Mass flux density, momentum, and flow velocity

Kiyoshi Maruyama

Department of Earth and Ocean Sciences, National Defense Academy,

Yokosuka, Kanagawa 239-8686, Japan

December 9, 2025

Abstract

In geophysical fluid dynamics, it is believed that, under the Boussinesq approximation, the equation of continuity need not be satisfied. This note proves that the mass flux density of a fluid divided by the density is the momentum per unit mass of the fluid. This indicates, since the velocity of a fluid is defined as the momentum per unit mass, that the density and the velocity of a fluid must always satisfy the equation of continuity. The belief in geophysical fluid dynamics is thus shown to be groundless.

1. Introduction

When there is no source of mass, the density ρ and the velocity $\boldsymbol{u}$ of a fluid must satisfy the equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0. \tag{1}$$

No other terms can appear in this equation of continuity; this is stated clearly by Landau & Lifshitz (1987, the last paragraph of § 49).

Nevertheless, in geophysical fluid dynamics, it is believed that the equation of continuity (1) need not be fulfilled under the Boussinesq approximation (see e.g. Pedlosky 1987, § 1.4; Cushman-Roisin & Beckers 2011, § 3.7). This implies that, at least in geophysical fluid dynamics, the above fundamental proposition of fluid mechanics is not fully acknowledged.

It may therefore be worthwhile to prove the proposition, though it is almost obvious. This can be done according to a sketchy account by Landau & Lifshitz.

2. Mass flux density and momentum

We consider the motion of a fluid whose density is ρ, and denote the mass flux density of the fluid by $\boldsymbol{j}$. When there is no source of mass, ρ and $\boldsymbol{j}$ satisfy the following equation (Landau & Lifshitz 1987, § 49):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{j} = 0. \tag{2}$$

Let us introduce the velocity $\boldsymbol{q}$ defined by

$$\boldsymbol{q} = \boldsymbol{j}/\rho, \tag{3}$$

and let $\boldsymbol{p}$ denote the momentum per unit mass of the fluid. Our objective is to show, on the basis of (2), that $\boldsymbol{q} = \boldsymbol{p}$.

To this end, we consider at an arbitrary instant $t = t_0$ an arbitrary volume Ω in the fluid. Next, let Ω_t be a volume composed of points moving in the fluid with the velocity $\boldsymbol{q}$; we assume that Ω_t coincides with Ω at $t = t_0$.

The mass M contained in Ω_t is given by

$$M = \int_{\Omega_t} \rho dV. \tag{4}$$

The rate of change of M can be calculated as follows (see Aris 1962, § 4.22):

$$\begin{aligned}\frac{dM}{dt} &= \int_{\Omega_t} \left(\frac{\partial \rho}{\partial t} + \boldsymbol{q} \cdot \nabla \rho + \rho \nabla \cdot \boldsymbol{q} \right) dV \\ &= \int_{\Omega_t} \left(\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{j} \right) dV. \end{aligned} \tag{5}$$

Substituting (2) into (5), we obtain

$$\frac{dM}{dt} = 0. \tag{6}$$

This indicates that the volume Ω_t moving with the velocity $\boldsymbol{q}$ defined by (3) is a material volume.

Let us next consider the following quantity:

$$\int_{\Omega_t} \rho \boldsymbol{r} dV. \tag{7}$$

Here $\boldsymbol{r}$ denotes the position in the space containing the fluid. We can calculate, using (2), the rate of change of this quantity as follows (see Aris 1962, § 4.3):

$$\begin{aligned}\frac{d}{dt} \int_{\Omega_t} \rho \boldsymbol{r} dV &= \int_{\Omega_t} \rho \left\{ \frac{\partial \boldsymbol{r}}{\partial t} + (\boldsymbol{q} \cdot \nabla) \boldsymbol{r} \right\} dV \\ &= \int_{\Omega_t} \rho \boldsymbol{q} dV. \end{aligned} \tag{8}$$

On the other hand, we observe that

$$\int_{\Omega_t} \rho \boldsymbol{r} dV = M \boldsymbol{r}_{\mathrm{cm}}, \tag{9}$$

where $\boldsymbol{r}_{\mathrm{cm}}$ denotes the position of the center of mass of Ω_t. Hence, by virtue of (6), we obtain

$$\frac{d}{dt} \int_{\Omega_t} \rho \boldsymbol{r} dV = M \frac{d\boldsymbol{r}_{\mathrm{cm}}}{dt}. \tag{10}$$

The right-hand side of (10) is the momentum of the material volume Ω_t. Thus, in terms of the momentum per unit mass $\boldsymbol{p}$, (10) can be written as

$$\frac{d}{dt} \int_{\Omega_t} \rho \boldsymbol{r} dV = \int_{\Omega_t} \rho \boldsymbol{p} dV. \tag{11}$$

Comparing (8) and (11), we find that

$$\int_{\Omega_t} \rho \boldsymbol{q} dV = \int_{\Omega_t} \rho \boldsymbol{p} dV. \tag{12}$$

Since Ω_t coincides with Ω at $t = t_0$, it follows from (12) that, at $t = t_0$,

$$\int_{\Omega} \rho(\boldsymbol{q} - \boldsymbol{p}) dV = 0. \tag{13}$$

Noting that Ω and t_0 are arbitrary and that $\rho > 0$, we get

$$\boldsymbol{q} = \boldsymbol{p}. \tag{14}$$

This is the desired result: it shows that the mass flux density of a fluid divided by the density is the momentum per unit mass of the fluid, and hence that the mass flux density of a fluid is the momentum per unit volume of the fluid.

3. Conclusion

In fluid mechanics, the velocity $\boldsymbol{u}$ of a fluid is defined as the momentum per unit mass (Landau & Lifshitz 1987, § 49). Accordingly, we see from (14) that

$$\boldsymbol{q} = \boldsymbol{u}. \tag{15}$$

Thus, from (2), (3), and (15), it follows that, when there is no source of mass, the density ρ and the velocity $\boldsymbol{u}$ of a fluid must always satisfy the equation of continuity (1); this is true even under the Boussinesq approximation.

Finally, let us suppose that, when there is no source of mass, the equation of continuity is violated for some fluid:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) \neq 0. \tag{16}$$

Considering the above result, this implies the following: ρ is not the density of the fluid, or $\boldsymbol{u}$ is not the momentum per unit mass of the fluid. The recognition of this fact is crucial in discussing the energetics of a fluid under the Boussinesq approximation (Maruyama 2014).

References

[1] Aris, R. 1962 *Vectors, Tensors, and the Basic Equations of Fluid Mechanics.* Dover.

[2] Cushman-Roisin, B. & Beckers, J. M. 2011 *Introduction to Geophysical Fluid Dynamics: Physical and Numerical Aspects,* 2nd ed. Academic Press.

[3] Landau, L. D. & Lifshitz, E. M. 1987 *Fluid Mechanics,* 2nd ed. Butterworth-Heinemann.

[4] Maruyama, K. 2014 Energetics of a fluid under the Boussinesq approximation. `arXiv:1405.1921 [physics.flu-dyn]`.

[5] Pedlosky, J. 1987 *Geophysical Fluid Dynamics,* 2nd ed. Springer.