Extension of the isothermal anelastic approximation to a two-component fluid

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Abstract

This paper extends a variant of the anelastic approximation to a two-component fluid. The energetics and the applicability of this extended approximation are also discussed, together with the relation to the Boussinesq approximation extended to a two-component fluid.

1. Introduction

Ogura & Phillips (1962) devised an approximation called the anelastic approximation in order to study the motion under gravity of a deep layer of ideal gas with an isentropic basic state. Maruyama (2021a) reconstructed the approximation in such a manner that it can be applied to any kind of fluid with an isentropic basic state.

Maruyama (2021b) further introduced a variant of the anelastic approximation which deals with a fluid having an isothermal basic state. As a limiting case of this isothermal anelastic approximation, the well-known Boussinesq approximation can be obtained

The objective of this paper is to extend the isothermal anelastic approximation to a two-component fluid: the energetics of the extended approximation and the conditions for its applicability are also fully clarified; its relation to the Boussinesq approximation extended to a two-component fluid (Maruyama 2019) is discussed as well.

2. Extended isothermal anelastic approximation

We consider the motion of an inviscid fluid consisting of two components, $\mathcal{A}$ and $\mathcal{B}$, in a uniform gravitational field. The concentration of component $\mathcal{A}$ is denoted by c; the mass of $\mathcal{A}$ in a unit volume of the fluid is given by ρc, with ρ being the density of the fluid. The fluid is contained in a fixed finite domain Ω, and, in this domain, the z-axis is taken vertically upwards. We denote by $\boldsymbol{k}$ the unit vector in the positive z-direction.

2.1. Equation of motion

We first write down the equation of motion for the fluid as follows:

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla p/\rho - g\boldsymbol{k}, \tag{2.1}$$

where D/Dt stands for the material derivative, and $\boldsymbol{u}$ denotes the velocity of the fluid; p is the pressure of the fluid, and g the acceleration due to gravity.

Let φ denote the specific Gibbs free energy of the fluid: it satisfies the relation

$$d\varphi = -sdT + vdp + \mu dc, \tag{2.2}$$

in which s and T are the specific entropy and the temperature of the fluid, respectively; $v = 1/\rho$ denotes the specific volume of the fluid, and μ is the chemical potential of the fluid (see Landau & Lifshitz 1987, § 58). In the following, all thermodynamic quantities are regarded as known functions of φ, T and c.

Using (2.2), we can rewrite $\nabla p/\rho$ as follows:

$$\nabla p/\rho = \nabla\varphi + s\nabla T - \mu\nabla c. \tag{2.3}$$

The substitution of (2.3) into the equation of motion (2.1) yields

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla\varphi - s\nabla T + \mu\nabla c - g\boldsymbol{k}. \tag{2.4}$$

Now, let us decompose φ, T, and c as follows:

$$\varphi = \varphi_0 + \varphi', \quad T = T_0 + T', \quad c = c_0 + c'. \tag{2.5}$$

Here φ_0, T_0, and c_0 are defined by

$$\varphi_0 = -gz + \alpha_1, \quad T_0 = \alpha_2, \quad c_0 = \alpha_3, \tag{2.6}$$

with α_1, α_2, and α_3 being constants. Then (2.4) takes the following form:

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla\varphi' - s\nabla T' + \mu\nabla c'. \tag{2.7}$$

Under the decomposition (2.5), s and μ can also be decomposed as follows:

$$s = s_0 + s', \quad \mu = \mu_0 + \mu', \tag{2.8}$$

where s_0 and μ_0 are given by

$$s_0 = s(\varphi_0, T_0, c_0), \quad \mu_0 = \mu(\varphi_0, T_0, c_0). \tag{2.9}$$

We introduce here the following assumptions:

$$|s'/s_0| \ll 1, \quad |\mu'/\mu_0| \ll 1. \tag{2.10}$$

Then the equation of motion (2.7) may be approximated as

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla\varphi' - s_0\nabla T' + \mu_0\nabla c'. \tag{2.11}$$

Since $s_0\nabla T' = \nabla(s_0T') - T'\nabla s_0$ and $\mu_0\nabla c' = \nabla(\mu_0 c') - c'\nabla\mu_0$, we have

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla(\varphi' + s_0T' - \mu_0c') + T'\nabla s_0 - c'\nabla\mu_0. \tag{2.12}$$

This is the equation of motion under the extended isothermal anelastic approximation.

The second term on the right-hand side of (2.12) represents the buoyancy force that arises from changes in temperature. Regarded as a function of φ, T, and c, s satisfies

$$\begin{aligned}
&(\partial s/\partial\varphi)_{T,c} = -\beta,\\
&(\partial s/\partial T)_{\varphi,c} = T/c_p - \beta s,\\
&(\partial s/\partial c)_{\varphi,T} = \beta\mu - (\partial\mu/\partial T)_{p,c},
\end{aligned} \tag{2.13}$$

where $\beta = v^{-1}(\partial v/\partial T)_{p,c}$ is the thermal expansion coefficient, and c_p the specific heat at constant pressure. Hence, in view of (2.6) and (2.9), we can write

$$\nabla s_0 = (\partial s/\partial\varphi)_{T,c}|_{(\varphi_0,T_0,c_0)}\nabla\varphi_0 = \beta_0 g\boldsymbol{k}, \tag{2.14}$$

in which the following notation has been introduced:

$$\beta_0 = \beta(\varphi_0, T_0, c_0). \tag{2.15}$$

The second term on the right-hand side of (2.12) can therefore be rewritten as

$$T'\nabla s_0 = \beta_0 T' g\boldsymbol{k}. \tag{2.16}$$

The third term on the right-hand side of (2.12) represents the buoyancy force due to changes in concentration. The chemical potential μ in this term satisfies the relations

$$\begin{aligned}
&(\partial\mu/\partial\varphi)_{T,c} = \rho(\partial\mu/\partial p)_{T,c} = -\beta_c,\\
&(\partial\mu/\partial T)_{\varphi,c} = (\partial\mu/\partial T)_{p,c} + \rho s(\partial\mu/\partial p)_{T,c},\\
&(\partial\mu/\partial c)_{\varphi,T} = (\partial\mu/\partial c)_{T,p} - \rho\mu(\partial\mu/\partial p)_{T,c},
\end{aligned} \tag{2.17}$$

where $\beta_c = \rho^{-1}(\partial\rho/\partial c)_{T,p}$. Thus $\nabla\mu_0$ can be rewritten as

$$\nabla\mu_0 = (\partial\mu/\partial\varphi)_{T,c}|_{(h_0,s_0,c_0)}\nabla\varphi_0 = \beta_{c0} g\boldsymbol{k}, \tag{2.18}$$

where $\beta_{c0} = \beta_c(\varphi_0, T_0, c_0)$. As a result, we obtain the following expression:

$$-c'\nabla\mu_0 = -\beta_{c0}c' g\boldsymbol{k}. \tag{2.19}$$

2.2. Equation of continuity

Under the decomposition (2.5), the density ρ of the fluid can be written as

$$\rho = \rho_0 + \rho', \tag{2.20}$$

where ρ_0 is defined by

$$\rho_0 = \rho(\varphi_0, T_0, c_0). \tag{2.21}$$

We introduce here the following assumption:

$$|\rho'/\rho_0| \ll 1. \tag{2.22}$$

On this assumption, ρ may be approximated as follows:

$$\rho = \rho_0. \tag{2.23}$$

Substituting (2.23) into the equation of continuity $\partial\rho/\partial t + \nabla\cdot(\rho\boldsymbol{u}) = 0$, we get

$$\nabla\cdot(\rho_0\boldsymbol{u}) = 0. \tag{2.24}$$

This is the equation of continuity under the extended approximation.

2.3. Concentration equation

In the absence of diffusion, the equation for the rate of change of the concentration c of component $\mathcal{A}$ is given by (see Landau & Lifshitz 1987, § 58)

$$\rho\frac{Dc}{Dt} = 0. \tag{2.25}$$

Considering (2.5) and (2.23), we can approximate this equation as follows:

$$\rho_0\frac{Dc'}{Dt} = 0. \tag{2.26}$$

2.4. General equation of heat transfer

The general equation of heat transfer (see Landau & Lifshitz 1987, § 58) takes, when the conduction of heat is neglected, the following form:

$$\rho T\frac{Ds}{Dt} + \rho\mu\frac{Dc}{Dt} = 0. \tag{2.27}$$

To the first order of primed variables, this equation can be written as

$$\rho_0 T_0\boldsymbol{u}\cdot\nabla s_0 + \rho_0 T_0\frac{Ds'}{Dt} + \rho_0 T'\boldsymbol{u}\cdot\nabla s_0 + \rho_0\mu_0\frac{Dc'}{Dt} = 0, \tag{2.28}$$

where (2.23) has been used. Using (2.13), we can further rewrite s' in (2.28) as follows:

$$\begin{aligned} s' &= (\partial s/\partial\varphi)_{T,c}|_{(\varphi_0,T_0,c_0)}\varphi' + (\partial s/\partial T)_{\varphi,c}|_{(\varphi_0,T_0,c_0)}T' + (\partial s/\partial c)_{\varphi,T}|_{(\varphi_0,T_0,c_0)}c' \\ &= -\beta_0\varphi' + (c_{p0}/T_0 - \beta_0 s_0)T' + \left\{\beta_0\mu_0 - (\partial\mu/\partial T)_{p,c}|_{(\varphi_0,T_0,c_0)}\right\}c'. \end{aligned} \tag{2.29}$$

Here the following notation has been introduced:

$$c_{p0} = c_p(\varphi_0, T_0, c_0). \tag{2.30}$$

2.5. Alternative forms of the equation of motion

We have thus fully formulated the isothermal anelastic approximation extended to a two-component fluid. For later reference, however, it is useful to rewrite the equation of motion (2.12) in somewhat different forms.

We first note that, under the decomposition (2.5), the pressure p of the fluid can also be decomposed as follows:

$$p = p_0 + p', \tag{2.31}$$

where p_0 is defined by

$$p_0 = p(\varphi_0, T_0, c_0). \tag{2.32}$$

On the other hand, the following thermodynamic relations are obtained from (2.2):

$$(\partial p/\partial\varphi)_{T,c} = \rho, \quad (\partial p/\partial T)_{\varphi,c} = \rho s, \quad (\partial p/\partial c)_{\varphi,T} = -\rho\mu. \tag{2.33}$$

Hence p' can be expressed, to the first order of φ', T', and c', as follows:

$$\begin{aligned} p' &= (\partial p/\partial\varphi)_{T,c}|_{(\varphi_0,T_0,c_0)}\varphi' + (\partial p/\partial T)_{\varphi,c}|_{(\varphi_0,T_0,c_0)}T' + (\partial p/\partial c)_{\varphi,T}|_{(\varphi_0,T_0,c_0)}c' \\ &= \rho_0\varphi' + \rho_0 s_0 T' - \rho_0\mu_0 c'. \end{aligned} \tag{2.34}$$

This expression enables us to rewrite (2.12) in the following form:

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla(p'/\rho_0) + T'\nabla s_0 - c'\nabla\mu_0. \tag{2.35}$$

The first term on the right-hand side, however, can further be rewritten as

$$-\nabla(p'/\rho_0) = -\nabla p'/\rho_0 + (p'/\rho_0)(\nabla\rho_0/\rho_0). \tag{2.36}$$

The substitution of (2.36) into (2.35) yields

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla p'/\rho_0 + (p'/\rho_0)(\nabla\rho_0/\rho_0) + T'\nabla s_0 - c'\nabla\mu_0. \tag{2.37}$$

Moreover, since ρ_0 is defined by (2.21), we can write

$$\nabla\rho_0/\rho_0 = (\partial\rho/\partial\varphi)_{T,c}|_{(\varphi_0,T_0,c_0)}\nabla\varphi_0/\rho_0 = -(\partial\rho/\partial\varphi)_{T,c}|_{(\varphi_0,T_0,c_0)}(g/\rho_0)\boldsymbol{k}. \tag{2.38}$$

Thus, in view of (2.34), the following expression for $(p'/\rho_0)(\nabla\rho_0/\rho_0)$ is obtained:

$$\begin{aligned} (p'/\rho_0)(\nabla\rho_0/\rho_0) &= -(\varphi' + s_0T' - \mu_0c')(\partial\rho/\partial\varphi)_{T,c}|_{(\varphi_0,T_0,c_0)}(g/\rho_0)\boldsymbol{k} \\ &= -\big\{(\partial\rho/\partial\varphi)_{T,c}|_{(\varphi_0,T_0,c_0)}\varphi' \\ &\qquad + s_0(\partial\rho/\partial\varphi)_{T,c}|_{(\varphi_0,T_0,c_0)}T' \\ &\qquad\quad -\mu_0(\partial\rho/\partial\varphi)_{T,c}|_{(\varphi_0,T_0,c_0)}c'\big\}(g/\rho_0)\boldsymbol{k}. \end{aligned} \tag{2.39}$$

The terms $T'\nabla s_0$ and $-c'\nabla\mu_0$ in (2.37) can also be expressed as follows:

$$\begin{aligned} T'\nabla s_0 &= T'(\partial s/\partial\varphi)_{T,c}|_{(\varphi_0,T_0,c_0)}\nabla\varphi_0 = -\rho_0(\partial s/\partial\varphi)_{T,c}|_{(\varphi_0,T_0,c_0)}T'(g/\rho_0)\boldsymbol{k}, \\ -c'\nabla\mu_0 &= -c'(\partial\mu/\partial\varphi)_{T,c}|_{(\varphi_0,T_0,c_0)}\nabla\varphi_0 = \rho_0(\partial\mu/\partial\varphi)_{T,c}|_{(\varphi_0,T_0,c_0)}c'(g/\rho_0)\boldsymbol{k}. \end{aligned} \tag{2.40}$$

Adding these expressions to (2.39), we get

$$\begin{aligned}(p'/\rho_0)(\nabla\rho_0/\rho_0) + T'\nabla s_0 - c'\nabla\mu_0 \\ = -\Big[(\partial\rho/\partial\varphi)_{T,c}|_{(\varphi_0,T_0,c_0)}\varphi' \\ + \{\partial(\rho s)/\partial\varphi\}_{T,c}|_{(\varphi_0,T_0,c_0)}T' \\ -\{\partial(\rho\mu)/\partial\varphi\}_{T,c}|_{(\varphi_0,T_0,c_0)}c'\Big]\,(g/\rho_0)\boldsymbol{k}.\end{aligned} \tag{2.41}$$

From (2.33), however, we observe that the following thermodynamic relations hold:

$$\{\partial(\rho s)/\partial\varphi\}_{T,c} = (\partial\rho/\partial T)_{\varphi,c}, \quad -\{\partial(\rho\mu)/\partial\varphi\}_{T,c} = (\partial\rho/\partial c)_{\varphi,T}. \tag{2.42}$$

These relations enable us to rewrite (2.41) as follows:

$$\begin{aligned}(p'/\rho_0)(\nabla\rho_0/\rho_0) + T'\nabla s_0 - c'\nabla\mu_0 \\ = -\Big\{(\partial\rho/\partial\varphi)_{T,c}|_{(\varphi_0,T_0,c_0)}\varphi' \\ + (\partial\rho/\partial T)_{\varphi,c}|_{(\varphi_0,T_0,c_0)}T' \\ +(\partial\rho/\partial c)_{\varphi,T}|_{(\varphi_0,T_0,c_0)}c'\Big\}\,(g/\rho_0)\boldsymbol{k}.\end{aligned} \tag{2.43}$$

On the other hand, ρ' in (2.20) is, to the first order of φ', T', and c', given by

$$\rho' = (\partial\rho/\partial\varphi)_{T,c}|_{(\varphi_0,T_0,c_0)}\varphi' + (\partial\rho/\partial T)_{\varphi,c}|_{(\varphi_0,T_0,c_0)}T' + (\partial\rho/\partial c)_{\varphi,T}|_{(\varphi_0,T_0,c_0)}c'. \tag{2.44}$$

We see, therefore, that the equation of motion (2.12) can be expressed in the form

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla p'/\rho_0 - (\rho' g/\rho_0)\boldsymbol{k}. \tag{2.45}$$

However, it must be emphasized that, in spite of this result, the fluid density under the present approximation is given by ρ_0, not by $\rho_0 + \rho'$.

2.6. Energetics of the extended approximation

Let us next proceed to study, under the extended isothermal anelastic approximation, the energy balance of the fluid. We first consider the internal energy of the fluid.

Let e denote the specific internal energy of the fluid. We then have the relation

$$e = \varphi - p/\rho + Ts. \tag{2.46}$$

Accordingly, using (2.5), (2.8), (2.23), and (2.31), we obtain, to the first order of primed variables, the following expression for e:

$$e = (\varphi_0 - p_0/\rho_0 + T_0 s_0) + (\varphi' - p'/\rho_0 + T_0 s' + s_0 T'). \tag{2.47}$$

Considering (2.34), however, we can further rewrite e in the following form:

$$e = (\varphi_0 - p_0/\rho_0 + T_0 s_0) + T_0 s' + \mu_0 c'. \tag{2.48}$$

Taking the material derivative of (2.48), and multiplying the result by ρ_0, we get

$$\rho_0\frac{De}{Dt} = \rho_0\boldsymbol{u}\cdot\nabla(\varphi_0 - p_0/\rho_0) + \rho_0 T_0\boldsymbol{u}\cdot\nabla s_0 + \rho_0 T_0\frac{Ds'}{Dt} + \rho_0 c'\boldsymbol{u}\cdot\nabla\mu_0 + \rho_0\mu_0\frac{Dc'}{Dt}. \quad (2.49)$$

This equation, by virtue of (2.24) and (2.28), reduces to

$$\rho_0\frac{De}{Dt} = \nabla\cdot\{\rho_0(\varphi_0 - p_0/\rho_0)\boldsymbol{u}\} - \rho_0 T'\boldsymbol{u}\cdot\nabla s_0 + \rho_0 c'\boldsymbol{u}\cdot\nabla\mu_0. \quad (2.50)$$

However, since $\rho_0 De/Dt = \rho_0\partial e/\partial t + \rho_0\boldsymbol{u}\cdot\nabla e = \partial(\rho_0 e)/\partial t + \nabla\cdot(\rho_0 e\boldsymbol{u})$, we have

$$\frac{\partial}{\partial t}(\rho_0 e) + \nabla\cdot\{\rho_0(e - \varphi_0 + p_0/\rho_0)\boldsymbol{u}\} = -\rho_0 T'\boldsymbol{u}\cdot\nabla s_0 + \rho_0 c'\boldsymbol{u}\cdot\nabla\mu_0. \quad (2.51)$$

Furthermore, in view of (2.48), this equation can be put into the following form:

$$\frac{\partial}{\partial t}(\rho_0 e) + \nabla\cdot\big[\rho_0\{T_0(s_0 + s') + \mu_0 c'\}\boldsymbol{u}\big] = -\rho_0 T'\boldsymbol{u}\cdot\nabla s_0 + \rho_0 c'\boldsymbol{u}\cdot\nabla\mu_0. \quad (2.52)$$

Integrating (2.52) over the domain Ω containing the fluid, we obtain

$$\frac{d}{dt}\int_\Omega \rho_0 e\, dV = -\int_\Omega \big(\rho_0 T'\boldsymbol{u}\cdot\nabla s_0 - \rho_0 c'\boldsymbol{u}\cdot\nabla\mu_0\big)\, dV, \quad (2.53)$$

where it has been assumed that the normal component of $\boldsymbol{u}$ vanishes on the boundary of Ω. This is the equation for the rate of change of the internal energy of the fluid.

The potential energy of the fluid, on the other hand, is invariable:

$$\frac{d}{dt}\int_\Omega \rho_0 gz\, dV = 0. \quad (2.54)$$

This is a logical consequence of the approximation (2.23).

The equation representing the rate of change of the kinetic energy of the fluid can be derived from (2.35): taking the inner product of (2.35) with $\rho_0\boldsymbol{u}$, we get

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho_0|\boldsymbol{u}|^2) + \nabla\cdot\big\{\rho_0(\tfrac{1}{2}|\boldsymbol{u}|^2 + p'/\rho_0)\boldsymbol{u}\big\} = \rho_0 T'\boldsymbol{u}\cdot\nabla s_0 - \rho_0 c'\boldsymbol{u}\cdot\nabla\mu_0; \quad (2.55)$$

integrating (2.55) over the domain Ω, on the assumption that the normal component of $\boldsymbol{u}$ vanishes on the boundary of Ω, we obtain

$$\frac{d}{dt}\int_\Omega \tfrac{1}{2}\rho_0|\boldsymbol{u}|^2 dV = \int_\Omega \big(\rho_0 T'\boldsymbol{u}\cdot\nabla s_0 - \rho_0 c'\boldsymbol{u}\cdot\nabla\mu_0\big)\, dV. \quad (2.56)$$

Maruyama (2021b) demonstrated that, under the isothermal anelastic approximation, the work done by the buoyancy force due to changes in temperature corresponds to the conversion between kinetic and internal energy. We see that this is also the case under the present extended approximation, comparing (2.56) with (2.53). Furthermore, it can be seen from the comparison of (2.56) with (2.53) that the work done by the buoyancy force due to changes in concentration also corresponds to the same energy conversion.

Finally, adding (2.53), (2.54), and (2.56), we have

$$\frac{d}{dt}\int_\Omega \rho_0\left(\tfrac{1}{2}|\boldsymbol{u}|^2 + gz + e\right) dV = 0. \quad (2.57)$$

This equation shows that the total energy of the fluid is conserved. Hence the extended isothermal anelastic approximation is consistent with the conservation law of energy.

2.7. Applicability of the extended approximation

The present approximation was formulated on the assumptions (2.10) and (2.22). In the following, we examine under what conditions these assumptions are justifiable. We first focus attention on the assumption (2.22).

When ρ is regarded as a function of φ, T, and c, the following relations hold:

$$\begin{aligned}
&(\partial\rho/\partial\varphi)_{T,c} = \rho(\gamma/a^2), \\
&(\partial\rho/\partial T)_{\varphi,c} = \rho\left\{s(\gamma/a^2) - \beta\right\}, \\
&(\partial\rho/\partial c)_{\varphi,T} = -\rho\left\{\rho(\partial\mu/\partial p)_{T,c} + \mu(\gamma/a^2)\right\},
\end{aligned} \tag{2.58}$$

where γ denotes the ratio of specific heats, and a the speed of sound. Substituting these relations into (2.44), we obtain

$$\begin{aligned}
\rho' = {}& \rho_0(\gamma_0/a_0^2)\varphi' - \rho_0\left\{s_0(\gamma_0/a_0^2) - \beta_0\right\}T' \\
&- \rho_0\left\{\rho_0(\partial\mu/\partial p)_{T,c}|_{(\varphi_0,T_0,c_0)} + \mu_0(\gamma_0/a_0^2)\right\}c',
\end{aligned} \tag{2.59}$$

in which γ_0 and a_0 are defined by

$$\gamma_0 = \gamma(\varphi_0, T_0, c_0), \quad a_0 = a(\varphi_0, T_0, c_0). \tag{2.60}$$

Now, let the characteristic scales of φ', T', and c' be respectively denoted by $\Delta\varphi'$, $\Delta T'$, and $\Delta c'$. Then, we can find the following estimate for $|\rho'/\rho_0|$:

$$\begin{aligned}
\left|\rho'/\rho_0\right| = {}& O\left\{\gamma_0(gH/a_0^2)(\Delta\varphi'/gH)\right\} \\
&+ O\left\{\gamma_0(gH/a_0^2)(s_0\Delta T'/gH)\right\} \\
&+ O(\beta_0\Delta T') \\
&+ O\left\{(\rho_0 gH/\mu_0)(\partial\mu/\partial p)_{T,c}|_{(\varphi_0,T_0,c_0)}(\mu_0\Delta c'/gH)\right\} \\
&+ O\left\{\gamma_0(gH/a_0^2)(\mu_0\Delta c'/gH)\right\}.
\end{aligned} \tag{2.61}$$

Here H stands for the vertical extent of the domain Ω containing the fluid: we assume that H satisfies the conditions

$$(gH)^{1/2}/a_0 \leq O(1), \quad \left|(\rho_0 gH/\mu_0)(\partial\mu/\partial p)_{T,c}|_{(\varphi_0,T_0,c_0)}\right| \leq O(1). \tag{2.62}$$

It then follows from (2.61) that, since $\gamma_0 = O(1)$, (2.22) holds under the conditions

$$\Delta\varphi'/gH \ll 1, \tag{2.63}$$

$$s_0\Delta T'/gH \ll 1, \tag{2.64}$$

$$\beta_0\Delta T' \ll 1, \tag{2.65}$$

$$|\mu_0|\Delta c'/gH \ll 1. \tag{2.66}$$

We can also find from (2.29) the following estimate for $|s'/s_0|$:

$$\begin{aligned}
|s'/s_0| = {}& O\{(c_{p0}/s_0)(\Gamma_0 H/\Delta T')(\Delta T'/T_0)(\Delta\varphi'/gH)\} \\
&+ O\{(c_{p0}/s_0)(\Delta T'/T_0)\} \\
&+ O(\beta_0\Delta T') \\
&+ O\{(c_{p0}/s_0)(\Gamma_0 H/\Delta T')(\Delta T'/T_0)(\mu_0\Delta c'/gH)\} \\
&+ O\{(c_{p0}/s_0)(T_0/\mu_0)(\partial\mu/\partial T)_{p,c}|_{(\varphi_0,T_0,c_0)}(\mu_0\Delta c'/c_{p0}T_0)\},
\end{aligned} \tag{2.67}$$

where $\Gamma_0 = \beta_0 T_0 g/c_{p0}$ is the adiabatic lapse rate. As pointed out by Maruyama (2021b), it is reasonable to expect that the following conditions are met:

$$c_{p0}/s_0 \leq O(1), \quad \Gamma_0 H/\Delta T' \leq O(1). \tag{2.68}$$

We also assume that

$$\big|(T_0/\mu_0)(\partial\mu/\partial T)_{p,c}|_{(\varphi_0,T_0,c_0)}\big| \leq O(1). \tag{2.69}$$

Then the first assumption of (2.10) is justifiable when the conditions

$$\Delta T'/T_0 \ll 1, \tag{2.70}$$

$$|\mu_0|\Delta c'/c_{p0}T_0 \ll 1 \tag{2.71}$$

hold together with (2.63), (2.65), and (2.66).

On the other hand, using (2.17), we can express μ' as follows:

$$\begin{aligned}\mu' &= \rho_0(\partial\mu/\partial p)_{T,c}|_{(\varphi_0,T_0,c_0)}\varphi' \\ &\quad + \{(\partial\mu/\partial T)_{p,c}|_{(\varphi_0,T_0,c_0)} + \rho_0 s_0(\partial\mu/\partial p)_{T,c}|_{(\varphi_0,T_0,c_0)}\}T' \\ &\quad\quad + \{(\partial\mu/\partial c)_{T,p}|_{(\varphi_0,T_0,c_0)} - \rho_0\mu_0(\partial\mu/\partial p)_{T,c}|_{(\varphi_0,T_0,c_0)}\}c'.\end{aligned} \tag{2.72}$$

This expression yields the following estimate for $|\mu'/\mu_0|$:

$$\begin{aligned}|\mu'/\mu_0| &= O\{(\rho_0 gH/\mu_0)(\partial\mu/\partial p)_{T,c}|_{(\varphi_0,T_0,c_0)}(\Delta\varphi'/gH)\} \\ &\quad + O\{(T_0/\mu_0)(\partial\mu/\partial T)_{p,c}|_{(\varphi_0,T_0,c_0)}(\Delta T'/T_0)\} \\ &\quad\quad + O\{(\rho_0 gH/\mu_0)(\partial\mu/\partial p)_{T,c}|_{(\varphi_0,T_0,c_0)}(s_0\Delta T'/gH)\} \\ &\quad\quad\quad + O\{(gH\mu_0^{-1}/\mu_0)(\partial\mu/\partial c)_{T,p}|_{(\varphi_0,T_0,c_0)}(\mu_0\Delta c'/gH)\} \\ &\quad\quad\quad\quad + O\{(\rho_0 gH/\mu_0)(\partial\mu/\partial p)_{T,c}|_{(\varphi_0,T_0,c_0)}(\mu_0\Delta c'/gH)\}.\end{aligned} \tag{2.73}$$

Accordingly, the second assumption of (2.10) is justifiable when the condition

$$\big|(gH\mu_0^{-1}/\mu_0)(\partial\mu/\partial c)_{T,p}|_{(\varphi_0,T_0,c_0)}\big| \leq O(1) \tag{2.74}$$

is fulfilled in addition to the above conditions.

It should be noted here, however, that the following inequality follows from (2.7):

$$|\nabla\varphi'| \leq |\partial\boldsymbol{u}/\partial t| + |(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}| + |s\nabla T'| + |\mu\nabla c'|. \tag{2.75}$$

Let L denote the length scale characteristic of the motion of the fluid; we then have

$$|\nabla\varphi'| = O(\Delta\varphi'/L), \quad |\nabla T'| = O(\Delta T'/L), \quad |\nabla c'| = O(\Delta c'/L). \tag{2.76}$$

Moreover, if the velocity scale characteristic of the motion is denoted by U, we get

$$|\partial\boldsymbol{u}/\partial t| = O(U/\tau), \quad |(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}| = O(U^2/L), \tag{2.77}$$

in which τ stands for the time scale characteristic of the motion. Now, suppose that α_1, α_2, and α_3 in (2.6) can be chosen so that the following conditions are met:

$$s/s_0 \leq O(1), \quad |\mu/\mu_0| \leq O(1). \tag{2.78}$$

Then it follows from (2.75), (2.76), and (2.77) that, when the conditions

$$U/(gH)^{1/2} \ll 1, \quad (L/\tau)/(gH)^{1/2} \ll 1 \tag{2.79}$$

hold together with (2.64) and (2.66), the condition (2.63) is satisfied automatically.

It can also be shown that (2.65) is redundant (Maruyama 2021b). Thus the extended isothermal anelastic approximation is applicable under the following conditions: (2.62), (2.64), (2.66), (2.68), (2.69), (2.70), (2.71), (2.74), (2.78), and (2.79).

Finally, note that the present approximation is inapplicable to ideal gases. This is, as shown by Maruyama (2021b), because of the condition (2.64).

3. Summary and Discussion

The isothermal anelastic approximation, a variant of the anelastic approximation, has been extended to a two-component fluid. The energetics of the extended approximation and the conditions under which it is applicable have also been elucidated.

3.1. Relation to the Boussinesq approximation extended to a two-component fluid

The Boussinesq approximation is often used to study the motion of a fluid thermally stratified in a uniform gravitational field. This approximation is obtainable as a limiting case of the isothermal anelastic approximation (Maruyama 2021b).

Maruyama (2019) extended, in a manner consistent with the conservation law of mass, the Boussinesq approximation to a two-component fluid. It might be expected that this extended Boussinesq approximation is obtainable from the present extended isothermal anelastic approximation. As explained below, however, this expectation is wrong.

In § 2.6, we have shown that, under the present approximation, the work done by the buoyancy force due to changes in concentration corresponds to the conversion between kinetic and *internal* energy. In contrast, Maruyama (2019) showed that the work under the extended Boussinesq approximation corresponds to the conversion between kinetic and *potential* energy. As a result of this contrast, we arrive at the following conclusion: the extended Boussinesq approximation cannot be reproduced as a limiting case of the present extended isothermal anelastic approximation.

References

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