Rational derivation of the Boussinesq approximation

Kiyoshi Maruyama

Department of Earth and Ocean Sciences, National Defense Academy,

Yokosuka, Kanagawa 239-8686, Japan

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Abstract

This note derives the Boussinesq approximation in a manner consistent with the conservation law of mass. It is shown that the governing equations of a fluid under the approximation can be obtained on the basis of the assumption that the fluid is incompressible, in the sense that the density of the fluid is constant. The equation of motion, in particular, can be formulated with the help of the conservation law of energy. The conditions for the approximation to be valid are also discussed.

1. Introduction

The Boussinesq approximation is widely used to describe the motion of a fluid with a nonuniform temperature distribution. In deriving the approximation, it is traditionally assumed that the density of the fluid is a linear function only of the temperature of the fluid (see e.g. Landau & Lifshitz 1987, § 56; Mihaljan 1962). It can be shown, however, that this assumption violates the conservation law of mass (Maruyama 2014). This is a fatal defect of the traditional method for deriving the approximation.

Thus, in view of the importance of the approximation, this note introduces a method to derive the approximation in a manner consistent with the conservation law of mass: the fluid is assumed to be incompressible in the sense that its density is constant. It is shown that the need for the buoyancy force in the approximation can be inferred on the basis of the conservation law of energy. The conditions necessary for the approximation to be valid (Spiegel & Veronis 1960) are also revised accordingly.

2. Governing equations under the Boussinesq approximation

We consider the motion of a fluid in a uniform gravitational field; it is assumed that the fluid occupies a fixed domain Ω. We set up in the domain a system of rectangular coordinates (x_1, x_2, x_3) with the x_3-axis taken vertically upwards. The unit vectors in the positive x_1-, x_2-, and x_3-directions are respectively denoted by $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$. In the following, Latin indices are consistently used to represent the numbers 1, 2, and 3; the summation convention is also implied.

2.1. Basic assumptions

First of all, we assume that the fluid is incompressible in the sense that the density ρ of the fluid is constant:

$$\rho = \rho_0. \tag{2.1}$$

The thermal expansion coefficient β of the fluid, however, is assumed not to vanish:

$$\beta = v^{-1}(\partial v/\partial T)_p \neq 0, \tag{2.2}$$

where $v = \rho^{-1}$ is the specific volume of the fluid; T and p are the temperature and the pressure of the fluid, respectively. We assume that T can be expressed in the form

$$T = T_0 + T', \tag{2.3}$$

where T_0 is a constant reference temperature, and T' the small deviation from T_0. We also write p, denoting by p' the small perturbation pressure, as follows:

$$p = p_0 + p'. \tag{2.4}$$

Here the hydrostatic pressure p_0 is defined by

$$p_0 = -\rho_0 g x_3 + \text{constant}, \tag{2.5}$$

in which g denotes the acceleration due to gravity.

2.2. Equation of continuity

Now, let us consider the equation of continuity

$$\rho^{-1} D\rho/Dt + \nabla \cdot \boldsymbol{u} = 0, \tag{2.6}$$

in which D/Dt denotes the material derivative, and $\boldsymbol{u}$ is the velocity of the fluid. The substitution of (2.1) into (2.6) yields

$$\nabla \cdot \boldsymbol{u} = 0. \tag{2.7}$$

This is the equation of continuity under the Boussinesq approximation.

2.3. Equation of motion

Let us next consider the equation of motion: using (2.4), it can be written as

$$\rho \frac{D\boldsymbol{u}}{Dt} = -\nabla p' + \frac{\partial \tau_{ij}}{\partial x_j}\boldsymbol{e}_i - (\rho - \rho_0) g \boldsymbol{e}_3, \tag{2.8}$$

where τ_{ij} are the components of the viscous stress tensor. Substituting (2.1) into (2.8), we obtain

$$\rho_0 \frac{D\boldsymbol{u}}{Dt} = -\nabla p' + \frac{\partial \tau_{ij}}{\partial x_j}\boldsymbol{e}_i. \tag{2.9}$$

This equation, however, must contain an additional term in order to be consistent with the conservation law of energy; our aim in the following is to demonstrate this fact.

To this end, we first need to consider the total energy of the fluid:

$$\int_\Omega \rho_0 \left(\tfrac{1}{2}|\boldsymbol{u}|^2 + g x_3 + e\right) dV. \tag{2.10}$$

Here e denotes the specific internal energy of the fluid. The conservation law of energy requires that this energy should satisfy the equation

$$\frac{d}{dt}\int_\Omega \rho_0 \left(\tfrac{1}{2}|\boldsymbol{u}|^2 + g x_3 + e\right) dV = \int_\Sigma u_i \tau_{ij} n_j dS - \int_\Sigma \boldsymbol{q}\cdot\boldsymbol{n} dS, \tag{2.11}$$

in which Σ denotes the boundary of Ω, and $\boldsymbol{n}$ the unit outward normal on Σ; u_i and n_j are the components of $\boldsymbol{u}$ and $\boldsymbol{n}$, respectively; $\boldsymbol{q}$ is the heat flux density due to thermal conduction. This equation states that the total energy of the fluid changes owing to the work done by the viscous force acting on Σ and owing to the heat transfer across Σ.

On the other hand, it is obvious that

$$\frac{d}{dt}\int_\Omega \rho_0 g x_3 \, dV = 0. \tag{2.12}$$

Thus the potential energy of the fluid is invariable.

We next proceed to derive, after Maruyama (2014), the equation that describes the rate of change of the internal energy of the fluid. This equation can be obtained from the general equation of heat transfer (see Landau & Lifshitz 1987, § 49):

$$\rho T \frac{Ds}{Dt} = \tau_{ij}\frac{\partial u_i}{\partial x_j} - \nabla\cdot\boldsymbol{q}, \tag{2.13}$$

where s denotes the specific entropy of the fluid, and the first term on the right-hand side represents the heating due to viscous dissipation.

Now, regarding s as a function of T and p, we obtain

$$\frac{Ds}{Dt} = \left(\frac{\partial s}{\partial T}\right)_p \frac{DT}{Dt} + \left(\frac{\partial s}{\partial p}\right)_T \frac{Dp}{Dt}. \tag{2.14}$$

Here, as is well known in thermodynamics, $(\partial s/\partial T)_p$ and $(\partial s/\partial p)_T$ are given by

$$\left(\frac{\partial s}{\partial T}\right)_p = \frac{c_p}{T}, \quad \left(\frac{\partial s}{\partial p}\right)_T = -\left(\frac{\partial v}{\partial T}\right)_p = -v\beta, \tag{2.15}$$

where c_p is the specific heat at constant pressure. These coefficients may be evaluated at $T = T_0$ and $p = p_0$, for T and p deviate only slightly from T_0 and p_0. We then have, since $v = v_0 = \rho_0^{-1}$,

$$\frac{Ds}{Dt} = \frac{c_{p0}}{T_0}\frac{DT'}{Dt} - v_0\beta_0\left(\frac{Dp_0}{Dt} + \frac{Dp'}{Dt}\right), \tag{2.16}$$

where (2.3) and (2.4) have been used; we have also introduced the notation

$$c_{p0} = c_p(T_0, p_0), \quad \beta_0 = \beta(T_0, p_0). \tag{2.17}$$

It should be noted that c_{p0} and β_0 may depend on x_3 through p_0.

Similarly, when s is regarded as a function of e and v, we get

$$\frac{Ds}{Dt} = \frac{1}{T_0}\frac{De}{Dt} + \frac{p_0}{T_0}\frac{Dv}{Dt}. \tag{2.18}$$

However, since $Dv/Dt = D\rho^{-1}/Dt = 0$, it follows that

$$\frac{Ds}{Dt} = \frac{1}{T_0}\frac{De}{Dt}. \tag{2.19}$$

As a consequence, using (2.1), (2.3), (2.16), and (2.19), we have

$$\begin{aligned}\rho T\frac{Ds}{Dt} &= \rho_0 T_0\frac{Ds}{Dt} + \rho_0 T'\frac{Ds}{Dt} \\ &= \rho_0\frac{De}{Dt} + \rho_0 T'\left\{\frac{c_{p0}}{T_0}\frac{DT'}{Dt} - v_0\beta_0\left(\frac{Dp_0}{Dt} + \frac{Dp'}{Dt}\right)\right\}.\end{aligned} \tag{2.20}$$

However, since $Dp_0/Dt = -\rho_0 g u_3$, (2.20) gives, to the first order of primed variables,

$$\rho T\frac{Ds}{Dt} = \rho_0\frac{De}{Dt} + \rho_0\beta_0 T' g u_3. \tag{2.21}$$

Substituting this into (2.13), we obtain

$$\rho_0\frac{De}{Dt} = \tau_{ij}\frac{\partial u_i}{\partial x_j} - \nabla\cdot\boldsymbol{q} - \rho_0\beta_0 T' g u_3. \tag{2.22}$$

The integration of (2.22) over Ω yields

$$\frac{d}{dt}\int_\Omega \rho_0 e dV = \int_\Omega \tau_{ij}\frac{\partial u_i}{\partial x_j}dV - \int_\Sigma \boldsymbol{q}\cdot\boldsymbol{n}dS - \int_\Omega \rho_0\beta_0 T' g u_3 dV. \tag{2.23}$$

This is the desired equation for the rate of change of the internal energy.

We can now derive the equation describing the rate of change of the kinetic energy of the fluid by subtracting (2.12) and (2.23) from (2.11):

$$\frac{d}{dt}\int_{\Omega}\tfrac{1}{2}\rho_0|\boldsymbol{u}|^2 dV = \int_{\Sigma} u_i \tau_{ij} n_j dS - \int_{\Omega}\tau_{ij}\frac{\partial u_i}{\partial x_j}dV + \int_{\Omega}\rho_0\beta_0 T' g u_3 dV. \tag{2.24}$$

We observe from this equation that (2.9) must contain the term

$$\rho_0\beta_0 T' g\boldsymbol{e}_3. \tag{2.25}$$

The force represented by this term is referred to as the buoyancy force. Thus we get

$$\rho_0\frac{D\boldsymbol{u}}{Dt} = -\nabla p' + \frac{\partial \tau_{ij}}{\partial x_j}\boldsymbol{e}_i + \rho_0\beta_0 T' g\boldsymbol{e}_3. \tag{2.26}$$

This is the equation of motion under the Boussinesq approximation. In (2.26), τ_{ij} are given, for example, by

$$\tau_{ij} = \eta\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{2.27}$$

in which η is the dynamic viscosity. It should be emphasized that the work done by the buoyancy force corresponds to the conversion between kinetic and internal energy; this indicates that the origin of the buoyancy force is the pressure gradient force.

Before closing this subsection, it is worth noting that, as shown in the appendix, the need for the buoyancy force can also be inferred from a simple thought experiment.

2.4. Temperature equation

From (2.16) and (2.19), we have

$$\rho_0\frac{De}{Dt} = \rho_0 c_{p0}\frac{DT'}{Dt} - \beta_0 T_0\left(\frac{Dp_0}{Dt} + \frac{Dp'}{Dt}\right). \tag{2.28}$$

The substitution of (2.28) into (2.22) yields the following temperature equation:

$$\rho_0 c_{p0}\frac{DT'}{Dt} = \tau_{ij}\frac{\partial u_i}{\partial x_j} - \nabla\cdot\boldsymbol{q} + \left\{\beta_0 T_0\left(\frac{Dp_0}{Dt} + \frac{Dp'}{Dt}\right) - \rho_0\beta_0 T' g u_3\right\}. \tag{2.29}$$

Here $\boldsymbol{q}$ is given, for example, by Fourier's law:

$$\boldsymbol{q} = -k\nabla T', \tag{2.30}$$

where k denotes the thermal conductivity. This is the temperature equation under the Boussinesq approximation consistent with the conservation law of energy.

3. Summary and discussion

The Boussinesq approximation has been reconstructed in a manner consistent with the conservation law of mass. It has been shown that the governing equations of a fluid under the approximation can be obtained on the basis of the following assumption: the fluid is incompressible in the sense that its density is constant. The equation of motion can in particular be formulated with the help of the conservation law of energy.

3.1. Applicability of the Boussinesq approximation

We are now in a position to discuss what conditions are necessary for the Boussinesq approximation to be applicable to a specific motion of a compressible fluid. We assume that the motion possesses the following characteristic scales: a time scale τ, a velocity scale U, and a length scale L. Let also the scale of the temperature difference of the fluid be denoted by $\Delta T'$. As in § 2, T_0 denotes a constant reference temperature; ρ_0 in the following should be interpreted as a constant reference density.

We first recall (2.3). This expression states that the temperature of the fluid varies only slightly from T_0. Hence $\Delta T'$ must be very small compared with T_0:

$$\Delta T'/T_0 \ll 1. \tag{3.1}$$

In (2.4), on the other hand, the variation of the perturbation pressure p' is taken to be very small compared with that of the hydrostatic pressure p_0. Thus we must have

$$|\nabla p'|/|\nabla p_0| \ll 1. \tag{3.2}$$

Here, in view of (2.5), $|\nabla p_0|$ is given by

$$|\nabla p_0| = \rho_0 g. \tag{3.3}$$

As for $|\nabla p'|$, the following inequality can be obtained from (2.8):

$$|\nabla p'| \leq |\rho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}| + |\rho(\partial\boldsymbol{u}/\partial t)| + |(\partial\tau_{ij}/\partial x_j)\boldsymbol{e}_i| + |(\rho-\rho_0)g\boldsymbol{e}_3|. \tag{3.4}$$

It therefore follows that, if each of the terms on the right-hand side of (3.4) is very small compared with $\rho_0 g$, (3.2) is satisfied. In the following, we examine each term in turn.

We can estimate the first term on the right-hand side of (3.4) as follows:

$$|\rho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}| = O(\rho_0 U^2/L). \tag{3.5}$$

This term, therefore, is very small compared with $\rho_0 g$ when

$$U/(gL)^{1/2} \ll 1. \tag{3.6}$$

The second term can also be estimated as

$$|\rho(\partial\boldsymbol{u}/\partial t)| = O(\rho_0 U/\tau). \tag{3.7}$$

Hence this term is very small compared with $\rho_0 g$ when, together with (3.6),

$$(L/\tau)/(gL)^{1/2} \ll 1 \tag{3.8}$$

applies. Regarding the third term, we observe that, if (2.27) may be employed,

$$|(\partial\tau_{ij}/\partial x_j)\boldsymbol{e}_i| = O(\eta U/L^2). \tag{3.9}$$

Then this term also is very small compared with $\rho_0 g$ when (3.6) and

$$\nu/\{(gL)^{1/2}L\} \ll 1 \tag{3.10}$$

are satisfied. Here $\nu = \eta/\rho_0$ is the kinematic viscosity.

In order to estimate the last term of (3.4), we note here the thermodynamic relation

$$d\rho = (\gamma/a^2)dp - \rho\beta dT, \tag{3.11}$$

where γ is the ratio of specific heats, and a the speed of sound. This allows us to write

$$|\rho - \rho_0| = O(\gamma\Delta p/a^2) + O(\rho_0\beta_0\Delta T'). \tag{3.12}$$

Here Δp denotes the scale of the pressure variation. Now, let the vertical extent of the fluid be denoted by H. Then it is reasonable to put

$$\Delta p = \rho_0 g H. \tag{3.13}$$

In consequence, the last term of (3.4) can be estimated as follows:

$$|(\rho - \rho_0)g\boldsymbol{e}_3| = O(\gamma\rho_0 g^2 H/a^2) + O(\rho_0\beta_0\Delta T' g). \tag{3.14}$$

Accordingly, since $\gamma = O(1)$, this term is very small compared with $\rho_0 g$ when

$$(gH)^{1/2}/a \ll 1, \quad \beta_0\Delta T' \ll 1. \tag{3.15}$$

It is interesting to note that, for an ideal gas, the second condition in (3.15) is identical to (3.1). This is because $\beta_0 = T_0^{-1}$ for an ideal gas.

The basic assumption (2.1) is also justifiable under (3.15); as is apparent from (3.12) and (3.13), $|\rho - \rho_0|/\rho_0 \ll 1$ under (3.15). The remaining assumption (2.2) is commonly satisfied. Thus the conditions for the Boussinesq approximation to be applicable to the motion have all been formulated: (3.1), (3.6), (3.8), (3.10), and (3.15).

3.2. Further approximations to the temperature equation

Although the temperature equation (2.29) is consistent with the conservation law of energy, it is somewhat hard to deal with. Thus, for practical purposes, it is desirable to introduce further approximations to the equation.

Let us consider again the fluid motion in § 3.1. We can approximate the terms in the braces on the right-hand side of (2.29), ignoring terms containing primed variables, as

$$\left\{\beta_0 T_0\left(\frac{Dp_0}{Dt} + \frac{Dp'}{Dt}\right) - \rho_0\beta_0 T' g u_3\right\} \approx \beta_0 T_0\frac{Dp_0}{Dt} = -\rho_0\beta_0 T_0 g u_3. \tag{3.16}$$

The term representing the heating due to viscous dissipation, with τ_{ij} given by (2.27), can also be ignored in comparison with the term $\rho_0\beta_0 T_0 g u_3$ when the condition

$$(1/\beta_0 T_0)\left[\nu/\{(gL)^{1/2}L\}\right] \ll 1 \tag{3.17}$$

holds in addition to (3.6). In this case, (2.29) takes the form

$$\rho_0 c_{p0} \frac{DT'}{Dt} = -\nabla \cdot \boldsymbol{q} - \rho_0 \beta_0 T_0 g u_3. \tag{3.18}$$

However, the last term of (3.18) can further be ignored when

$$\Gamma_0 H / \Delta T' \ll 1, \tag{3.19}$$

where $\Gamma_0 = \beta_0 T_0 g / c_{p0}$ is the adiabatic lapse rate. We then obtain

$$\rho_0 c_{p0} \frac{DT'}{Dt} = -\nabla \cdot \boldsymbol{q}. \tag{3.20}$$

This is in fact the temperature equation in the original Boussinesq approximation.

3.3. Potential temperature under the Boussinesq approximation

When the temperature equation is approximated by (3.18), we can employ, in place of T', the potential temperature θ defined by

$$\theta = \theta_0 + T', \quad \theta_0 = T_0 + \int_{\zeta_a}^{x_3} \Gamma_0(x_3') dx_3'. \tag{3.21}$$

Here the integral is taken from an arbitrary reference level $x_3' = \zeta_a$. In terms of θ, the temperature equation (3.18) is written as

$$\rho_0 c_{p0} \frac{D\theta}{Dt} = -\nabla \cdot \boldsymbol{q}. \tag{3.22}$$

The equation of motion (2.26) can also be written as

$$\rho_0 \frac{D\boldsymbol{u}}{Dt} = -\nabla \hat{p}' + \frac{\partial \tau_{ij}}{\partial x_j} \boldsymbol{e}_i + \rho_0 \beta_0 \theta g \boldsymbol{e}_3, \tag{3.23}$$

where $\hat{p}'$ is the modified perturbation pressure defied by

$$\hat{p}' = p' + \rho_0 g \int_{\zeta_a}^{x_3} \beta_0(x_3') \theta_0(x_3') dx_3'. \tag{3.24}$$

Comparing (3.22) with (3.20), and (3.23) with (2.26), we observe that θ plays formally the same role as T' in the original Boussinesq approximation.

However, it is also important to pay attention to the difference between θ and T'. Let us consider, as an example, Fourier's law (2.30). It is expressed in terms of θ as follows:

$$\boldsymbol{q} = -k \nabla \theta + k \Gamma_0 \boldsymbol{e}_3. \tag{3.25}$$

Hence the heat flux density $\boldsymbol{q}$, which is proportional to $\nabla T'$, is not proportional to $\nabla \theta$.

This fact becomes significant, for example, when an adiabatic condition is imposed on a fixed boundary $x_3 = \zeta_b$. Expressing the condition in terms of T', we have

$$(\partial T'/\partial x_3)|_{x_3=\zeta_b} = 0. \tag{3.26}$$

On the other hand, the same condition requires that

$$(\partial\theta/\partial x_3)|_{x_3=\zeta_b} = \Gamma_0(\zeta_b). \tag{3.27}$$

It therefore follows that θ must satisfy a condition formally different from that for T'.

As is evident from the above discussion, θ is not a mere substitute for T'. This is not surprising considering the following fact: potential temperature is a measure of specific entropy (see e.g. Gill 1982); it is, despite its name, a quantity essentially different from temperature.

Appendix. Another derivation of the buoyancy force

We consider the same physical situation as that in § 2, but the fluid is now assumed to be at rest. Let us first take a fluid element of unit volume in the fluid: it is referred to as element A, and its temperature is T_a. We next consider another fluid element of unit volume referred to as element B: its position relative to element A is $\boldsymbol{e}_i dx_i$, and its temperature is T_b. The fluid elements have the same mass ρ_0, and T_a and T_b are not so different. Our aim in this appendix is to analyze the quasistatic process of adiabatically interchanging the positions of the fluid elements in order to derive the buoyancy force in the Boussinesq approximation.

Let us focus attention on element A. When its position is adiabatically interchanged with that of element B, the specific entropy s remains unchanged. Hence the change in temperature dT_a of element A can be calculated from

$$dT_a = \left(\frac{\partial T}{\partial p}\right)_s dp_a. \tag{A.1}$$

Here dp_a denotes the change in pressure experienced by element A when it is displaced vertically by dx_3; since the pressure is hydrostatic, dp_a is given by

$$dp_a = -\rho_0 g dx_3. \tag{A.2}$$

We can also write, using (2.15), the coefficient on the right-hand side of (A.1) as

$$\left(\frac{\partial T}{\partial p}\right)_s = -\left(\frac{\partial s}{\partial p}\right)_T \bigg/ \left(\frac{\partial s}{\partial T}\right)_p = \frac{v_0\beta T_a}{c_p}, \tag{A.3}$$

where β and c_p are regarded as constant. Thus, from (A.1), (A.2), and (A.3), we have

$$dT_a = -\frac{\beta T_a g}{c_p} dx_3. \tag{A.4}$$

As a result, the temperature of element A after the process is found to be

$$T_a - \frac{\beta T_a g}{c_p} dx_3. \tag{A.5}$$

Similarly, the temperature of element B after the process can be obtained as follows:

$$T_b + \frac{\beta T_b g}{c_p} dx_3. \tag{A.6}$$

On the basis of these results, we wish to analyze the change in the energy of the system consisting of the two fluid elements.

The kinetic energy of the system remains zero in the above quasistatic process since both the fluid elements are stationary throughout the process. The potential energy of the system is also unchanged: the potential energy of element A increases by $\rho_0 g dx_3$, but that of element B decreases by the same amount. However, as explained below, the internal energy of the system changes through the process.

Let us consider again element A after the above quasistatic process. In view of (A.5), its temperature becomes T_b if the following amount of heat is removed:

$$\rho_0 c_p \left(T_a - T_b - \frac{\beta T_a g}{c_p} dx_3 \right). \tag{A.7}$$

Similarly, the temperature of element B becomes T_a if we remove an amount of heat

$$\rho_0 c_p \left(T_b - T_a + \frac{\beta T_b g}{c_p} dx_3 \right). \tag{A.8}$$

Thus, removing from the two fluid elements after the process an amount of heat

$$\rho_0 c_p \left(T_a - T_b - \frac{\beta T_a g}{c_p} dx_3 \right) + \rho_0 c_p \left(T_b - T_a + \frac{\beta T_b g}{c_p} dx_3 \right), \tag{A.9}$$

we can restore the initial state of the process: the position of a fluid element of unit volume with $T = T_b$ is $\boldsymbol{e}_i dx_i$ relative to one with $T = T_a$. This proves that the internal energy of the system of the two fluid elements increases through the process by

$$\rho_0 \beta (T_b - T_a) g dx_3. \tag{A.10}$$

The above result can be interpreted on the basis of the first law of thermodynamics: since the process is adiabatic, the increase (A.10) in the internal energy of the system is equal to the work done on the system in the process. It therefore follows that, in order to carry out the process, we need to do work by applying forces on the system.

Let $\tilde{\boldsymbol{F}}_a$ and $\tilde{\boldsymbol{F}}_b$ denote the forces applied in the process to element A and to element B, respectively. The work done by the forces in the process is then given by

$$(\tilde{\boldsymbol{F}}_a - \tilde{\boldsymbol{F}}_b) \cdot \boldsymbol{e}_i dx_i. \tag{A.11}$$

This work must be equal to (A.10); we obtain, after some manipulation,

$$\{(\tilde{\boldsymbol{F}}_a - \tilde{\boldsymbol{F}}_b) - \rho_0\beta(T_b - T_a)g\boldsymbol{e}_3\} \cdot \boldsymbol{e}_i dx_i = 0. \tag{A.12}$$

Since the direction of $\boldsymbol{e}_i dx_i$ is arbitrary, we observe that

$$(\tilde{\boldsymbol{F}}_a - \tilde{\boldsymbol{F}}_b) = \rho_0\beta(T_b - T_a)g\boldsymbol{e}_3. \tag{A.13}$$

This result can be accounted for if $\tilde{\boldsymbol{F}}_a$ and $\tilde{\boldsymbol{F}}_b$ are given by

$$\tilde{\boldsymbol{F}}_a = -\rho_0\beta(T_a - T_r)g\boldsymbol{e}_3, \quad \tilde{\boldsymbol{F}}_b = -\rho_0\beta(T_b - T_r)g\boldsymbol{e}_3, \tag{A.14}$$

where T_r is an arbitrary constant temperature.

We recall here that the above process is quasistatic. Thus the fluid elements must be stationary throughout the process. This implies that the following forces, which are in balance with the forces (A.14), must be acting on the fluid elements inherently:

$$\boldsymbol{F}_a = \rho_0\beta(T_a - T_r)g\boldsymbol{e}_3, \quad \boldsymbol{F}_b = \rho_0\beta(T_b - T_r)g\boldsymbol{e}_3. \tag{A.15}$$

Here $\boldsymbol{F}_a$ and $\boldsymbol{F}_b$ denote the forces acting on element A and on element B, respectively.

This conclusion can be generalized as follows: an incompressible fluid element of unit volume with temperature T is acted on, in a uniform gravitational field, by the force

$$\boldsymbol{F} = \rho_0\beta(T - T_r)g\boldsymbol{e}_3. \tag{A.16}$$

This is the buoyancy force in the Boussinesq approximation. Note that the arbitrariness of T_r corresponds to that of the reference temperature in the approximation.

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