_{2,2}$	0.24920	0.24691	0.24347	0.23916	0.23428	0.20646	0.17696	0.15962
$B_{2,4}$	0.00021	0.00041	0.00089	0.00172	0.00231	0.00717	0.01273	0.01592
$B_{2,6}$	0	0	0	0.00001	0.00001	0.00017	0.00066	0.00122
$B_{4,4}$	0.12491	0.12467	0.12426	0.12370	0.12301	0.11787	0.10999	0.10403
$B_{4,6}$	0.00002	0.00009	0.00019	0.00038	0.00057	0.00183	0.00393	0.00546
$B_{6,6}$	0.08331	0.08327	0.08312	..08296	0.08273	0.08111	0.077L5	0.07576

Table 3

q	0.25	1	2.5	4
g	2	4	$\sqrt{40}$	8
F_r	0.5000	0.3536	0.2812	0.2500
λ_0	0.53006	0.33596	0.25608	0.22568
λ_2	0.23202	0.19364	0.15043	0.12770
λ_4	0.12297	0.11737	0.10793	0.10006
λ_6	0.084	0.08104	0.07789	0.07503
a_0^*	0.88814	0.80137	0.64228	0.55116
a_2^*	0.17815	0.44815	0.66966	0.73834
a_4^*	0.00177	0.01776	0.08237	0.18163
a_6^*	0.00002	0.00021	0.00238	0.00821
a_8^*	0	0	0.00002	0.00019
b_0^*	0.93649	1.09883	1.28569	1.37913
b_2^*	0.37250	0.47674	0.38995	0.35853
b_4^*	0.37250	0.10260	0.18623	0.20719
b_6^*	0.00105	0.00684	0.02797	0.04676
b_8^*	0	0.00021	0.00259	0.00547
b_{10}^*	0	0	0.00007	0.00025

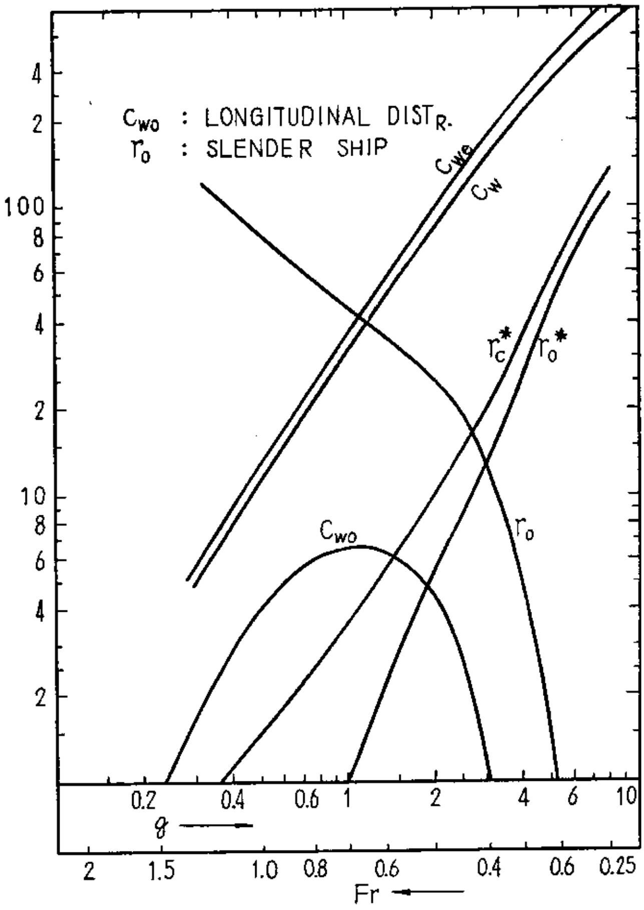

Fig. 1.

For a comparison, if we consider the elliptic distribution which represents approximately an elliptic cylinder, we obtain putting $\alpha_0=-\alpha_2=2/\pi$,

$$Cw_e=(64g^2/\pi[B_{0,0}-2B_{0,2}+B_{2,2}]\,, \tag{1.1.21}$$

1.2 Limit in high speed

When the velocity is very high, that is, g is very small, the problem becomes very simple as the kernel of the integral (1.1.8) plays simply as logarithmic.

We have approximately

$$\left.\begin{aligned}&B_{0,0}=\log(8/\gamma g)+(g^2/16)[\log(8/\gamma g)-1/4]\,,\\&B_{0,2}=(g/8)^2[\log(8/\gamma g)-1/2]\,,\\&B_{2,2}=1/4-g^2/192\,,\end{aligned}\right\} \tag{1.2.1}$$

where $\log \gamma = 0.5772\ldots$, Euler's constant, and then

$$\left.\begin{array}{l} \alpha_0 = 2/\pi, \quad \alpha_{2n} \doteqdot 0 \quad \text{for} \quad n \geqq 1 \\ Cw \doteqdot (64g^2/\pi) \log (8/\gamma g)\,. \end{array}\right\} \tag{1.2.2}$$

For the elliptic distribution, we obtain from (1.1.21)

$$Cw_e \doteqdot (64g^2/\pi)[\log (8/\gamma g) + 1/4]\,. \tag{1.2.3}$$

Hence, the difference between this and the optimum distribution is not so great.

This value may be compared with the one of the longitudinal similar distribution in the reference[7], namely,

$$R_l/(\rho \bar{B}^2 V^2/L^2 2) = Cw_0 \doteqdot 8g^2 L^2/(\pi V^4) \log (8V^2/\gamma g L)\,, \tag{1.2.4}$$

where V means the advance velocity and L the length of the distribution and $2\bar{B}$ the total sum of the distribution.

Putting the breadth T and the velocity V, we may rewrite (1.2.2) as

$$R_t/(\rho \bar{B}^2 V^2/L^2 2) = Cw \doteqdot 16g^2 T^2/(\pi V^4) \log (16V^2/\gamma g T)\,. \tag{1.2.5}$$

If we preserve $\bar{B}$ and V, the ratio of the wave-making resistance between them may be

$$R_t/R_l \doteqdot 2 \log (16V^2/\gamma g T)/\log (8V^2/\gamma g L)\,. \tag{1.2.6}$$

This ratio equals to unity at

$$T/L \doteqdot 4.2(V/\sqrt{gL})\,, \tag{1.2.7}$$

Namely, R_l will be greater than R_t when T is greater than $4.2(V/\sqrt{gL})L$.

Since the Froude number in our case is very high, this means the extra-ordinary aspect ratio of the transverse distribution.

1.3 Stream line at the great depth

The foregoing analysis treats only doublet distributions but not actual solid bodies.

To obtain a corresponding body shape from a doublet distribution is a difficult problem especially near the water surface, but we may calculate easily stream lines at great depth where the disturbance of the water wave does not contribute very much.

Now, let us consider the $x-y$ plane and the doublet distribution $\bar{B}H(y)$ on the segment $|y| \leqq 1$.

Define the complex velocity potential as

$$f(z) = \bar{B}/\pi \int_{-1}^{1} \frac{H(y')dy'}{z - iy'}\,, \tag{1.3.1}$$

where $z = x + iy$, and introduce the transformation

$$\left.\begin{aligned} &z=i\cosh u, \quad u=t+i\theta\,, \\ &y=\cosh t\cos\theta, \quad y'=\cos\theta'\,, \\ &x=-\sinh t\sin\theta\,. \end{aligned}\right\} \tag{1.3.2}$$

Then, assuming the expansion (1.1.16), we have

$$f[z(t+i\theta)]=(-i\bar{B}/\sinh u)\sum_{n=0}^{\infty}\alpha_{2n}\exp(-2nu)\,.$$

Hence, the integral equation to determine the stream line is to be

$$\left.\begin{aligned} &\cosh t\cos\theta=\bar{B}/(\sinh^2 t+\sin^2\theta)[\sinh t\cos\theta\, h_1(t,\theta) \\ &\qquad -\cosh t\sin\theta\, h_2(t,\theta)]\,, \\ &\text{with} \\ &h_1(t,\theta)=\sum_{n=0}^{\infty}\alpha_{2n}e^{-2nt}\cos 2n\theta\,, \\ &h_2(t,\theta)=\sum_{n=1}^{\infty}\alpha_{2n}e^{-2nt}\sin 2n\theta\,. \end{aligned}\right\} \tag{1.3.3}$$

For example, if we put $\alpha_0=-\alpha_2=2/\pi$ and $\alpha_{2n}=0$ for $n\geqq 1$, this equation degenerates to

$$\cosh t\cos\theta=\frac{4}{\pi}\bar{B}e^{-t}\cos\theta\,,$$

and solved as

$$t=(1/2)\log\left(\frac{8}{\pi}B-1\right). \tag{1.3.4}$$

This equation gives a real positive value of t whenever

$$\bar{B}\geqq\pi/4=0.7854\,. \tag{1.3.5}$$

And the stream line gives an ellipse with the longer and shorter radius respectively

$$\frac{4\bar{B}}{\pi\sqrt{\frac{8}{\pi}\bar{B}-1}}\,, \qquad \frac{\left(\frac{4}{\pi}\bar{B}-1\right)}{\sqrt{\frac{8}{\pi}\bar{B}-1}} \tag{1.3.6}$$

and the area

$$2\bar{B}\left(\frac{\bar{B}-\frac{\pi}{4}}{\bar{B}-\frac{\pi}{8}}\right)<2\bar{B}\,.$$

If the condition (1.3.5) does not hold, such distribution can not represent a solid shape, namely, the transverse distribution necesitates very large $\bar{B}$. This is a different point with the longitudinal one in which case we can always obtain a solid shape with an arbitrary small $\bar{B}$.

Now, as we obtain in the preceding, the optimum distribution has almost always very large first term and negligible other terms.

Thence, we consider nextly the case in which the first term equals to $2/\pi$ and other ones vanish.

The equation (1.3.3) gives

$$\sinh^2 t+\sin^2\theta=\frac{2}{\pi}\bar{B}\tanh t\,. \tag{1.3.7}$$

To represent a solid body, this equation must have a real positive root for t at $\theta=\pi/2$, namely, since we may write (1.3.7) as

$$x^6=\left(\frac{2}{\pi}\bar{B}\right)^2(1-x^2)\,,\quad \text{for}\quad x=\cosh t\,.$$

To have a real value of t from this equation, it must be

$$\bar{B}\geqq\pi\sqrt{27}/4=4.081\,. \tag{1.3.8}$$

Some stream lines are shown in Fig. 2.

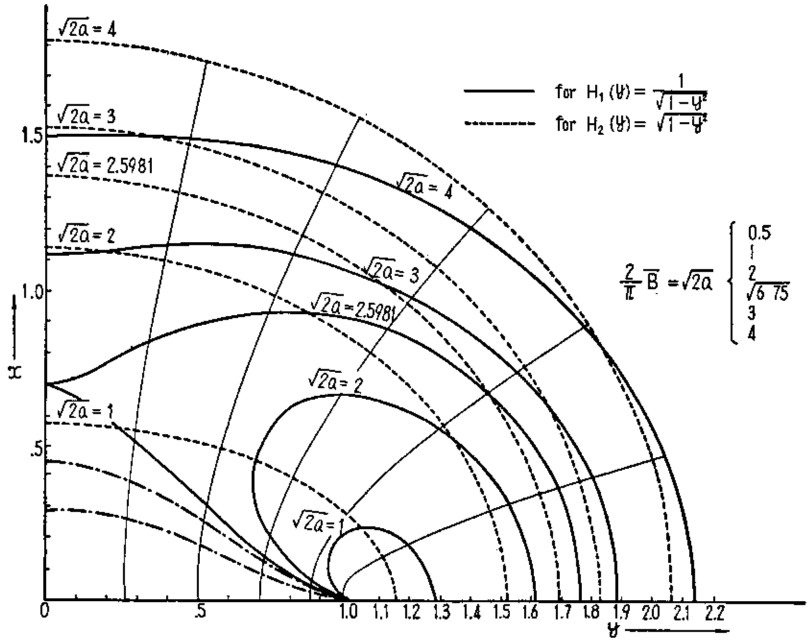

Fig. 2. Stream lines.

As we see in these examples, we can not take small value for $\bar{B}$, so that the comparison with the longitudinal distribution as in the preceding paragragh may be understood in very restricted sense.

Namely, although its result says that the breadth must be very large for reducing the wave resistance, in our case, but we can not do so.

This fact suggests that the wave resistance of the transverse distribution is almost always greater than the corresponding longitudinal distribution.

2.1 Line distribution

Nextly, let us consider an extreme case in which the distribution concentrates on a line $|y| \leqq 1$ at $z=0$.

The minimum solution was given in the reference 7), here we repeat it.

The wave resistance is given as (1.1.1), defining the total sum of the distribution

$$\lim_{z\to 0}\int_z^0\int_{-1}^1 h(y, z)\, dy\, dz=\nabla \tag{2.1.1}$$

we may rewrite it as

$$r^*\equiv R/(\rho g\nabla^2/8)=2\int_{-1}^1 H(y)G(y)\, dy\,, \tag{2.1.2}$$

with

$$G(y)=\frac{g^3}{2}\left(1-\frac{2}{g^2}\frac{d^2}{dy^2}\right)G^*(y)\,, \tag{2.1.3}$$

where G^* is defined by (1.1.8) and $H(y)$ and (1.1.6) and (1.1.9)[7].

The solution of the minimum problem is the same as in § 1.1 and it gives the distribution (1.1.13) and the wave resistance

$$r_1^*=2g^3\lambda=2g^3/(\pi A)\,, \tag{2.1.4}$$

with λ and A of (1.1.14).

One of the interesting result of the reference 7) is the existence of the quasi-wave free distribution, that is, if

$$G^*(y)=C\cosh(gy/\sqrt{2})\,, \tag{2.1.5}$$

then

$$G(y)=0 \quad \text{and} \quad r^*=0\,.$$

Such solution is given as follows[7].

Since we have an expansion

$$\cosh(gy/\sqrt{2})=\sum_{n=0}^{\infty}\varepsilon_n I_{2n}(g/\sqrt{2})\cos 2n\theta=\sum_{n=0}^{\infty}(-1)^n C_{2n}ce_{2n}(\theta, -q)\,,$$

where

$$C_{2n}=\frac{2(-1)^n A_0^{(2n)}}{ce_{2n}(0, q)}Ce_{2n}(z_0, -q)\,, \quad z_0=\sinh^{-1}1\,, \tag{2.1.7}$$

we have the solution equating (2.1.5) to (1.1.10) in the form as (1.1.9), namely,

$$a_{2n}=C(-1)^n C_{2n}/\lambda_{2n}\,. \qquad (2.1.7)$$

The costant C may be determined arbitrary in general, but we put it a definite value by the condition (1.1.6), that is,

$$C=2/(\pi D)\,,\quad D=\sum_{n=0}^{\infty} C_{2n}A_0^{(2n)}/\lambda_{2n}\,, \qquad (2.1.8)$$

Now, we write this solution as

$$\left.\begin{aligned} &H_b(y)=\varphi_b(\theta)/\sin\theta \\ &\varphi_b(\theta)=\sum_{n=0}^{\infty} b_{2n}^* ce_{2n}(\theta,\,-q)\,, \\ &b_{2n}^*=2(-1)^n C_{2n}/(\pi D\lambda_{2n})\,, \end{aligned}\right\} \qquad (2.1.9)$$

and the former solution as

$$\left.\begin{aligned} &H_a(y)=\varphi_a(\theta)/\sin\theta\,, \\ &\varphi_a(\theta)=\sum_{n=0}^{\infty} a_{2n}^*\, ce_{2n}(\theta,\,-q)\,, \end{aligned}\right\} \qquad (2.1.10)$$

with a_{2n}^* by (1.1.13).

Although $H(\pm 1)$ tends infinity in these solutions, but we may not imagine such distribution in practice.

Thence, let us put the condition

$$H(\pm 1)=0 \quad \text{or} \quad \varphi(0)=\varphi(\pi)=0\,, \qquad (2.1.11)$$

Combining H_a and H_b as

$$\left.\begin{aligned} &H_c(y)=aH_a(y)-bH_b(y)\,, \\ &\int_{-1}^{1} H_c(y)dy=2\,, \end{aligned}\right\} \qquad (2.1.12)$$

this condition is satisfied when

$$b=a-1\,,\quad a=\varphi_b(0)/[\varphi_b(0)-\varphi_a(0)]\,. \qquad (2.1.13)$$

Then, the wave resistance will be

$$r_c^*=2g^3\lambda a=2g^3a/(\pi A)\,. \qquad (2.1.14)$$

Lastly, if we expand $\varphi_b(\theta)$ of (2.1.9) in Fourier series of

$$\varphi_b(\theta)=\sum_{n=0}^{\infty}\beta_{2n}\cos 2n\theta\,, \qquad (2.1.15)$$

the coefficients are determined by the next equations

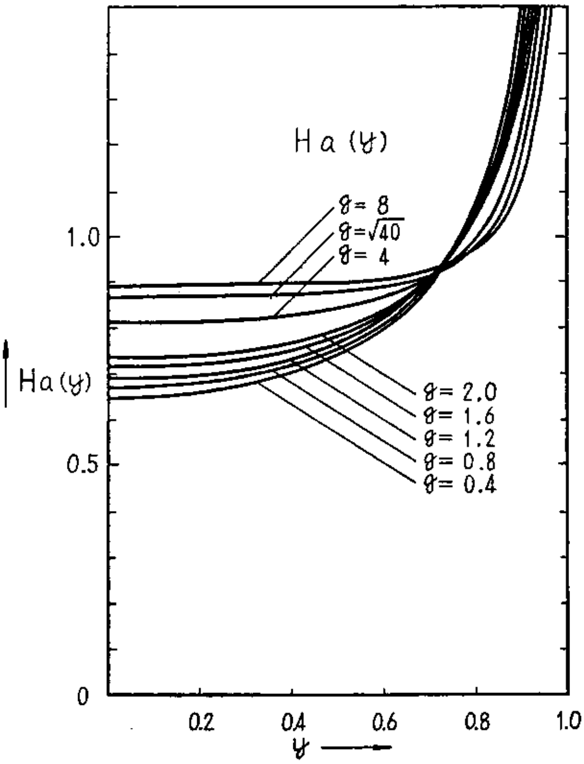

Fig. 3 (a)

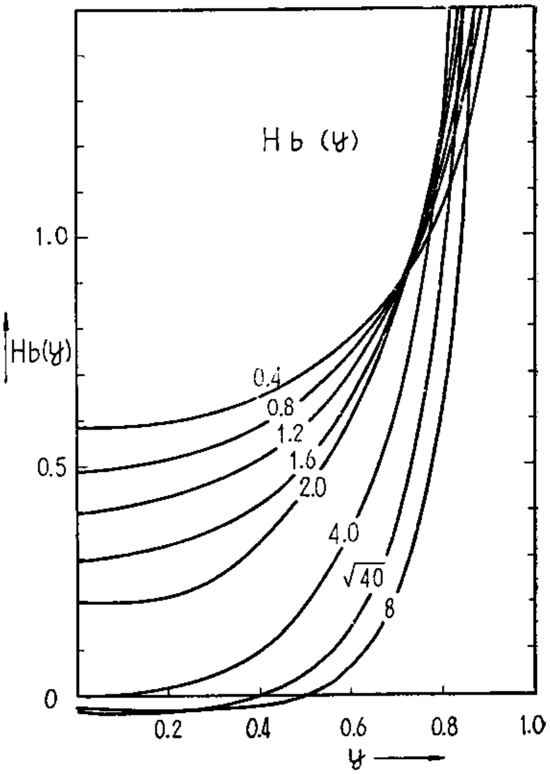

Fig. 3 (b)

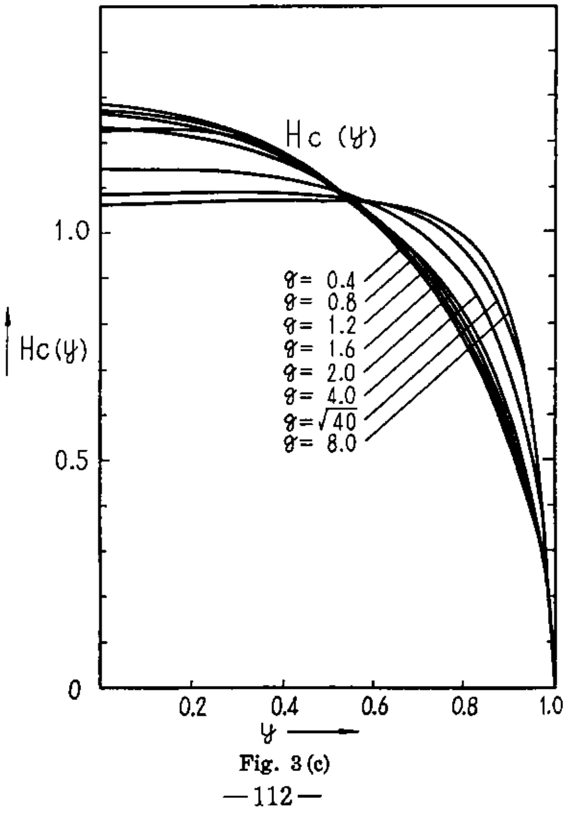

Fig. 3 (c)

$$\beta_0=2/\pi \quad \text{and} \quad \sum_{n=0}^{\infty}\beta_{2n}B_{2n,2m}=CI_{2m}(g/\sqrt{2}) \quad \text{for} \quad m=0, 1, 2, \cdots \tag{2.1.16}$$

Obviously, we may also calculate them by the Fourier expansions of Mathieu functions using the coefficients b_{2n}^*.

The numerical results are shown in Tables 1, 2, 3, Figs. 1 and 3.

2.2 Limit in high speed

Firstly, in the high speed, the minimum solution nearly equals to the quasi-wave-free solution.

Namely, considering up to the second order terms, we have from (1.1.19)

$$\varphi_a(\theta)\doteqdot(2/\pi)[1-(g^2/16)\log(8/\gamma g)\cos 2\theta], \tag{2.2.1}$$

and from (2.1.15)

$$\varphi_b(\theta)\doteqdot(2/\pi)[1+(3g^2/16)\log(8/\gamma g)\cos 2\theta]. \tag{2.2.2}$$

Hence, using these values, we obtain by the definition (2.1.12) and (2.1.13),

$$\varphi_c(\theta)\doteqdot(4/\pi)\sin^2\theta, \quad Hc(\theta)=(4/\pi)\sin\theta, \tag{2.2.3}$$

and the wave resistance is to be

$$r_0^*\doteqdot(2g^3/\pi)\log(8/\gamma g), \tag{2.2.4}$$

$$r_c^*\doteqdot 8g/\pi. \tag{2.2.5}$$

The last value corresponds to the induced drag of a wing as H. Maruo elucidated1).

Namely, introducing the lift P, the lift coefficient C_L, the aspect ratio λ and the drag lift ratio ε as

$$\left.\begin{aligned} &P=\rho g\nabla, \\ &C_L=P/\{(\rho/2)V^2BL), \\ &\lambda=B/L, \\ &\varepsilon=R/P, \end{aligned}\right\} \tag{2.2.6}$$

since we can write

$$r^*=R/(\rho g\nabla^2/B^3)=2\varepsilon\lambda/(C_LV^2/gB), \tag{2.2.7}$$

the value of (2.2.5) is the same one as

$$\varepsilon_0\doteqdot 2C_L/\pi\lambda, \tag{2.2.8}$$

where B means the breadth and L the chord length.

Then, in the other hand, the value of (2.2.4) will be much smaller than this value, that is,

$$\varepsilon \doteqdot \varepsilon_0 (g^2 B^2/16 V^4) \log (16 V^2/\gamma g B) . \tag{2.2.9}$$

Regardless to say here, it is out of question if we can imagine a planing surface as represented by (2.2.1).

It is also interesting to compare with the two dimensional wave resistance of a pressure distribution P per unit breadth, with infinite aspect rario, given as

$$\varepsilon_2 = R/P \doteqdot gP/(\rho V^4) = (gL/2V^2) C_L , \tag{2.2.10}$$

where

$$C_L = P \Big/ \left(\frac{\rho}{2} V^2 L \right) .$$

As we see easily, the value of (2.2.9) is also smaller the last one when the velocity is high.

With respect to the comparison with the longitudinal distribution, H. Maruo gave many examples and explaned that the wave-resistance of the transverse distribution might be smaller than the longitudinal ones in the high speed[1].

Hence, we have no more explanation than his.

2.3 Considerations on the velocity field

If we can add to his work, it is to the point if the distribution as (2.2.1) can really represent a ship shape.

Now, consider the velocity is very great, then the velocity potential degenerates to the one of the lifting line, namely,

$$\varphi(x, y, z) = \frac{1}{2\pi} \int_{-1}^{1} H(y') \frac{z}{z^2 + (y-y')^2} \left[1 - \frac{x}{\sqrt{x^2 + (y-y')^2 + z^2}} \right] dy' . \tag{2.3.1}$$

At the $y-z$ plane for $x=0$, this potential goes to the same form as (1.3.1), so that figures of those stream line, Fig. 2, may give contours of the same upwards velocity in this case.

The stream lines and the equi-potential lines are given in Fig. 4 (a) and (b) for the characteristic two loadings as

$$H_1(y) = 1/\sqrt{1-y^2} , \tag{2.3.2}$$

$$H_2(y) = \sqrt{1-y^2} , \tag{2.3.3}$$

The difference between both cases is quite clear and in fact the induced up-wards velocity for the case of H_1 vanishes at $z=0$, so that we might have no induced drag.

However, since we have very strong induced velocity just outside of the end points, we might have a concentrated drag at these points.

In any way, it would be out of question unless we could establish its physical reality.

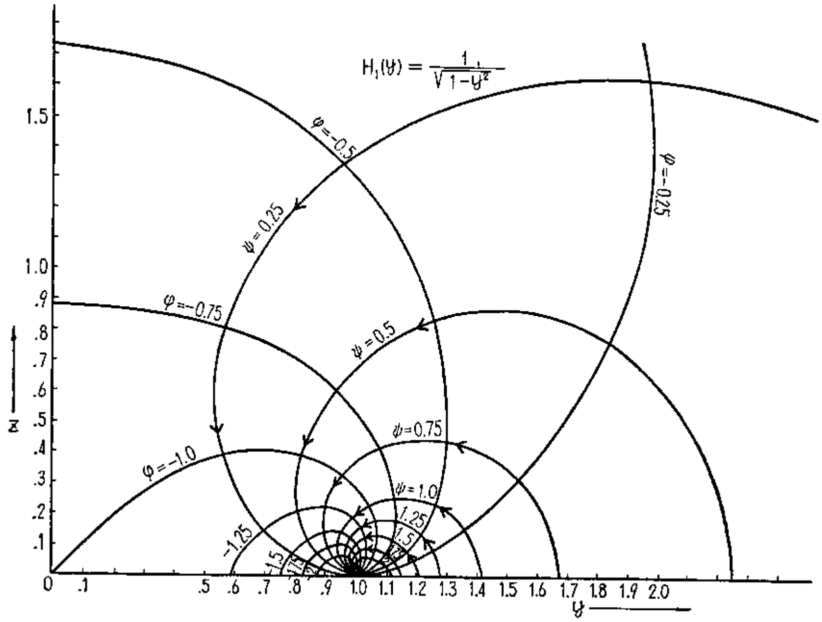

Fig. 4 (a). Contour of the Velocity Potential and Stream function.

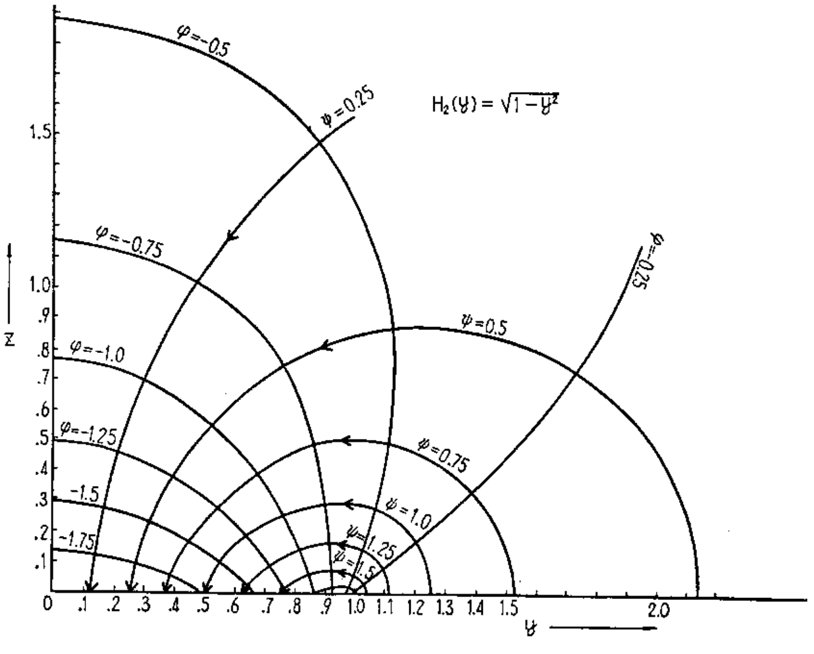

Fig. 4 (b). Contour of the Velocity Potential and Stream sunction.

3 Wave-free distribution

Lastly, we consider the wave-free distribution over the area on the $y-z$ plane.

In the sense of (1.1.2), let us define

$$F(k, q)=\iint_S h(y, z) \exp. (kz-iqy)dS, \tag{3.1}$$

where S means the area over which the doublet with the axis in the x-direction $h(y, z)$ distributes and $q=\sqrt{k(k-g)}$.

If we introduce the function $m(y, z)$ by the equation

$$h(y, z)=\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}+g\frac{\partial}{\partial z}\right)m(y, z), \tag{3.2}$$

this function may be determined uniquely except an arbitrary boundary values.

Hence, putting the condition, say,

$$m(y, z)=0 \text{ on } C, \text{ the boundary curve of } S, \tag{3.3}$$

and integrating (3.1) partially, we have

$$F(k, q)=\int_c \exp. (kz-iqy)\frac{\partial}{\partial \nu}m(y, z)\, ds, \tag{3.4}$$

where ν means the outwards normal to the cuve C.

Hence, if

$$\frac{\partial}{\partial \nu}m(y, z)=0 \text{ on } C, \tag{3.5}$$

we have

$$F(k, q)=0, \tag{3.6}$$

and the wave resistance vanishes.

Many wave-free distributions belong to this species. For example, let S be the strip of the breadth 2 and extending from the water surface to the infinite depth, and $m(y, z)$ be able to write

$$m(y, z)=Y(y)Z(z). \tag{3.7}$$

In this case, the boundary conditions on the lower edge at infinity need not be satisfied and all functions satisfying the conditions

$$Y(\pm 1)=Y'(\pm 1)=Z(0)=Z'(0)=0, \tag{3.8}$$

will become the wave-free distributions.

Nextly, the formula (3.4) tells us directly the wave resistance of $m(y, z)$ equals to the one of the line distribution $\frac{\partial}{\partial \nu}m(y, z)$ on the curve C.

That is to say, if we put the negative distribution of the last quantity on the same curve, the wave resistance will be clearly cancelled out.

Lastly, assuming the partition (3.7) over the same area, (3.1) can be written

$$F(k, q)=Y^*(q)Z^*(k)\,, \tag{3.9}$$

with

$$Y^*(q)=\int_{-1}^{1} Y(y)\exp.(-iqy)dy\,,\quad q=\sqrt{k(k-g)}\,, \tag{3.10}$$

$$Z^*(k)=\int_{-\infty}^{0} Z(z)\exp.(kz)dz\,,\quad k>g\,. \tag{3.11}$$

Are there the function for which the integral (3.10) or (3.11) vanishes respectively ?

The answer is none for the appropriate class of the functions $Y(y)$ and $Z(z)$ by the theory of the Fourier and Laplace transformation.

All these present considerations are confined in themselves to the doublet distribution with its axis to the x-direction, but if we introduce more complicated singularities, we may obtain more interesting examples.

Conclusion

We have solved the minimum problem of the wave-making resistance of the line and area distribution perpendicular to the uniform flow and obtained the conclusions as follows;

For the area distribution over the strip extending to the infinite depth,

1. The optimum distribution is determined uniquely and it has infinite strength at both ends.
2. As shown in figures, its stream line at great depth does not close when its strength is not sufficiently strong.
3. The minimum wave resistance is very much higher than the one of the longitudinal distribution and its coefficient becomes greater by decreasing velocity.

For the line distribution on the water surface,

1. There is a quasi-wave free distribution but it tends also infinity at both ends.
2. Such distribution may not be realized physically.
3. The minimum problem has a solution except such one and also has infinite strength at ends.
4. Combining both distributions as the way in which both end strength vanish, we can obtain the same conclusion as H. Maruo did, namely, it tends to the elliptic distribution in very high speed.

In the last paragraph, we give a few types of the wave free distribution.

It may be remarked that the freedom of such distribution is narrower than the one in the case of the longitudinal distribution.

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