Energetics of a fluid under the Boussinesq approximation

Kiyoshi Maruyama
Department of Earth and Ocean Sciences, National Defense Academy,
Yokosuka, Kanagawa 239-8686, Japan

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Abstract

This paper presents a theory describing the energy budget of a fluid under the Boussinesq approximation: the theory is developed in a manner consistent with the conservation law of mass. It shows that no potential energy is available under the Boussinesq approximation, and also reveals that the work done by the buoyancy force due to changes in temperature corresponds to the conversion between kinetic and internal energy. This energy conversion, however, makes only an ignorable contribution to the distribution of temperature under the approximation.

The Boussinesq approximation is, in physical oceanography, extended so that the motion of seawater can be studied. This paper considers this extended approximation as well. Under the extended approximation, the work done by the buoyancy force due to changes in salinity corresponds to the conversion between kinetic and potential energy. It also turns out that the conservation law of mass does not allow the condition $\nabla \cdot \boldsymbol{u} = 0$ on the fluid velocity $\boldsymbol{u}$ to be imposed under the extended approximation; the condition to be imposed instead is presented.

1. Introduction

The Boussinesq approximation is frequently used to study the motion of a fluid with a non-uniform temperature distribution. Under this approximation, Winters et al. (1995) presented a theoretical framework for analyzing the energy budget of a density-stratified flow. Since then, in physical oceanography, the energetics of the overturning circulation of the oceans has been studied within this theoretical framework (Huang 1998; Hughes, Hogg & Griffiths 2009). This framework, however, is physically inadmissible as explained below.

Winters et al. assumed in their paper that the density ρ appearing in their definition of the potential energy of a fluid fulfilled the following equation:

$$\partial\rho/\partial t + \boldsymbol{u} \cdot \nabla\rho = -\nabla \cdot \boldsymbol{j}_{\mathrm{d}}, \tag{1}$$

where $\boldsymbol{u}$ is the fluid velocity satisfying the condition $\nabla \cdot \boldsymbol{u} = 0$, and $\boldsymbol{j}_{\mathrm{d}}$ denotes the diffusion flux density of mass allowed in their paper to exist. This equation can be rewritten, using $\nabla \cdot \boldsymbol{u} = 0$, in the form

$$\partial\rho/\partial t + \nabla \cdot (\rho\boldsymbol{u} + \boldsymbol{j}_{\mathrm{d}}) = 0. \tag{2}$$

It follows from this expression that $\rho \boldsymbol{u} + \boldsymbol{j}_{\mathrm{d}}$ is the momentum of unit volume of fluid (see Landau & Lifshitz 1987, § 49; Maruyama 2015). Hence, so long as $\boldsymbol{j}_{\mathrm{d}} \neq 0$, we arrive at the irrational conclusion that $\boldsymbol{u}$ is not the momentum of unit mass of fluid.

The origin of this irrational conclusion is as follows. To derive the equation of motion for a fluid under the Boussinesq approximation, the density ρ of the fluid is customarily assumed to be given by (see e.g. Landau & Lifshitz 1987, § 56; Mihaljan 1962)

$$\rho = \rho_0 - \rho_0 \beta T', \tag{3}$$

where ρ_0 is a constant reference density, β is the thermal expansion coefficient, and T' is the temperature of the fluid measured from a reference value. This assumption, combined with the condition $\nabla \cdot \boldsymbol{u} = 0$, gives rise to the diffusion flux of mass; we are consequently led to the above irrational conclusion.

The assumption (3), however, should be interpreted as a mere expedient to derive the equation of motion for the following reason. Suppose that the fluid is contained in a domain with a fixed volume V. The mass M of the fluid is then given, if the variation of β is negligible, as follows:

$$M = \rho_0 V - \rho_0 \beta V \overline{T'}, \tag{4}$$

where $\overline{T'}$ denotes the averaged temperature of the fluid. From this expression, it follows that the mass M varies when the fluid is heated or cooled. Evidently, this conclusion violates the conservation law of mass; the law requires, instead of (3), that $\rho = \rho_0$.

It is now evident that the definition by Winters et al. of the potential energy of a fluid is physically unacceptable. Since their theoretical framework contains no mathematical error, it may be used for analyzing numerical simulations (see e.g. Gayen et al. 2013); it cannot be used, nevertheless, for analyzing real flows. Thus the aim of this paper is to present a physically reasonable theory that can describe the energy budget of a fluid under the Boussinesq approximation. It is also extended so that the motion of seawater can be dealt with.

2. Energetics under the Boussinesq approximation

Let us consider the motion under gravity of a viscous fluid contained in a fixed domain Ω. We set up in the domain a system of rectangular coordinates (x_1, x_2, x_3) with the x_3-axis taken vertically upwards. The unit vectors in the positive x_1-, x_2-, and x_3-directions are respectively denoted by $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$. In the following, Latin indices are used to represent the numbers 1, 2, and 3; the summation convention is also implied.

As explained in § 1, in view of the conservation law of mass, the density ρ of the fluid should be regarded under the Boussinesq approximation as constant:

$$\rho = \rho_0. \tag{5}$$

This is consistent with the usual assumption under the approximation that the velocity $\boldsymbol{u}$ of the fluid is solenoidal:

$$\nabla \cdot \boldsymbol{u} = 0. \tag{6}$$

Nevertheless, the thermal expansion coefficient β of the fluid is assumed not to vanish:

$$\beta = v^{-1}(\partial v/\partial T)_p \neq 0. \tag{7}$$

Here $v = 1/\rho$ denotes the specific volume of the fluid; T and p are respectively the temperature and the pressure of the fluid.

We first wish to find an equation for the rate of change of the internal energy of the fluid. This can be derived from the general equation of heat transfer (see Landau & Lifshitz 1987, § 49):

$$\rho T \frac{Ds}{Dt} = \tau_{ij}\frac{\partial u_i}{\partial x_j} - \nabla \cdot \boldsymbol{q}, \tag{8}$$

in which D/Dt denotes the material derivative, s is the specific entropy of the fluid, τ_{ij} are the components of the viscous stress tensor, u_i are those of $\boldsymbol{u}$, and $\boldsymbol{q}$ is the heat flux density; the first term on the right represents the heating due to viscous dissipation.

Under the Boussinesq approximation, the temperature and the pressure are assumed to vary only slightly in the fluid (see Landau & Lifshitz 1987, § 56). Accordingly, T can be written in the form

$$T = T_0 + T', \tag{9}$$

where T_0 is a constant reference temperature, and T' is the small variation from T_0. We also write, denoting by p' the small perturbation pressure,

$$p = p_0 + p'. \tag{10}$$

Here p_0, which satisfies the hydrostatic equation, is defined by

$$p_0 = -\rho_0 g x_3 + \text{constant}, \tag{11}$$

where g is the acceleration due to gravity.

Now let us express Ds/Dt in terms of T and p:

$$\frac{Ds}{Dt} = \left(\frac{\partial s}{\partial T}\right)_p \frac{DT}{Dt} + \left(\frac{\partial s}{\partial p}\right)_T \frac{Dp}{Dt}. \tag{12}$$

Here $(\partial s/\partial T)_p$ and $(\partial s/\partial p)_T$ are given by (see e.g. Batchelor 1967, § 1.5)

$$\left(\frac{\partial s}{\partial T}\right)_p = \frac{c_p}{T}, \quad \left(\frac{\partial s}{\partial p}\right)_T = -\left(\frac{\partial v}{\partial T}\right)_p = -v\beta, \tag{13}$$

in which c_p is the specific heat at constant pressure. Since T and p vary only slightly, these coefficients may be evaluated at $T = T_0$, $p = p_0$. Thus we get, since $v = v_0 = 1/\rho_0$,

$$\frac{Ds}{Dt} = \frac{c_{p0}}{T_0}\frac{DT'}{Dt} - v_0\beta_0\left(\frac{Dp_0}{Dt} + \frac{Dp'}{Dt}\right), \tag{14}$$

where (9) and (10) have been used; we have also introduced the notation

$$c_{p0} = c_p(T_0, p_0), \quad \beta_0 = \beta(T_0, p_0). \tag{15}$$

Note that c_{p0} and β_0 may, in general, depend on x_3 through p_0.

It is also possible to express Ds/Dt in terms of the specific internal energy e and the specific volume v in a similar manner:

$$\frac{Ds}{Dt} = \frac{1}{T_0}\frac{De}{Dt} + \frac{p_0}{T_0}\frac{Dv}{Dt}. \tag{16}$$

We note, however, that $Dv/Dt = v\nabla\cdot\boldsymbol{u} = 0$. Hence it follows that

$$\frac{Ds}{Dt} = \frac{1}{T_0}\frac{De}{Dt}. \tag{17}$$

Using (5), (9), (14), and (17), we find

$$\begin{aligned} \rho T\frac{Ds}{Dt} &= \rho_0 T_0\frac{Ds}{Dt} + \rho_0 T'\frac{Ds}{Dt} \\ &= \rho_0\frac{De}{Dt} + \rho_0 T'\left\{\frac{c_{p0}}{T_0}\frac{DT'}{Dt} - v_0\beta_0\left(\frac{Dp_0}{Dt} + \frac{Dp'}{Dt}\right)\right\}. \end{aligned} \tag{18}$$

Since $Dp_0/Dt = -\rho_0 g u_3$, we obtain, to the first order of primed variables,

$$\rho T\frac{Ds}{Dt} = \rho_0\frac{De}{Dt} + \rho_0\beta_0 T' g u_3. \tag{19}$$

The substitution of this result into (8) yields

$$\rho_0\frac{De}{Dt} = \tau_{ij}\frac{\partial u_i}{\partial x_j} - \nabla\cdot\boldsymbol{q} - \rho_0\beta_0 T' g u_3. \tag{20}$$

Integrating (20) over Ω, whose boundary is denoted by Σ, we have

$$\frac{d}{dt}\int_\Omega \rho_0 e dV = \int_\Omega \tau_{ij}\frac{\partial u_i}{\partial x_j}dV - \int_\Sigma \boldsymbol{q}\cdot\boldsymbol{n} dS - \int_\Omega \rho_0\beta_0 T' g u_3 dV. \tag{21}$$

Here $\boldsymbol{n}$ is the unit outward normal on Σ, and we have assumed that $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on Σ. This is the desired equation for the rate of change of the internal energy.

Let us next consider the kinetic energy of the fluid. Under the Boussinesq approximation, the equation of motion is given by (see Landau & Lifshitz 1987, § 56)

$$\rho_0\frac{D\boldsymbol{u}}{Dt} = -\nabla p' + \frac{\partial\tau_{ij}}{\partial x_j}\boldsymbol{e}_i + \rho_0\beta_0 T' g\boldsymbol{e}_3. \tag{22}$$

From this equation, we can obtain the following equation for the rate of change of the kinetic energy:

$$\frac{d}{dt}\int_\Omega \tfrac{1}{2}\rho_0|\boldsymbol{u}|^2 dV = \int_\Sigma u_i\tau_{ij}n_j dS - \int_\Omega \tau_{ij}\frac{\partial u_i}{\partial x_j}dV + \int_\Omega \rho_0\beta_0 T' g u_3 dV, \tag{23}$$

where n_j are the components of $\boldsymbol{n}$.

On the other hand, the potential energy of the fluid is invariable:

$$\frac{d}{dt}\int_\Omega \rho_0 g x_3 dV = 0. \tag{24}$$

Hence, under the Boussinesq approximation, no potential energy is available.

Adding (21), (23), and (24), we get

$$\frac{d}{dt}\int_{\Omega}\rho_0\left(\tfrac{1}{2}|\boldsymbol{u}|^2+gx_3+e\right)dV=\int_{\Sigma}u_i\tau_{ij}n_j dS-\int_{\Sigma}\boldsymbol{q}\cdot\boldsymbol{n}dS. \tag{25}$$

This expression states that the total energy of the fluid changes owing to the work done by the viscous force acting on Σ and owing also to the heat transfer across Σ. We have thus obtained a physically reasonable result.

We now focus our attention on the equation (23). Of the three terms on its right-hand side, the last term represents the work done by the buoyancy force: this force is represented by the last term of (22). The last term of (23) appears, with an opposite sign, in (21) too. Hence the work done by the buoyancy force is seen to correspond to the conversion between kinetic and internal energy.

Finally, let us return to the equation (20). Rewriting this equation with the help of (14) and (17), we obtain

$$\rho_0 c_{p0}\frac{DT'}{Dt}=\tau_{ij}\frac{\partial u_i}{\partial x_j}-\nabla\cdot\boldsymbol{q}+\left\{\beta_0 T_0\left(\frac{Dp_0}{Dt}+\frac{Dp'}{Dt}\right)-\rho_0\beta_0 T' g u_3\right\}. \tag{26}$$

The sum of the terms in the braces, however, is neglected under the Boussinesq approximation together with the term representing the heating due to viscous dissipation (see Landau & Lifshitz 1987, § 56). Thus we have

$$\frac{DT'}{Dt}=-\frac{1}{\rho_0 c_{p0}}\nabla\cdot\boldsymbol{q}. \tag{27}$$

This equation, combined with e.g. Fourier's law, determines the distribution of temperature under the Boussinesq approximation. However, as is apparent from the derivation, this temperature equation takes no account of the contribution of the conversion between kinetic and internal energy discussed above.

3. Energetics under the oceanographic Boussinesq approximation

In physical oceanography, the Boussinesq approximation is extended to study the motion of seawater (see e.g. McWilliams 2006, § 2.2). Our aim in this section is to investigate the energy budget of a fluid under this extended approximation which is called in this paper the oceanographic Boussinesq approximation.

We consider the same situation as that in § 2, but the fluid to be studied is now regarded as seawater. The salinity c, the mass of dissolved salt per unit mass of fluid, is assumed to change only slightly in the fluid. Thus we write

$$c=c_0+c', \tag{28}$$

where c_0 is some constant reference salinity, and c' is the small deviation from c_0. The equation for c is as follows (see Landau & Lifshitz 1987, § 58):

$$\rho\frac{Dc}{Dt}=-\nabla\cdot\boldsymbol{i}, \tag{29}$$

where $\boldsymbol{i}$ denotes the diffusion flux density of salt; it should be noted that the flux density of salt and that of water are given by $\rho c\boldsymbol{u}+\boldsymbol{i}$ and $\rho(1-c)\boldsymbol{u}-\boldsymbol{i}$, respectively. We also need to introduce the chemical potential μ of the fluid (see Landau & Lifshitz 1987, § 58). It is regarded as a function of T, p, and c:

$$\mu = \mu(T, p, c). \tag{30}$$

When c' is uniform throughout the fluid, the density ρ should be equated to a constant as in § 2. This constant, however, may be a function of c'. Hence, in view of the smallness of c', we put

$$\rho = \rho_0 + \rho_0 \beta_c c', \tag{31}$$

in which the haline contraction coefficient $\beta_c = \rho^{-1}(\partial \rho / \partial c)_{T,p}$ is regarded as a constant. Then, from the equation of continuity

$$\rho^{-1} \frac{D\rho}{Dt} + \nabla \cdot \boldsymbol{u} = 0, \tag{32}$$

it follows that in general $\nabla \cdot \boldsymbol{u} \neq 0$. Using (31) and (32), we can obtain

$$\frac{d}{dt} \int_\Omega \rho g x_3 dV = \int_\Omega (\rho_0 g u_3 + \rho_0 \beta_c c' g u_3) dV. \tag{33}$$

Here we have assumed, for simplicity, that $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Σ. This is the potential energy equation for the present problem.

The internal energy equation can now be found from the following general equation of heat transfer (see Landau & Lifshitz 1987, § 58):

$$\rho T \frac{Ds}{Dt} = \tau_{ij} \frac{\partial u_i}{\partial x_j} - \nabla \cdot \boldsymbol{q} + \mu \nabla \cdot \boldsymbol{i}. \tag{34}$$

If s is regarded as a function of T, p, and c, we get, using the thermodynamic relation $(\partial s / \partial c)_{T,p} = -(\partial \mu / \partial T)_{p,c}$ (see Landau & Lifshitz 1987, § 59),

$$\frac{Ds}{Dt} = \frac{c_{p0}}{T_0} \frac{DT'}{Dt} - v_0 \beta_0 \left(\frac{Dp_0}{Dt} + \frac{Dp'}{Dt} \right) - \left(\frac{\partial \mu}{\partial T} \right)_{p,c} \frac{Dc}{Dt}, \tag{35}$$

in place of (14). Here $c_{p0} = c_p(T_0, p_0, c_0)$ and $\beta_0 = \beta(T_0, p_0, c_0)$; $(\partial \mu / \partial T)_{p,c}$ should also be evaluated at $T = T_0$, $p = p_0$, $c = c_0$. On the other hand, since $(\partial s / \partial c)_{e,v} = -\mu / T$, we have, in place of (17),

$$\frac{Ds}{Dt} = \frac{1}{T_0} \frac{De}{Dt} + \frac{p_0}{T_0} \frac{Dv}{Dt} - \frac{\mu_0}{T_0} \frac{Dc}{Dt}, \tag{36}$$

where $\mu_0 = \mu(T_0, p_0, c_0)$. Thus, neglecting terms above the first order in primed variables, we can obtain the following equation corresponding to (20):

$$\rho \frac{De}{Dt} = -p_0 \nabla \cdot \boldsymbol{u} + \tau_{ij} \frac{\partial u_i}{\partial x_j} - \nabla \cdot \boldsymbol{q} - \rho_0 \beta_0 T' g u_3, \tag{37}$$

where $\rho Dv/Dt = \nabla \cdot \boldsymbol{u}$ and (29) have been used. We note in addition that

$$-p_0 \nabla \cdot \boldsymbol{u} = -\nabla \cdot (p_0 \boldsymbol{u}) + \boldsymbol{u} \cdot \nabla p_0 = -\nabla \cdot (p_0 \boldsymbol{u}) - \rho_0 g u_3. \tag{38}$$

The internal energy equation is therefore obtained as follows:

$$\frac{d}{dt} \int_\Omega \rho e dV = \int_\Omega \tau_{ij} \frac{\partial u_i}{\partial x_j} dV - \int_\Sigma \boldsymbol{q} \cdot \boldsymbol{n} dS - \int_\Omega (\rho_0 g u_3 + \rho_0 \beta_0 T' g u_3) dV. \tag{39}$$

Next, the equation of motion is now given by (see McWilliams 2006, § 2.2)

$$\rho_0 \frac{D\boldsymbol{u}}{Dt} = -\nabla p' + \left(\frac{\rho_0}{\rho}\right) \frac{\partial \tau_{ij}}{\partial x_j} \boldsymbol{e}_i + \rho_0 \beta_0 T' g \boldsymbol{e}_3 - \rho_0 \beta_c c' g \boldsymbol{e}_3, \tag{40}$$

in which the last term represents the buoyancy force due to changes in salinity, and the factor ρ_0/ρ is usually identified with unity. Neglecting terms above the first order in primed variables, we can find from (40) the kinetic energy equation

$$\begin{aligned} \frac{d}{dt} \int_\Omega \tfrac{1}{2} \rho |\boldsymbol{u}|^2 dV = \int_\Sigma u_i \tau_{ij} n_j dS - \int_\Omega \tau_{ij} \frac{\partial u_i}{\partial x_j} dV \\ + \int_\Omega (\rho_0 \beta_0 T' g u_3 - \rho_0 \beta_c c' g u_3) dV. \end{aligned} \tag{41}$$

In consequence, adding (33), (39), and (41), we get

$$\frac{d}{dt} \int_\Omega \rho \left(\tfrac{1}{2} |\boldsymbol{u}|^2 + g x_3 + e\right) dV = \int_\Sigma u_i \tau_{ij} n_j dS - \int_\Sigma \boldsymbol{q} \cdot \boldsymbol{n} dS. \tag{42}$$

This total energy equation has exactly the same form as (25); we have obtained a physically reasonable result again.

The second integrand in the last term of (41) represents the work done by the buoyancy force due to changes in salinity. Comparing (41) with (33), we observe that this work, in contrast to the work done by the buoyancy force due to changes in temperature, corresponds to the conversion between kinetic and potential energy.[†] On the other hand, we recognize that the first integrand on the right-hand side of (33) corresponds to the conversion between potential and internal energy; it appears, with an opposite sign, in the last term of (39).

Finally, it is interesting to note the following fact: suppose first that $\beta_0 = 0$ in the above discussion; then, if $-c'$ is regarded as the temperature deviation, the resulting energy budget is formally the same as that of Mihaljan (1962).

4. Summary and discussion

The energy budget of a fluid has been investigated, in a manner consistent with the conservation law of mass, under the Boussinesq approximation. It has been shown that no potential energy is available under the approximation. It has also turned out that the work done by the buoyancy force due to changes in temperature corresponds to the conversion between kinetic and internal energy. This energy conversion, however, is altogether neglected in the determination of the distribution of temperature under the approximation.

In contrast, under the oceanographic Boussinesq approximation, the work done by the buoyancy force due to changes in salinity represents the conversion between kinetic and potential energy. In addition, energy conversion also arises between potential and internal energy under the approximation.

In the discussion of the oceanographic Boussinesq approximation, we found that in general $\nabla \cdot \boldsymbol{u} \neq 0$. If we had assumed that $\nabla \cdot \boldsymbol{u} = 0$ together with (31), then the diffusion flux of mass would have arisen. The occurrence of this flux

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[†] This work, however, comes to correspond to the conversion between kinetic and *internal* energy in the limit $c' \to 0$ (see the appendix; see also Maruyama 2022, 2025).

is physically unacceptable, as stated in § 1. Thus it follows that we cannot take the condition $\nabla \cdot \boldsymbol{u} = 0$ under the oceanographic Boussinesq approximation.

Let us examine what condition should then be taken in place of $\nabla \cdot \boldsymbol{u} = 0$. The substitution of (28) and (31) into (29) leads to

$$\rho_0 \frac{Dc'}{Dt} = -\nabla \cdot \boldsymbol{i}, \tag{43}$$

where terms above the first order in primed variables have been neglected. We can similarly find from (32) the following equation:

$$\beta_c \frac{Dc'}{Dt} = -\nabla \cdot \boldsymbol{u}. \tag{44}$$

From (43) and (44), we get

$$\nabla \cdot \boldsymbol{u} = (\beta_c/\rho_0) \nabla \cdot \boldsymbol{i}. \tag{45}$$

We have thus obtained the condition to be imposed in place of $\nabla \cdot \boldsymbol{u} = 0$ under the oceanographic Boussinesq approximation; this condition guarantees that $\boldsymbol{u}$ is the momentum of unit mass of fluid.

Appendix. Work done by the buoyancy force due to changes in salinity

In § 3, we observed that the work done by the buoyancy force due to changes in salinity corresponds to the conversion between kinetic and *potential* energy. However, as explained below, this work comes to correspond to the conversion between kinetic and *internal* energy as the salinity deviation c' tends to zero.

Let us go back to the expression (31). It shows that, in the limit $c' \to 0$,

$$\rho \to \rho_0. \tag{A1}$$

Accordingly, in the same limit, we have

$$\frac{d}{dt} \int_\Omega \rho g x_3 dV \to 0. \tag{A2}$$

Then it follows from (33) that

$$\int_\Omega \rho_0 g u_3 dV \to -\int_\Omega \rho_0 \beta_c c' g u_3 dV. \tag{A3}$$

We can therefore approximate (39), in the limit $c' \to 0$, as follows:

$$\frac{d}{dt} \int_\Omega \rho e dV = \int_\Omega \tau_{ij} \frac{\partial u_i}{\partial x_j} dV - \int_\Sigma \boldsymbol{q} \cdot \boldsymbol{n} dS - \int_\Omega (\rho_0 \beta_0 T' g u_3 - \rho_0 \beta_c c' g u_3) dV. \tag{A4}$$

Comparing this with (41), we recognize that, in the limit $c' \to 0$, the work done by the buoyancy force due to changes in salinity corresponds to the conversion between kinetic and internal energy.

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