Low Mach number approximation

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Abstract

The low Mach number approximation, a soundproof approximation for ideal gases, is reconstructed in a manner not specific to ideal gases; its energetics is examined on the basis of this reconstruction. Consequently, it is shown that the conservation law of energy is not satisfied under the approximation; this implies that it excludes sound waves at the cost of the consistency with the conservation law of energy.

1. Introduction

The low Mach number approximation was devised by Rehm & Baum (1978) in order to study thermally driven flows of ideal gases. This approximation allows temperature, and hence density, to vary significantly. Nevertheless, it is a *soundproof* approximation: it excludes sound waves from the solutions of the governing equations, contrary to the expectation that significant variations in density will inevitably create sound waves.

This note reconstructs the low Mach number approximation in a manner not specific to ideal gases. On the basis of this reconstruction, the energetics of the approximation is studied with the aim of clarifying why the approximation can exclude sound waves.

2. Reconstruction of the low Mach number approximation

We consider the motion of a fluid driven by differences in temperature in a uniform gravitational field. The fluid occupies a fixed finite domain Ω. In Ω, the z-axis is taken vertically upwards: the unit vector in the positive z-direction is denoted by $\boldsymbol{k}$.

2.1. Equation of motion

Let T, ρ, and p denote respectively the temperature, the density, and the pressure of the fluid. We then have the following thermodynamic relation:

$$\beta dT = -\rho^{-1}d\rho + (\gamma/\rho a^2)dp, \tag{2.1}$$

in which $\beta = -\rho^{-1}(\partial\rho/\partial T)_p$ is the thermal expansion coefficient, γ the ratio of specific heats, and a the speed of sound. Next, let ΔT denote the scale that characterizes the variation of T. We assume that ΔT is *large* in the sense that the condition

$$\beta\Delta T = O(1) \tag{2.2}$$

holds. Then, from (2.1), we obtain the following estimates:

$$\rho^{-1}\Delta\rho = O(1), \quad (\gamma/\rho a^2)\Delta p = O(1). \tag{2.3}$$

Here $\Delta\rho$ and Δp are the variation scales of ρ and p, respectively.

Let us next turn our attention to the equation of motion of the fluid. Neglecting the viscosity of the fluid for simplicity, we can express the equation as follows:

$$\rho\frac{D\boldsymbol{u}}{Dt} = -\nabla p - \rho g\boldsymbol{k}, \tag{2.4}$$

in which D/Dt stands for the material derivative, $\boldsymbol{u}$ the velocity of the fluid, and g the acceleration due to gravity.

Now, let H denote the vertical extent of the domain Ω containing the fluid. Since the motion under consideration is driven by differences in temperature, the fluid is expected to flow primarily in the vertical direction. We may therefore take H as the length scale of the motion. Furthermore, if U denotes the velocity scale of the motion, the time scale of the motion will be given by H/U. We then obtain the estimate $|\rho D\boldsymbol{u}/Dt| = O(\rho U^2/H)$. However, since $\rho D\boldsymbol{u}/Dt$ must be of the same order of magnitude as $\rho g\boldsymbol{k}$, we have

$$U/(gH)^{1/2} = O(1). \tag{2.5}$$

On the other hand, we have $|\nabla p| = O(\Delta p/H)$. Thus the requirement that ∇p be of the same order of magnitude as $\rho g\boldsymbol{k}$ leads to

$$\Delta p/\rho gH = O(1). \tag{2.6}$$

The above argument shows that two distinct variation scales of p exist. Accordingly, we decompose p as follows:

$$p = \Pi + \pi, \tag{2.7}$$

where, if the variation scales of Π and π are denoted respectively by $\Delta\Pi$ and $\Delta\pi$,

$$(\gamma/\rho a^2)\Delta\Pi = O(1), \quad \Delta\pi/\rho gH = O(1). \tag{2.8}$$

The substitution of (2.7) into (2.4) yields

$$\rho\frac{D\boldsymbol{u}}{Dt} = -\nabla\Pi - \nabla\pi - \rho g\boldsymbol{k}. \tag{2.9}$$

However, since $|\nabla\Pi| = O(\Delta\Pi/H)$, we see from (2.8) that $|\nabla\Pi| = O(\rho a^2/\gamma H)$. On the other hand, $|\rho D\boldsymbol{u}/Dt| = O(\rho g)$ and $|\nabla\pi| = O(\rho g)$. Thus we have

$$\begin{aligned} |\rho D\boldsymbol{u}/Dt|/|\nabla\Pi| &= O(\gamma gH/a^2), \\ |\nabla\pi|/|\nabla\Pi| &= O(\gamma gH/a^2), \\ |\rho g\boldsymbol{k}|/|\nabla\Pi| &= O(\gamma gH/a^2). \end{aligned} \tag{2.10}$$

We introduce here the assumption that the domain Ω is *shallow* in the following sense:

$$gH/a^2 \ll 1. \tag{2.11}$$

Then, since $\gamma = O(1)$, (2.9) can be approximated by $\nabla\Pi = 0$; we therefore obtain

$$\Pi = \Pi(t). \tag{2.12}$$

Substituting this into (2.9), we get the following second approximation:

$$\rho\frac{D\boldsymbol{u}}{Dt} = -\nabla\pi - \rho g\boldsymbol{k}. \tag{2.13}$$

This is the equation of motion under the low Mach number approximation.

2.2. Thermodynamic potential

A thermodynamic potential is a function of a set of state variables: once it is given as a function of an appropriate set of variables, other variables can be determined from it as a function of the set of variables. In this study, we assume that the specific enthalpy h of the fluid is given as a function of the specific entropy s and the pressure p:

$$h = h(s, p). \tag{2.14}$$

The essence of the low Mach number approximation, however, is to use Π, in place of $p = \Pi + \pi$, as the thermodynamic pressure: the idea of using Π in place of $p = \Pi + \pi$ is based on the following estimate that can be obtained from (2.8) and (2.11):

$$\Delta\pi/\Delta\Pi \ll 1. \tag{2.15}$$

By virtue of using Π in place of $p = \Pi + \pi$, sound waves are excluded.

We therefore replace (2.14) with

$$h = h(s, \Pi). \tag{2.16}$$

Then the density ρ and the temperature T of the fluid are given by

$$\rho = (\partial h/\partial\Pi)_s^{-1}, \quad T = (\partial h/\partial s)_\Pi. \tag{2.17}$$

2.3. General equation of heat transfer

When the viscosity of the fluid is neglected, the specific entropy s of the fluid satisfies the following general equation of heat transfer (see Landau & Lifshitz 1987, § 49):

$$\rho T\frac{Ds}{Dt} = -\nabla\cdot\boldsymbol{q}, \tag{2.18}$$

where $\boldsymbol{q}$ denotes the heat flux density due to thermal conduction.

2.4. Equation for the thermodynamic pressure

On the other hand, since the total mass M_0 of the fluid is constant, we can determine the thermodynamic pressure $\Pi = \Pi(t)$ from the following condition:

$$\int_\Omega \rho(s, \Pi) dV = M_0. \tag{2.19}$$

2.5. Equation of continuity

Finally, the equation of continuity is given as usual by

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \boldsymbol{u} = 0. \tag{2.20}$$

It is to be noted here that the condition (2.19) is not a trivial result of (2.20): it cannot be obtained from (2.20) without the boundary condition that the normal component of $\boldsymbol{u}$ vanishes on the boundary of Ω.

3. Energetics of the low Mach number approximation

We have thus completed the reconstruction of the low Mach number approximation. Our next aim is to examine the energetics of the approximation. The physical situation to be considered is the same as in the preceding section.

To begin with, we derive the equation for the rate of change of the internal energy of the fluid. Denoting by e the specific internal energy of the fluid, we can write

$$e = h - \rho^{-1}\Pi. \tag{3.1}$$

Thus, using (2.17), we obtain

$$\rho \frac{De}{Dt} = \rho T \frac{Ds}{Dt} + \rho^{-1} \frac{D\rho}{Dt} \Pi. \tag{3.2}$$

The substitution of (2.18), together with the use of (2.20) and (2.12), leads to

$$\frac{\partial(\rho e)}{\partial t} + \nabla \cdot (\rho e \boldsymbol{u}) = -\nabla \cdot \boldsymbol{q} - \nabla \cdot (\Pi \boldsymbol{u}). \tag{3.3}$$

Finally, integrating over the fixed domain Ω, we get the following equation for the rate of change of the internal energy of the fluid:

$$\frac{d}{dt} \int_\Omega \rho e \, dV = -\int_{\partial\Omega} \boldsymbol{q} \cdot \boldsymbol{n} \, dS, \tag{3.4}$$

where $\partial\Omega$ denotes the boundary of Ω, and $\boldsymbol{n}$ the unit outward normal to $\partial\Omega$.

On the other hand, the equation for the kinetic energy of the fluid is given by

$$\frac{d}{dt} \int_\Omega \tfrac{1}{2}\rho |\boldsymbol{u}|^2 dV = \int_\Omega \pi \nabla \cdot \boldsymbol{u} \, dV - \int_\Omega \rho g \boldsymbol{u} \cdot \boldsymbol{k} \, dV, \tag{3.5}$$

and that for the potential energy of the fluid by

$$\frac{d}{dt}\int_\Omega \rho g z\, dV = \int_\Omega \rho g \boldsymbol{u}\cdot\boldsymbol{k}\, dV. \tag{3.6}$$

Note here that the first term on the right-hand side of (3.5) represents the rate of work done by the spatially variable part π of the pressure $p = \Pi + \pi$.

As a result, adding all the equations (3.4) to (3.6), we get

$$\frac{d}{dt}\int_\Omega \rho\left(\tfrac{1}{2}|\boldsymbol{u}|^2 + gz + e\right) dV = \int_\Omega \pi\nabla\cdot\boldsymbol{u}\, dV - \int_{\partial\Omega} \boldsymbol{q}\cdot\boldsymbol{n}\, dS. \tag{3.7}$$

However, the conservation law of energy evidently requires that

$$\frac{d}{dt}\int_\Omega \rho\left(\tfrac{1}{2}|\boldsymbol{u}|^2 + gz + e\right) dV = -\int_{\partial\Omega} \boldsymbol{q}\cdot\boldsymbol{n}\, dS. \tag{3.8}$$

Thus the first term on the right-hand side of (3.7) must in fact be canceled out: it is not canceled out because of using Π, in place of $p = \Pi + \pi$, as the thermodynamic pressure. Consequently, we are led to the following conclusion: the conservation law of energy is not satisfied under the low Mach number approximation.

4. Conclusion

The low Mach number approximation, which is a soundproof approximation for ideal gases, has been reconstructed in a manner not specific to ideal gases; its energetics has been studied on the basis of this reconstruction. Consequently, it has become apparent that the conservation law of energy is not satisfied under the approximation on account of its essential methodology. This implies that the approximation excludes sound waves at the cost of the consistency with the conservation law of energy. This is in contrast to other soundproof approximations (see Maruyama 2019, 2021).

References

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